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Scaling for the critical percolation backbone

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We study the backbone connecting two given sites of a two-dimensional lattice separated by an arbitrary distance r in a system of size L at the percolation threshold. We find a scaling form for the average backbone mass: $\langle M_B\rangle \sim L^{d_B} G(r/L)$, where G can be well approximated by a power law for $0 \leqslant x \leqslant 1$: $G(x) \sim x^{\psi}$ with $\psi = 0.37 \pm 0.02$. This result implies that $\langle M_B\rangle \sim L^{d_B-\psi} r^{\psi}$ for the entire range $0 < r < L$. We also propose a scaling form for the probability distribution $P(M_B)$ of backbone mass for a given r. For $r \approx L$, $P(M_B)$ is peaked around L^{d_B}, whereas for $r \ll L$, $P(M_B)$ decreases as a power law, $M_B^{-\tau_B}$, with $\tau_B \simeq 1.20 \pm 0.03$. The exponents ψ and τ_B satisfy the relation $\psi = d_B(\tau_B - 1)$, and ψ is the codimension of the backbone, $\psi = d - d_B$. [S1063-651X(99)51408-6]

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I. INTRODUCTION AND MOTIVATION

The percolation problem is a classical model of phase transitions, as well as a useful model for describing connectivity phenomena, and in particular for describing porous media [1–3]. At the percolation threshold p_c, the mass of the largest cluster scales with the system size L as $M \sim L^{d_f}$. The fractal dimension d_f is related to the space dimension d and to the order parameter and correlation length exponents β and ν by $d_f = d - \beta/\nu$ [1–3]. In two dimensions, $d_f = 91/48$ is known exactly.

An interesting subset of the percolation cluster is the backbone that is obtained by removing the non-current-carrying bonds from the percolation cluster [4]. The structure of the backbone consists of blobs and links [1,5–7]. The backbone can in fact be further partitioned into subsets according to the magnitude of the electric current carried [8,9]. The backbone is relevant to transport properties [1–3] and fracture [10]. The fractal dimension d_B of the backbone can be defined via its typical mass M_B, which scales with the system size L as $M_B \sim L^{d_B}$. The backbone dimension is an independent exponent and its exact value is not known. A current numerical estimate [11] is $d_B = 1.6432 \pm 0.0008$.

The operational definition of the backbone has an interesting history [1–3]. Customarily, one defines the backbone using parallel bars, and looks for the percolation cluster (and the backbone) that connects the two sides of the system [4]. A different situation arises in oil field applications [12], where one studies the backbone connecting two wells separated by an *arbitrary* distance r. This situation is important for transport properties, since in oil recovery one injects water at one point and recovers oil at another point [12]. From a fundamental point of view, it is important to understand how the percolation properties depend on different boundary conditions.

We study here the backbone connecting two points separated by an arbitrary distance r in a two-dimensional system of linear size L. One goal is to understand the distribution of the backbone mass $M_B(r,L)$, and how its average value scales with r and L in the entire range $0 < r < L$.

II. MODEL

We choose two sites A and B belonging to the infinite percolating cluster on a two-dimensional square lattice (the fraction of bonds is $p = p_c = 1/2$). A and B are separated by a distance r and symmetrically located between the boundaries. Using the burning algorithm [13], we determine the backbone connecting these two points for values of L ranging from 100 to 1000. For each value of L, we consider a sequence of values of r with $2 \leqslant r \leqslant L-2$. In order to test the universality of the exponents, we perform our study on three lattices: square, honeycomb, and triangular lattice. For simplicity, we restrict our discussion here to the square lattice, as we find similar results for the other two lattices.

III. BACKBONE MASS PROBABILITY DISTRIBUTION

We begin by studying the backbone mass probability distribution $P(M_B)$. We show that $P(M_B)$ obeys a simple scaling form in the entire range of r/L,

$$P(M_B) \sim \frac{1}{r^{d_B}} F_{r/L}\left(\frac{M_B}{r^{d_B}}\right), \tag{1}$$

where F is a scaling function whose shape depends on the ratio r/L.

For $r \approx L$, it seems reasonable to assume that $P(M_B)$ will be peaked around its average value $\langle M_B\rangle \sim L^{d_B}$. The data collapse predicted by Eq. (1) is represented in Fig. 1(a). In this case, the scaling function F is peaked at approximately L^{d_B}.

However, the case $r \ll L$ is less clear. In fact, we expect for $r \ll L$ that the backbone mass fluctuates greatly from one realization to another, since its minimum value can be r and its maximum can be of order L^{d_f}. Figure 1(b) shows a log-log plot of $P(M_B)$ and the straight line suggests that $P(M_B) \sim M_B^{-\tau_B}$. It has a lower cutoff of order r (since the backbone must connect points A and B) and an upper cutoff of order L^{d_B}. We find good data collapse [Fig. 1(c)], which

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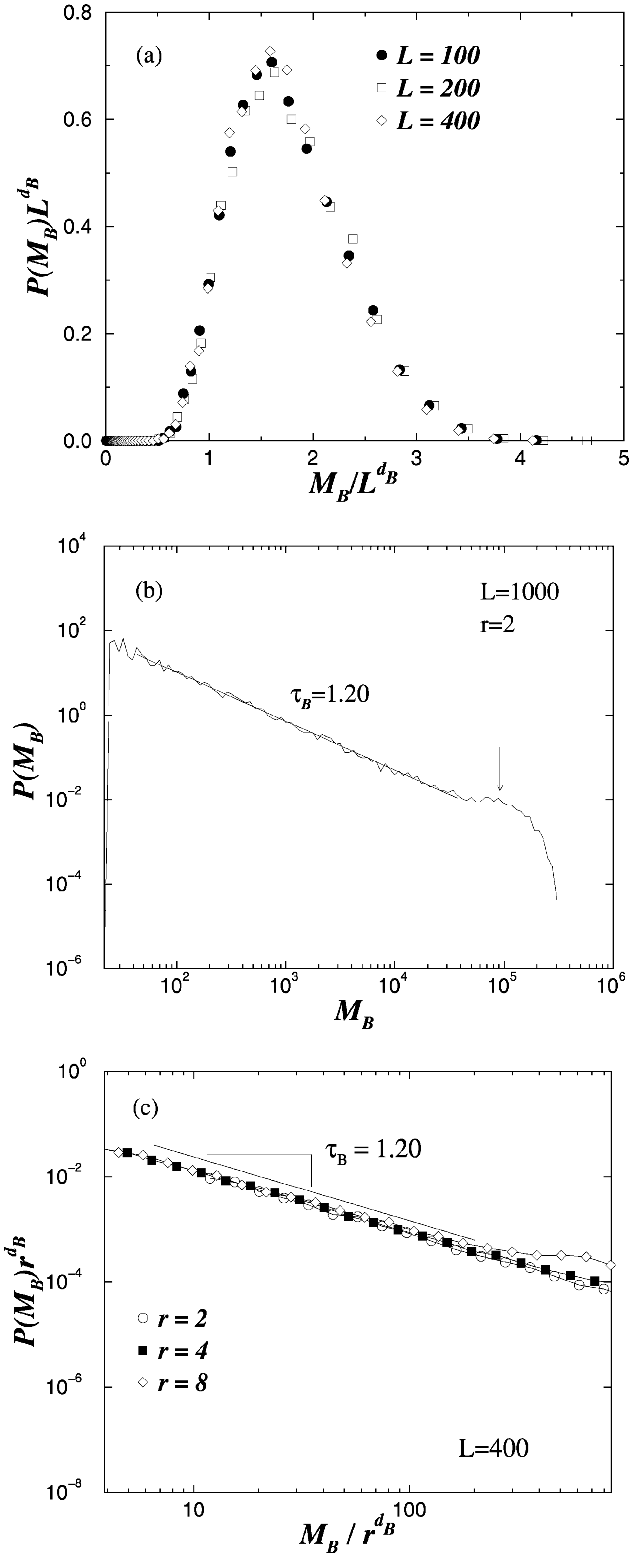

FIG. 1. (a) Data collapse of $P(M_B)$ using Eq. (1) for three different values of $r\approx L$. (b) Probability distribution of the backbone mass for $L=1000$ and $r=2$ (computed with 10^5 configurations). The exponent $\tau_B=1.20\pm0.03$ is obtained by a linear fit over the range $30<M_B<3\times10^4$. The arrow denotes the fact that M_B peaks at L^{d_B}. (c) Data collapse of $P(M_B)$ for $L=400$ using Eq. (1) for three different values of r.

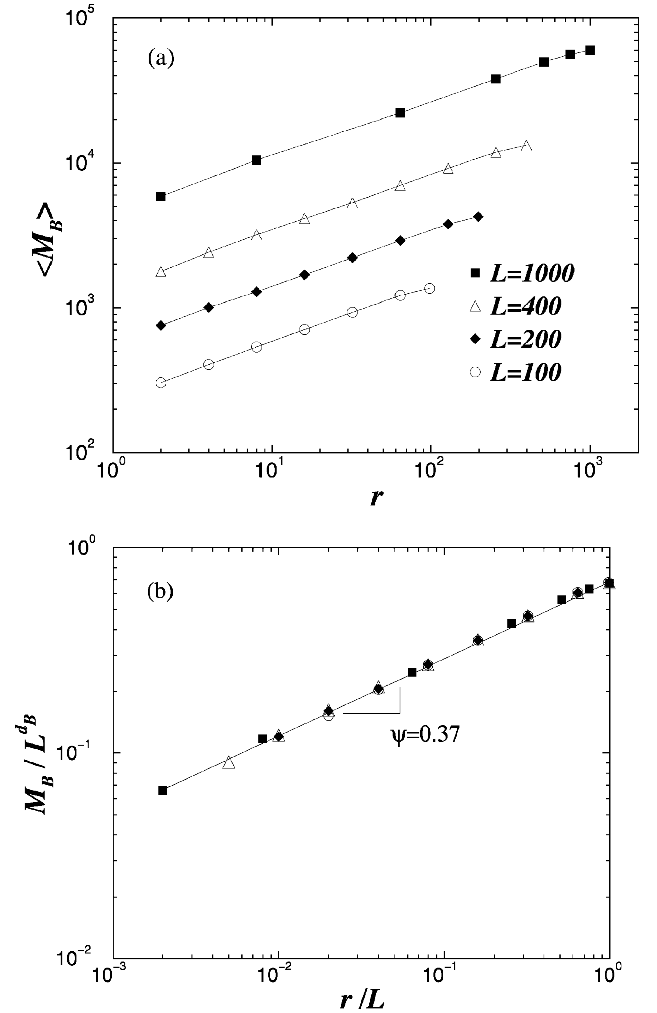

FIG. 2. (a) Log-log plot of the average backbone mass $\langle M_B\rangle$ vs r for four different values of L. (b) Data from Fig. 2(a) collapsed with the use of the scaling form proposed in Eq. (3). The error on ψ is typically 0.02.

indicates that the scaling function F_0 is a power law in the range from r^{d_B} to L^{d_B}, with exponent approximately $\tau_B=1.20\pm0.03$ (there is a cutoff at $M_B\sim L^{d_B}$ not shown here). We note that for larger values of r, a ''bump'' [indicated by an arrow on Fig. 1(b)] located at approximately L^{d_B} appears and assumes increasing importance when r approaches L. This bump appears also in Fig. 1(c) already for $r=8$.

The exponent τ_B is connected to the blob size distribution [5] since typically, the two sites belong to the same blob, and the sampling of backbones is equivalent to sampling of the blobs. In [5], there is a relation between the exponent τ governing the blob size distribution and the fractal dimension of the backbone $d/d_B=\tau-1$. The exponent τ_B governs the variation of the whole backbone mass, and is therefore obtained by integration of the blob size distribution. We thus have $\tau_B=\tau-1$, which implies

$$\frac{d}{d_B}=\tau_B, \tag{2}$$

This relation gives the estimate $\tau_B \simeq 1.22$, which is in good agreement with our numerical simulation.

IV. AVERAGE BACKBONE MASS

We now study the average backbone mass $\langle M_B \rangle$. From dimensional considerations, the r dependence can only be a function of r/L. We thus propose the following *Ansatz*:

$$\langle M_B(r,L) \rangle \sim L^{d_B} G\left(\frac{r}{L}\right). \tag{3}$$

In Fig. 2(a), we show a double logarithmic scale M_B versus r for different values of L. In order to test Eq. (3), we scale the data of Fig. 2(a). The data collapse is obtained using $d_B = 1.65$ and is shown on Fig. 2(b). This (log-log) plot supports the scaling Ansatz (3). Moreover, one can see that the scaling function G is, surprisingly, a pure power law on the entire range [0,1], with exponent $\psi = 0.37 \pm 0.02$. This result leads to the following interesting behavior for the average mass:

$$\langle M_B(r,L) \rangle \sim L^{d_B - \psi} r^{\psi}. \tag{4}$$

The results (1) and (3) are consistent, as we will show. The average mass is given by

$$\langle M_B(r,L) \rangle \sim \int_r^{L^{d_B}} F_{r/L}\left(\frac{M}{r^{d_B}}\right) \frac{dM}{r^{d_B}} M. \tag{5}$$

A. The case $r \approx L$

In the case where $r \approx L$, the function $F_1(x)$ is peaked around L^{d_B} and we obtain

$$\langle M_B(r,L) \rangle \sim L^{d_B}, \tag{6}$$

which is consistent with Eq. (3).

B. The case $r \ll L$

When $r \ll L$, the scaling function $F_{r/L}$ has now a power law behavior $F_0(x) \sim x^{-\tau_B}$ for $x > 1$, and $F_0(x) = 0$ for $x < 1$. The average mass is then given by

$$\langle M_B(r,L) \rangle \sim \int_{r^{d_B}}^{L^{d_B}} \left(\frac{M}{r^{d_B}}\right)^{-\tau_B} \frac{dM}{r^{d_B}} M. \tag{7}$$

Assuming that L/r is large enough, the integral in Eq. (5) can be approximated as $L^{d_B - \psi} r^{\psi}$, where

$$\psi = d_B(\tau_B - 1). \tag{8}$$

In our simulation $\tau_B = 1.20 \pm 0.03$, which leads to the value $\psi = 0.33 \pm 0.05$, in reasonable agreement with the value measured directly on the average mass.

Moreover, using Eq. (2) together with Eq. (8), we obtain

$$\psi = d - d_B, \tag{9}$$

which means that ψ is the codimension of the fractal backbone.

V. SUMMARY

To summarize, we find that for any value of r/L, the scaling form, Eq. (1), for the probability distribution is valid. The shape of the scaling function F depends on r/L, being a peaked distribution for $r \approx L$, and a power law for $r \ll L$. The average backbone mass varies with r and L according to Eq. (5). For fixed system size, it varies as $\langle M_B \rangle \sim r^{\psi}$ (for $0 < r < L$). The value of ψ is small ($\psi \approx 0.37$), indicating that the backbone mass does not change drastically as r changes. On the other hand, the exponent governing the variation of $\langle M_B \rangle$ with L for fixed r is expected to be larger, with $\langle M_B \rangle \sim L^{d_B - \psi}$. This exponent $d_B - \psi$ is not equal to the fractal dimension d_B of the backbone, but is smaller by an amount equal to ψ.

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