Optimal Paths in Disordered Complex Networks

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We study the optimal distance in networks, \( \ell_{\text{opt}} \), defined as the length of the path minimizing the total weight, in the presence of disorder. Disorder is introduced by assigning random weights to the links or nodes. For strong disorder, where the maximal weight along the path dominates the sum, we find that \( \ell_{\text{opt}} \sim N^{1/3} \) in both Erdős-Rényi (ER) and Watts-Strogatz (WS) networks. For scale free (SF) networks, with degree distribution \( P(k) \sim k^{-\lambda} \), we find that \( \ell_{\text{opt}} \) scales as \( N^{(\lambda-3)/(\lambda-1)} \) for \( 3 < \lambda < 4 \) and as \( N^{1/3} \) for \( \lambda \geq 4 \). Thus, for these networks, the small-world nature is destroyed. For \( 2 < \lambda < 3 \), our numerical results suggest that \( \ell_{\text{opt}} \) scales as \( \ln^{1-1/\lambda} N \). We also find numerically that for weak disorder \( \ell_{\text{opt}} \sim \ln N \) for both the ER and WS models as well as for SF networks.

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Recently much attention has been focused on the topic of complex networks which characterize many biological, social, and communication systems. The networks can be visualized by nodes representing individuals, organizations, or computers and by links between them representing their interactions.

The classical model for random networks is the Erdős-Rényi (ER) model. An important quantity characterizing networks is the average distance (minimal hopping) \( \ell_{\text{min}} \) between two nodes in the network of total \( N \) nodes. For the Erdős-Rényi network, and the related, more realistic Watts-Strogatz (WS) network, \( \ell_{\text{min}} \) scales as \( \ln N \), which leads to the concept of “six degrees of separation”.

In most studies, all links in the network are regarded as identical and thus the relevant parameter for information flow including efficient routing is \( \ell_{\text{min}} \). In practice, however, the weights (e.g., the quality or cost) of links are usually not equal, and thus the length of the optimal path minimizing the sum of weights is usually longer than the distance. In many cases, the selection of the path is controlled by the sum of weights (e.g., total cost) and this case corresponds to regular or weak disorder. However, in other cases, for example, when a transmission at a constant high rate is needed (e.g., in broadcasting video records over the Internet) the narrowest band link in the path between the transmitter and receiver controls the rate of transmission. This situation—in which one link controls the selection of the path—is called the strong disorder limit. In this Letter we show that disorder or inhomogeneity in the weight of links may increase the distance dramatically, destroying the “small-world” nature of the networks.

To implement the disorder, we assign a weight or “cost” to each link or node. For example, the weight could be the time \( \tau_i \) required to transit the link \( i \). The optimal path connecting nodes A and B is the one for which \( \sum_i \tau_i \) is a minimum. While in weak disorder all links contribute to the sum, in strong disorder one term dominates it. The strong disorder limit may be naturally realized in the vicinity of the absolute zero temperature if passing through a link is an activation process with a random activation energy \( \epsilon_i \) and \( \tau_i = \exp(\beta \epsilon_i) \), where \( \beta \) is inverse temperature. Let us assume that the energy spectrum is discrete and that the minimal difference between energy levels is \( \Delta \epsilon \). It can be easily shown that if \( \beta > \ln 2/\Delta \epsilon \), the value of \( \sum \tau_i \) is dominated by the largest term, \( \tau_{\text{max}} \). Thus if we have two different paths characterized by the sums \( \sum_i \tau_i \) and \( \sum_i \tau_i' \), such that \( \tau_{\text{max}} > \tau_{\text{max}}' \), it follows that \( \sum_i \tau_i > \sum_i \tau_i' \).

To generate ER graphs, we start with \( z N \) links and for each link randomly select from the total \( N(N-1)/2 \) possible pairs of nodes a pair that is connected by this link. The WS network is implemented by placing the \( N \) nodes on a circle. Initially, each node \( i \) is connected with \( z \) nodes \( i \pm 1, i \pm 2, \ldots, i \pm z \) and periodic boundaries are implemented. Thus each node has a degree \( 2z \) and the total number of links is \( zN \). Next we randomly remove a fraction \( p \) of the links and use them to connect randomly-selected pairs of nodes. When \( p = 1 \), we obtain a model very similar to the ER graph.

To generate scale-free (SF) graphs, we employ the Molloy-Reed algorithm, in which each node is first assigned a random integer \( k \) from a power law distribution \( P(k > k_0) \sim (k/k_0)^{-\lambda+1} \), where \( k_0 \) is the minimal number of links for each node. Next we randomly select a node and try to connect each of its \( k \) links with randomly selected \( k \) nodes that still have free positions for links.

We expect that the optimal path length in the weak disorder case will not be considerably different from the shortest path, as found for regular lattices and random graphs. Thus we expect that the scaling for the shortest path \( \ell_{\text{min}} \sim \ln N \) will also be valid for the optimal path in weak disorder, but with a different prefactor depending on the details of the graph.

In the case of strong disorder, we present the following theoretical arguments. Cieplak et al. showed that finding the optimal path between nodes A and B in the...
strong disorder limit is equivalent to the following procedure. First, we sort all \( M \) links of the network in the descending order of their weights, so that the first link in this list has the largest weight. Since the sum of the weights on any path between nodes A and B is dominated by a single link with the largest weight, the optimal path cannot go through the first link in the list, provided there is a path between A and B which avoids this link. Thus the first link in the list can be eliminated and now our problem is reduced to the problem of finding the minimal path on the network of \( M - 1 \) links. We can continue to remove links from the top of the list one by one until we pick a link whose removal destroys the connectivity between A and B. This means that all the remaining paths between A and B go through this singly-connecting or “red” link 13 and all these paths have the same largest weight corresponding to the “red” link. To continue optimization among these paths we must select the paths with the minimal second largest term, minimal third largest term and so on. So we must continue to remove links in the descending order of their weights unless they are “red” until a single path between A and B, consisting of only “red” links remains. Since the assigning of weights to the links is random so is their ordering. Hence the optimization procedure in the strong disorder limit is statistically equivalent to removing the links in random order unless the connectivity between nodes A and B is destroyed.

At the beginning of this process, the chances of losing connectivity by removing a random link are very low, so the process corresponds exactly to diluting the network, which is identical to the percolation model. Only when the concentration of the remaining links approaches the percolation threshold will the chances of removing a singly-connected “red” link become significant, indicating that the optimal path must be on the percolation backbone connecting A and B. Since the network is not embedded in space but has an infinite dimensionality, we expect from percolation theory that loops are not relevant at criticality 14. Thus, the shortest path must also be the optimal path.

We begin by considering the case of the ER graph that, at criticality, is equivalent to percolation on the Cayley tree or percolation at the upper critical dimension \( d_c = 6 \). For the ER graph, it is known that the mass of the incipient infinite cluster \( S \) scales as \( N^{2/3} \). This result can also be obtained in the framework of percolation theory for \( d_c = 6 \). Since \( S \sim R^d_f \) and \( N \sim R^d \) (where \( d_f \) is the fractal dimension and \( R \) the diameter of the cluster), it follows that \( S \sim N^{d_f/d} \) and for \( d_c = 6, d_f = 4 \) 15

\[
S \sim N^{2/3}.
\] (1)

It is also known 15 that, at criticality, at the upper critical dimension, \( S \sim \rho_{d_c} \) with \( d_c = 2, \) 15 and thus

\[
\ell_{\min} \sim \ell_{opt} \sim S^{1/\ell_c} \sim N^{2/3 \ell_c} \sim N^{\nu_{opt}},
\] (2)

where \( \nu_{opt} = 2/3d_c = 1/3 \). We expect that the WS model for large \( N \) and large \( p \) will be in the same universality class as ER.

For SF networks, we can also use the percolation results at criticality. It was found 16 that \( d_t = 2 \) for \( \lambda > 4, d_t = (\lambda - 2)/(\lambda - 3) \) for \( 3 < \lambda < 4, S \sim N^{2/3} \) for \( \lambda > 4, \) and \( S \sim N^{(\lambda - 2)/(\lambda - 1)} \) for \( 3 < \lambda \leq 4 \). Hence, we conclude that

\[
\ell_{min} \sim \ell_{opt} \sim \begin{cases} N^{1/3} & \lambda > 4 \\ N^{(\lambda - 3)/(\lambda - 1)} & 3 < \lambda \leq 4 \end{cases}.
\] (3)

To test these theoretical predictions, we perform numerical simulations in the strong disorder limit by randomly removing links (or nodes) for ER, WS, and SF networks and use the Dijkstra algorithm 17 for the weak disorder case. We also perform additional simulations for the case of strong disorder on ER networks using direct optimization with Dijkstra algorithm of the sum of weights and find results identical to the results obtained by randomly removing links, see Fig. 2a.

Results for weak disorder for WS graphs with different \( p \) are shown in Fig. 1. We propose a scaling formula for \( \ell_{opt} \) similar to the formula derived in 18, 19 for the minimal distance on the WS graphs with a different rewiring probability \( p \)

\[
\ell_{opt} \sim n_0(p, z) F\left(\frac{N}{n_0(p, z)}\right),
\] (4)

where \( n_0(p, z) \sim 1/pz \) is the characteristic graph size at which the crossover from large to small world behavior occurs, \( n_1(z) \sim z \) is a correction factor, and \( F(x) \) is the scaling function

\[
F(x) \sim \begin{cases} \ln x & x \to \infty \\ x & x \to 0 \end{cases}.
\] (5)
The scaling variable $x = N/n_0$ indicates the number of nodes with long range links. As $p \to 0$, this quantity scales as $Npz$. The quantity $n_0(p, z) \sim 1/pz$ indicates a typical short-range neighborhood of a node with long range links. We can think about this graph as an ER graph consisting of $N/n_0$ effective nodes, each representing a typical short-range chain-like neighborhood of size $n_0$. Thus we conclude that an optimal path connecting any two nodes is proportional to $\ln(N/n_0)$, as in an ER graph, times an average path length through a chain of short-range links. This average path is proportional to the length of this chain $n_0$ and inversely proportional to the average range $n_1(z)$ of a link in this chain. Ideally $n_1(z) = z$, but in reality it can significantly deviate from $z$ due to finite size effects. Figure 1 shows the scaled optimal path $\ell_{\text{opt}}/n_0$ versus the scaled variable $N/n_0$ for $z = 2$ and different values of $p$. The inset in Fig. 1 shows that $n_0 \sim 1/pz$ as $p \to 0$.

In contrast, for Eq. (1) to be in agreement with Eq. (2),

$$F(x) \sim \begin{cases} x^{1/3} & x \to \infty \\ x & x \to 0 \end{cases}. \quad (6)$$

For large enough $z$ and $p \to 1$, we recover ER network for which $\ell_{\text{opt}}$ does not depend on $z$. Thus we can assume $n_0(1, z) = 1$. Using similar scaling arguments as in case of weak disorder, we assume that as $p \to 0$, $\ell_{\text{opt}} \sim z^{-2/3}N$, and hence $n_1(z) \sim z^{2/3}$.

For SF networks, the behavior of the optimal path in the weak disorder limit is shown in Fig. 3 for different degree distribution exponents $\lambda$. Here we plot $\ell_{\text{opt}}$ as a function of $\ln N$. All the curves have linear asymptotes, but the slopes depend on $\lambda$,

$$\ell_{\text{opt}} \sim f(\lambda) \ln N. \quad (7)$$

This result is analogous to the behavior of the shortest path $\ell_{\text{min}} \sim \ln N$ for $3 < \lambda < 4$. However, for $2 < \lambda < 3$, for the strong disorder limit, we have (see Fig. 2),

$$\ell_{\text{opt}}(z) \sim f(\lambda) \ln N. \quad (7)$$

For large enough $z$ and $p \to 1$, we recover ER network for which $\ell_{\text{opt}}$ does not depend on $z$. Thus we can assume $n_0(1, z) = 1$. Using similar scaling arguments as in case of weak disorder, we assume that as $p \to 0$, $\ell_{\text{opt}} \sim z^{-2/3}N$, and hence $n_1(z) \sim z^{2/3}$.

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$\ell_{\text{opt}}$ scales as $\ln \ln N$ [20] while $\ell_{\text{opt}}$ is significantly larger and scales as $\ln N$ (Fig. 3b). Thus weak disorder does not change the universality class of the length of the optimal path except in the case of “ultra-small” worlds $2 < \lambda < 3$.

In contrast, strong disorder dramatically changes the universality class of the optimal path. Theoretical considerations [Eqs. 2 and 3] predict that in the case of WS and ER (Fig. 2) and SF graphs with $\lambda > 4$, $\ell_{\text{opt}} = N^{1/2}$, while for SF graphs with $3 < \lambda < 4$, $\ell_{\text{opt}} \sim N^{(\lambda-3)/(\lambda-1)}$. Figure 4a shows the linear behavior of $\ell_{\text{opt}}$ versus $N^{1/3}$ for $\lambda > 4$. The quality of the linear fit becomes poor for $\lambda \to 4$. At this value, the logarithmic divergence of the second moment of the expected length of the optimal path except in the case of “ultra-small” worlds $2 < \lambda < 3$.

In summary, we study the optimal distance in ER, WS, and SF networks in the presence of strong and weak disorder. We find that in ER and WS networks and for strong disorder, the optimal distance $\ell_{\text{opt}}$ scales as $N^{1/3}$. We also study the strong disorder limit in SF networks theoretically and by simulations and find that $\ell_{\text{opt}}$ scales as $N^{1/3}$ for $\lambda > 4$ and as $N^{(\lambda-3)/(\lambda-1)}$ for $3 < \lambda < 4$. Thus, the optimal distance increases dramatically in strong disorder when it is compared to the known small world result $\ell_{\text{min}} \sim \ln N$ and the “small world” nature for these networks is destroyed. Our simulations also suggest that for $2 < \lambda < 3$, $\ell_{\text{opt}}$ scales as $\ln^{\lambda-1} N$, which is also much faster than the “ultra-small world” result $\ell_{\text{min}} \sim \ln(\ln N)$ [20]. We also find numerically that in weak disorder $\ell_{\text{opt}} \sim \ln N$ in all types of networks studied.

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[8] We simulate weak disorder by selecting $0 \leq \tau_i < 1$ from a uniform distribution. We simulate strong disorder by selecting $0 \leq \epsilon_i < 1$ from a uniform distribution, using a 48-bit random number generator, so that there are no two identical values of $\epsilon_i$ in a system of any size that we study. In this case $\Delta \epsilon \geq 2^{-48}$ and we can select $\beta \geq 2^{48}$ to guarantee the strong disorder limit.