Application of statistical physics to random graph models of networks

Sameet Sreenivasan

Advisor: H. Eugene Stanley

Application of statistical physics to random graph models of networks

Three questions addressed in thesis:

1. Scaling of optimal path length on disordered networks ?
2. Resilience of networks to random failures ?
3. Structural bounds on communication in networks ?

Earlier work on networks:

Erdos and Renyi (1954)

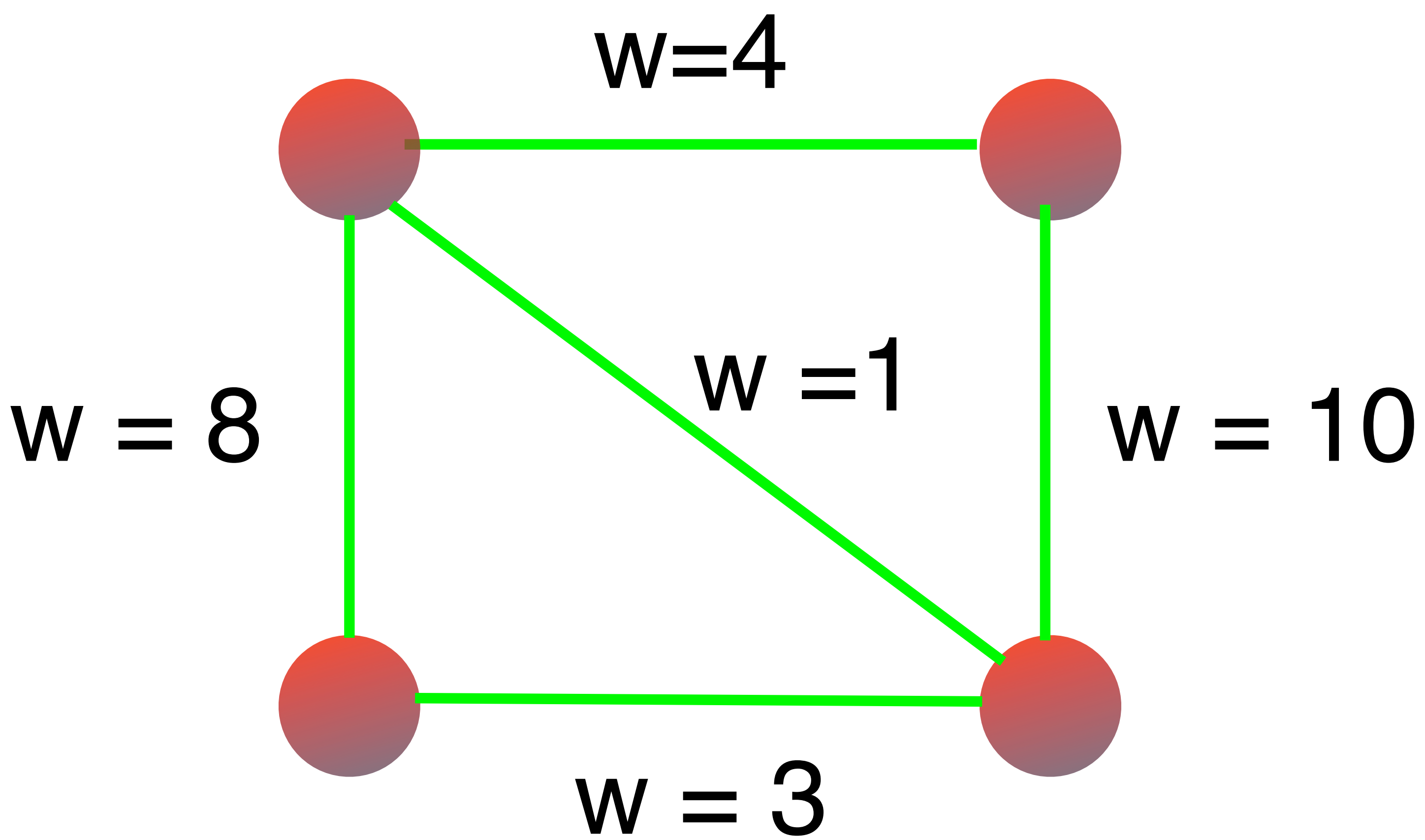
Broadbent and Hammersley (1959)

Watts and Strogatz (1998)

Barabasi and Albert (1999)

Definitions

Disordered Network: A network on which every link *i* has an associated link weight w_i .

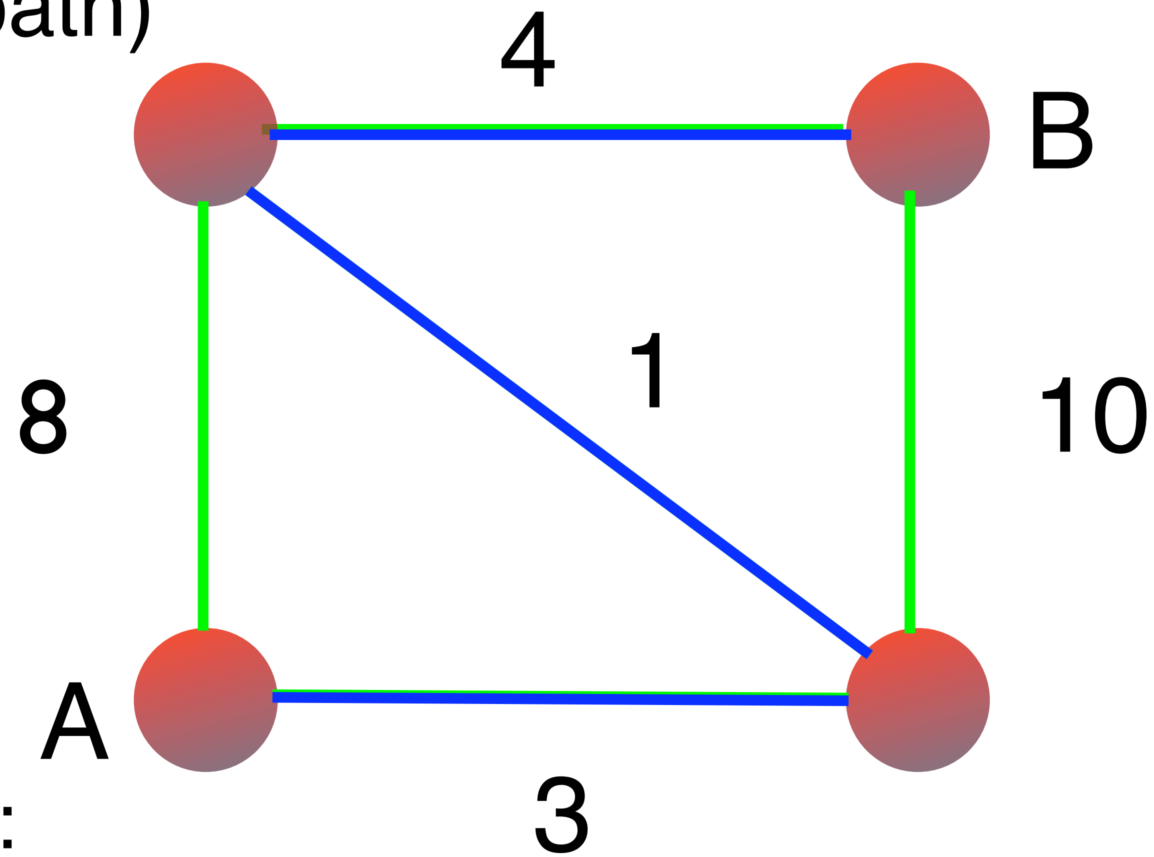


Example:

w_i = Time to travel through a link

Definitions

Optimal Path between a pair of nodes A and B ≡ The path between with the least total weight (sum of weights of links along the path)



Quantity of interest:

Average optimal path length ≡ ℓ_{opt} .

(Average over an ensemble of random networks for fixed N.)

Motivation:

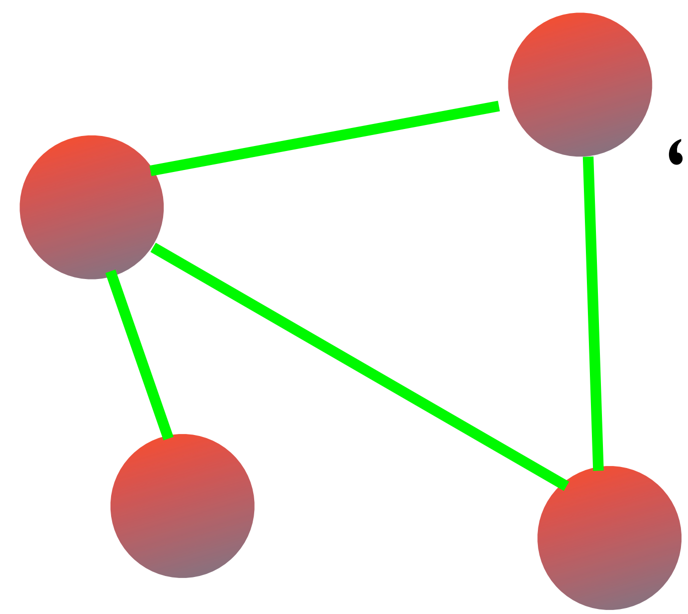
- Activated processes on disordered landscapes.
- Path allocation in routing problems.

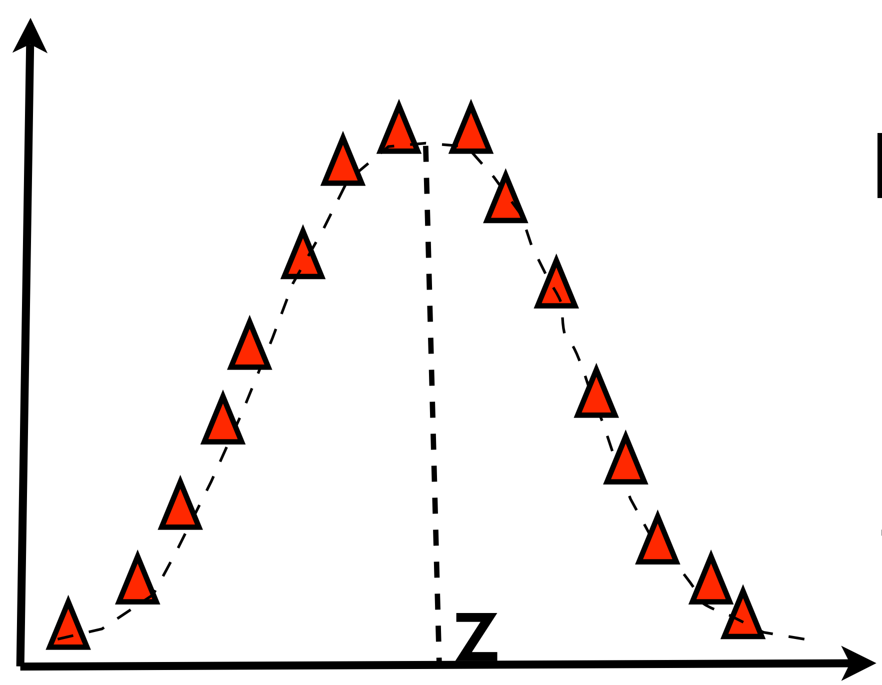
Definitions

Network Model Studied: Erdos-Renyi Random Graph

- Start with **N** nodes.
- Connect **L** pairs randomly picked out of the N(N-1)/2 possible pairs.

Resulting graph has **N** nodes and **L** links.





$P(k) = e^{-z} z^k / k!$

$z = 2L / N$

z = Average degree

Definitions

Assignment of link weights (disorder):

$w = e^{ar}$, $r \in [0,1]$ uniform

$a \equiv$ "disorder strength"

The parameter "a" controls the heterogeneity in the link weights.

The probability density function for the weights is :

$$P(w) = 1/(aw) \qquad w \in [1, e^a]$$

Previous results

Scaling of the average optimal path length ℓ_{opt} with network size N :

Two scaling regimes found* depending on the strength of the disorder “a”:

$\ell_{opt}(a,N) \sim \ln N$ Weak disorder regime ($a \ll a_x(N)$)

$\ell_{opt}(a,N) \sim N^{1/3}$ Strong Disorder regime ($a \gg a_x(N)$)

how to explain these scaling laws and crossover?

* L.A Braunstein, S.V. Buldyrev, R. Cohen, S. Havlin and H. E. Stanley. Phys. Rev. Lett. 91, 168701 (2003)

Our work on optimal paths in disordered networks

Questions asked :

1. Explain the $N^{1/3}$ scaling law for strong disorder regime.

2. Derive the crossover disorder strength $a_X(N)$.

3. Obtain a general scaling ansatz for the optimal path length .

The strong disorder limit

Suppose weights on the links are ordered according to their magnitudes:

$$w_1 < w_2 < w_3 < \ldots < w_L$$

Consider the ratio of two successive weights :

$$\frac{w_k}{w_{k-1}} = e^{a\,\Delta r} \quad \text{where , on avg. , } \Delta r \cong 1/L = 2/\langle k\rangle N$$

For fixed N, L, as $a \to \infty$, one link dominates with high probability,

$$w_k > w_{k-1} + w_{k-2} + \ldots w_1$$

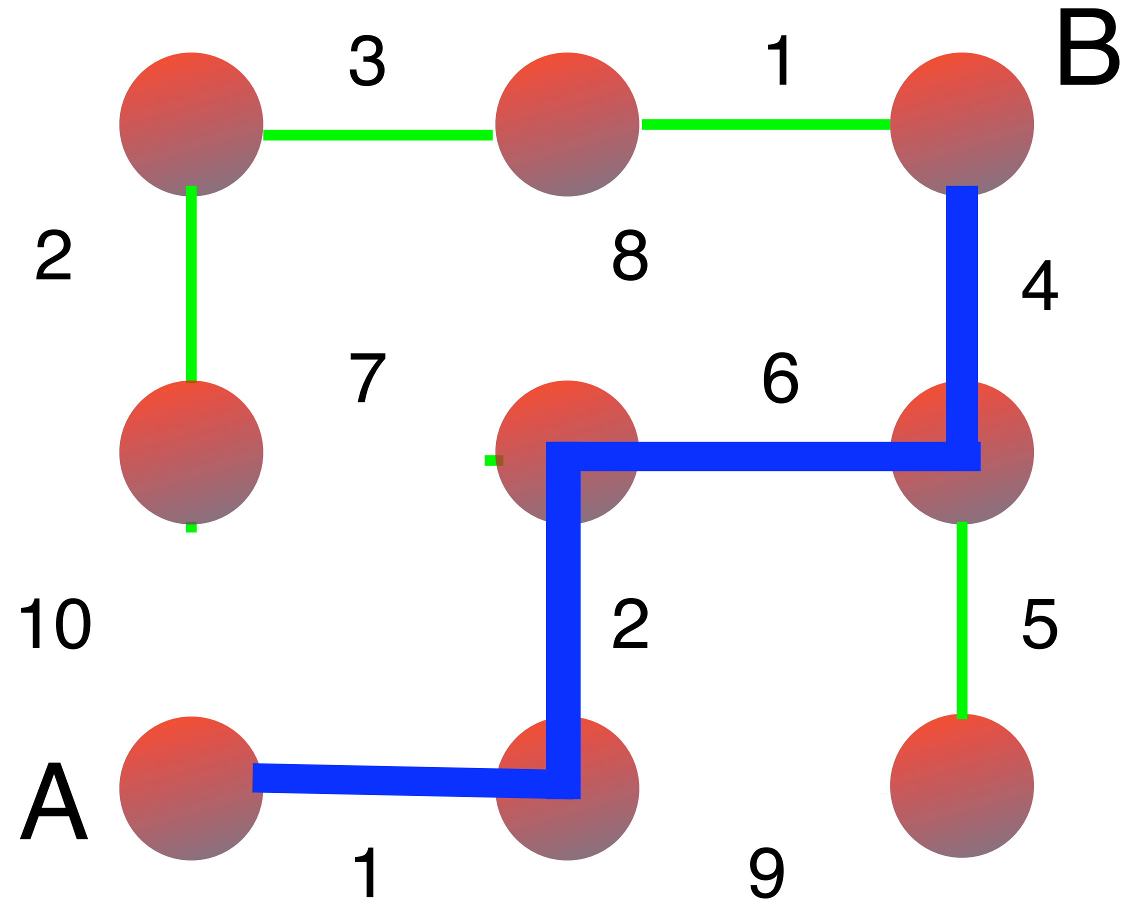
$a \to \infty \equiv$ **strong disorder limit**.

Obtaining optimal path in strong disorder limit

When $w_k > w_{k-1} + w_{k-2} + \ldots w_1$

The maximal weight w_{max} along a path P dominates the sum of the weights on the path:

"Bombing" algorithm

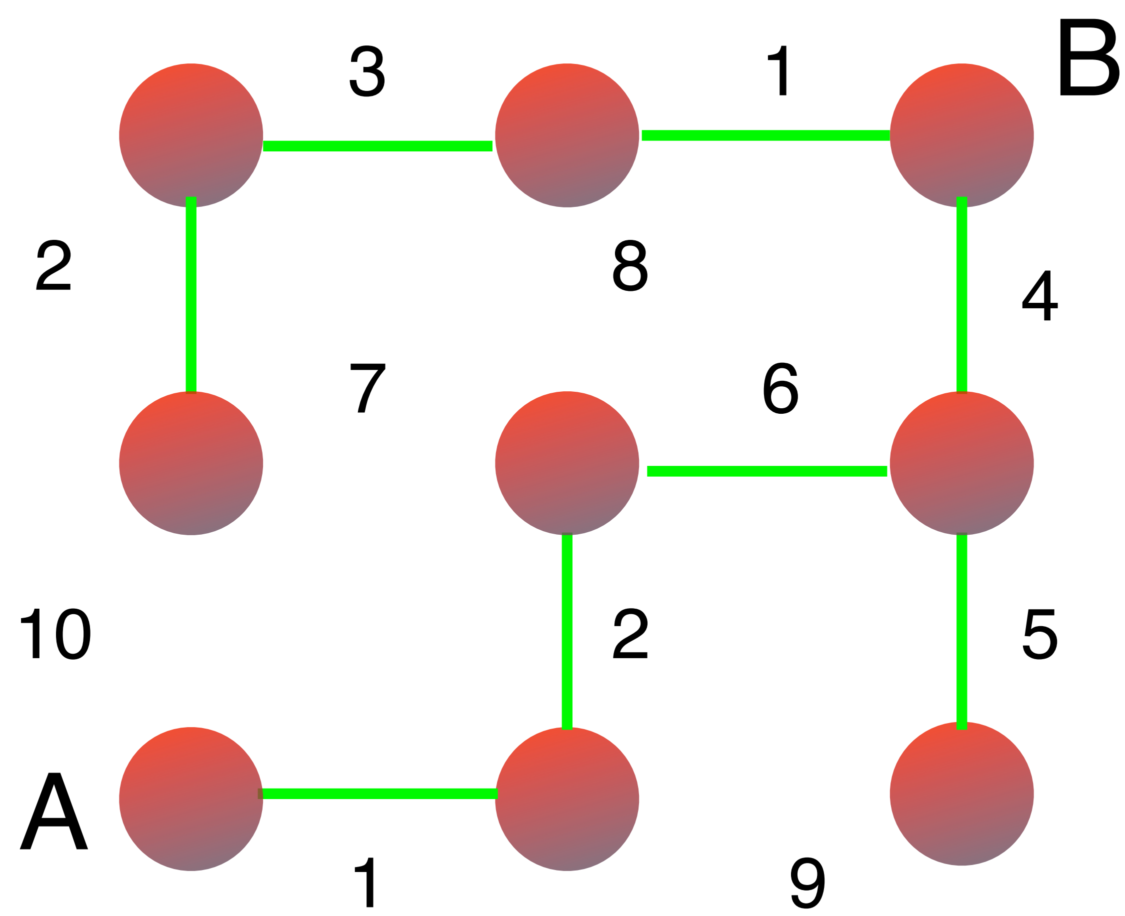


Optimal path between A and B is the **Min-Max** path ≡ path with the **lowest maximal weight** along the path.

The Minimum Spanning Tree

Minimum Spanning tree (MST) ≡ Union of optimal paths for all choices of (A,B)

MST ≡ Tree on the original network with the minimum total link weight



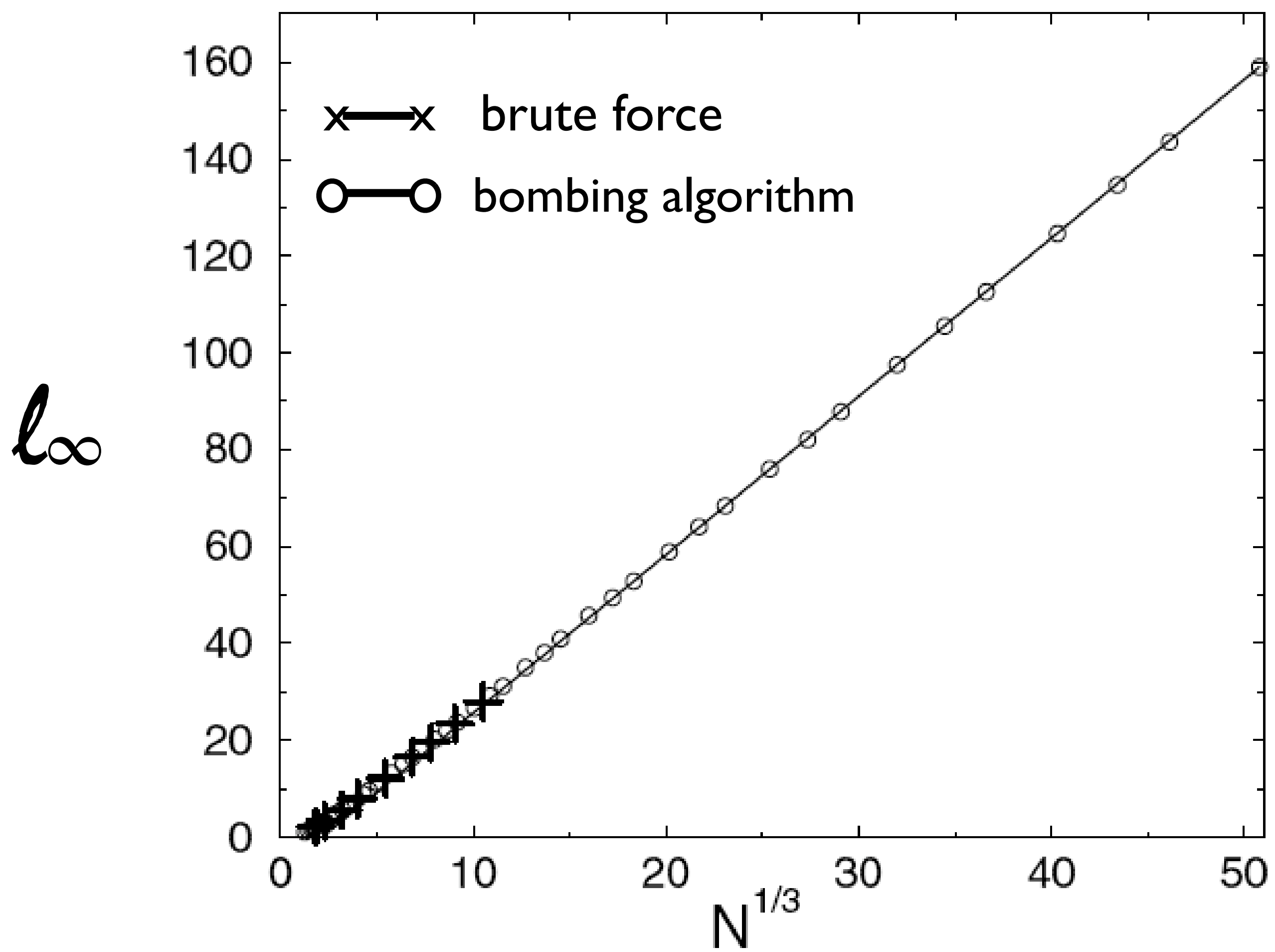
The average number of links between two nodes on the MST = average length of the Min-Max Path.

Average Min-Max path length denoted as

$$\ell_{\infty}(N) \equiv \lim_{a \to \infty} \ell_{opt}(a,N).$$

The $N^{1/3}$ scaling law (for strong disorder)

Test of the bombing algorithm

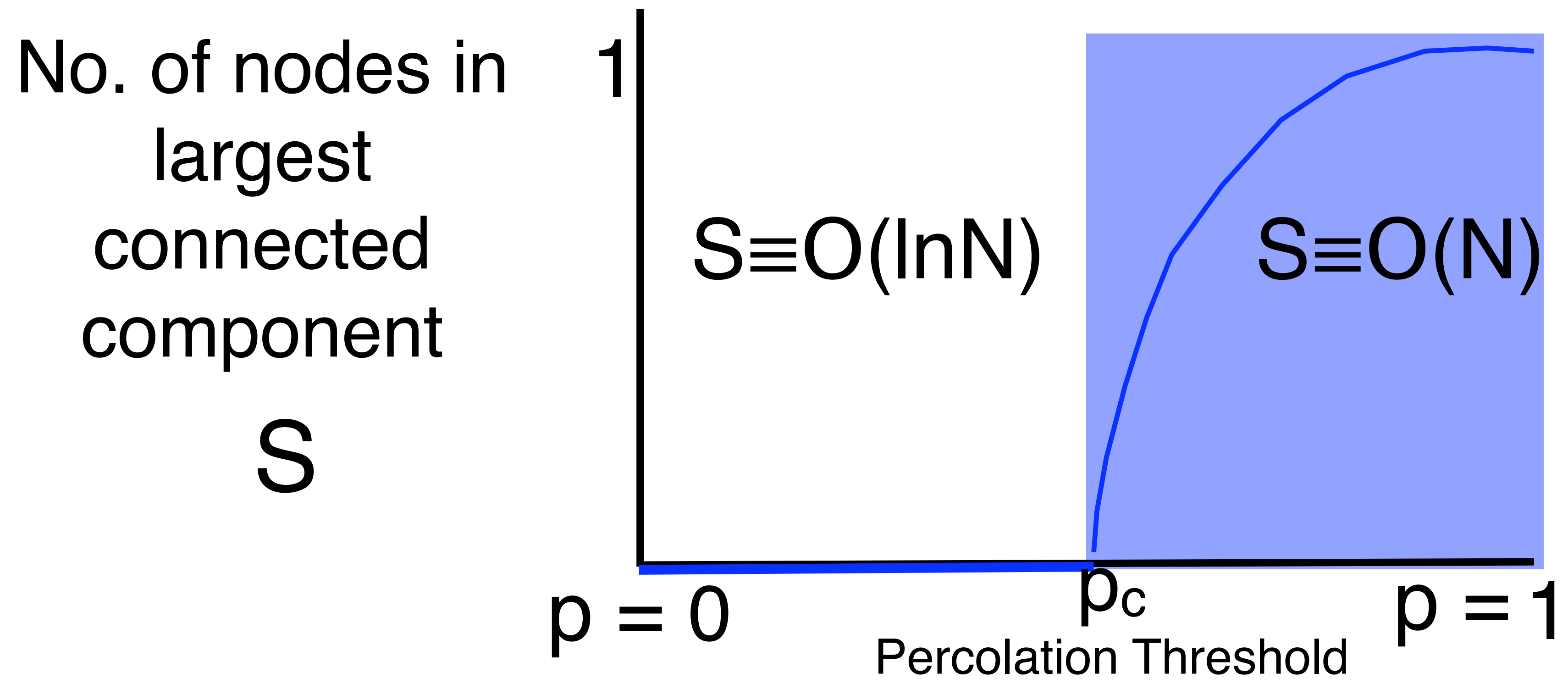


When $a \gg a_X$ (Strong Disorder regime),

Since the Bombing algorithm yields the MST, the $N^{1/3}$ scaling law arises due to optimal paths which lie on the Minimum Spanning Tree.

Connection of the Bombing Algorithm to Percolation

Percolation process ≡ Links on the network are "alive" with probability p



Percolation process:

1) Assign a random number $r \in [0,1]$ uniformly to links on the network

2) "Bomb" all links with $r > p$.

"Bombing" algorithm to find the optimal path:

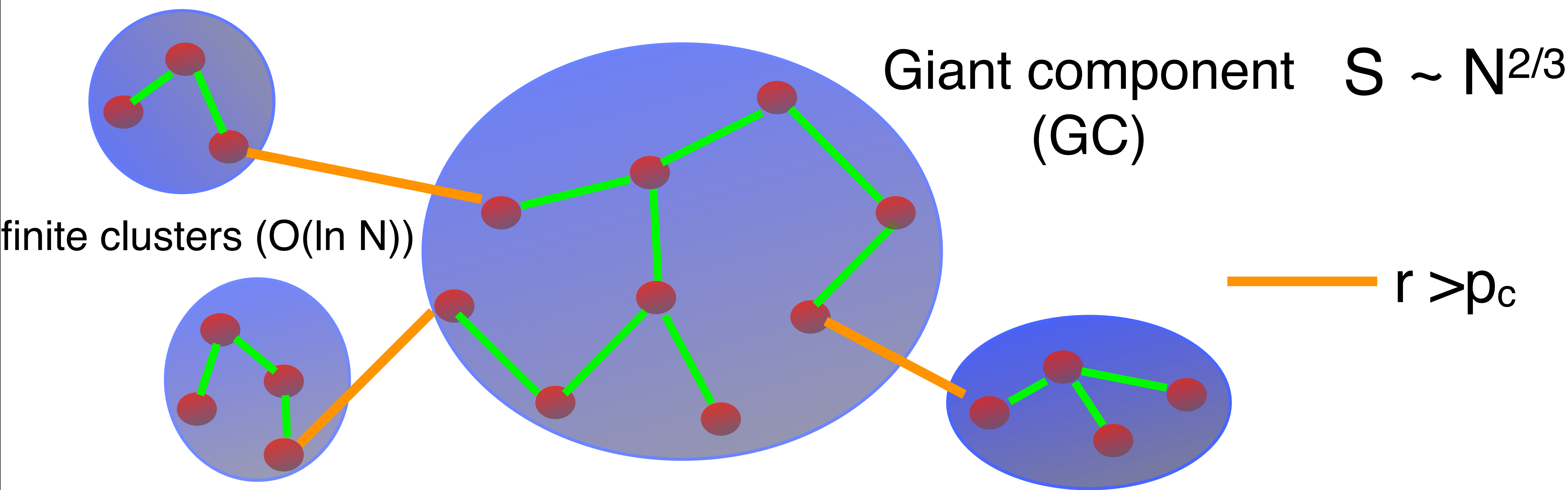
1) Order the links by their weights ($r \in [0,1]$) assigned.

2) "Bomb" links in descending order of weight assigned maintaining, connectivity.

"Bombing" algorithm is analogous to a percolation process.

Connection of MST to Percolation

At the percolation threshold p_c , there are O(N) clusters:

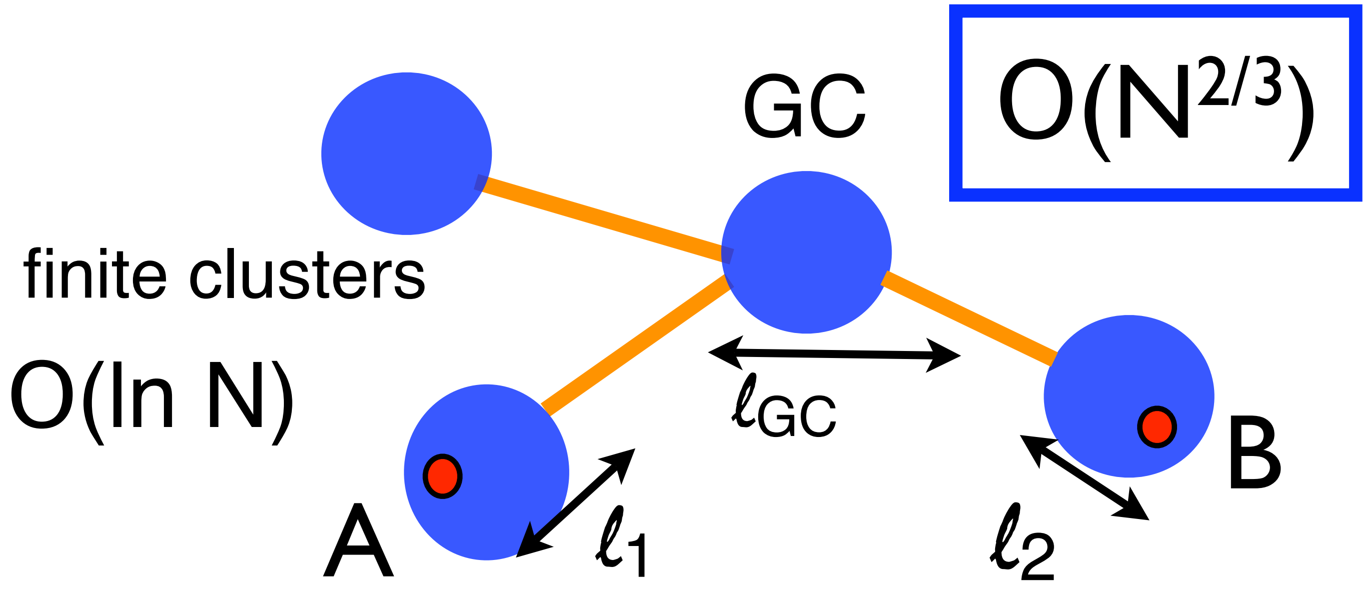


Clusters at p_c are trees for random graphs *

MST formed by the connecting percolation clusters with links with $r > p_c$

* S. Janson, D. E. Knuth, T. Luczek and B. Pittel, Rand. Struct. Alg. 4, 233 (1993),
E.Ben -Naim and P. Krapivsky, Phys. Rev E. 71, 026129 (2005).

Deriving $\ell_\infty \sim N^{1/3}$



$$\ell_{\infty(AB)} \approx \ell_1 + \ell_{GC} + \ell_2$$

Typical path length within a percolation cluster of size s : $\ell_s \sim s^{1/2}$

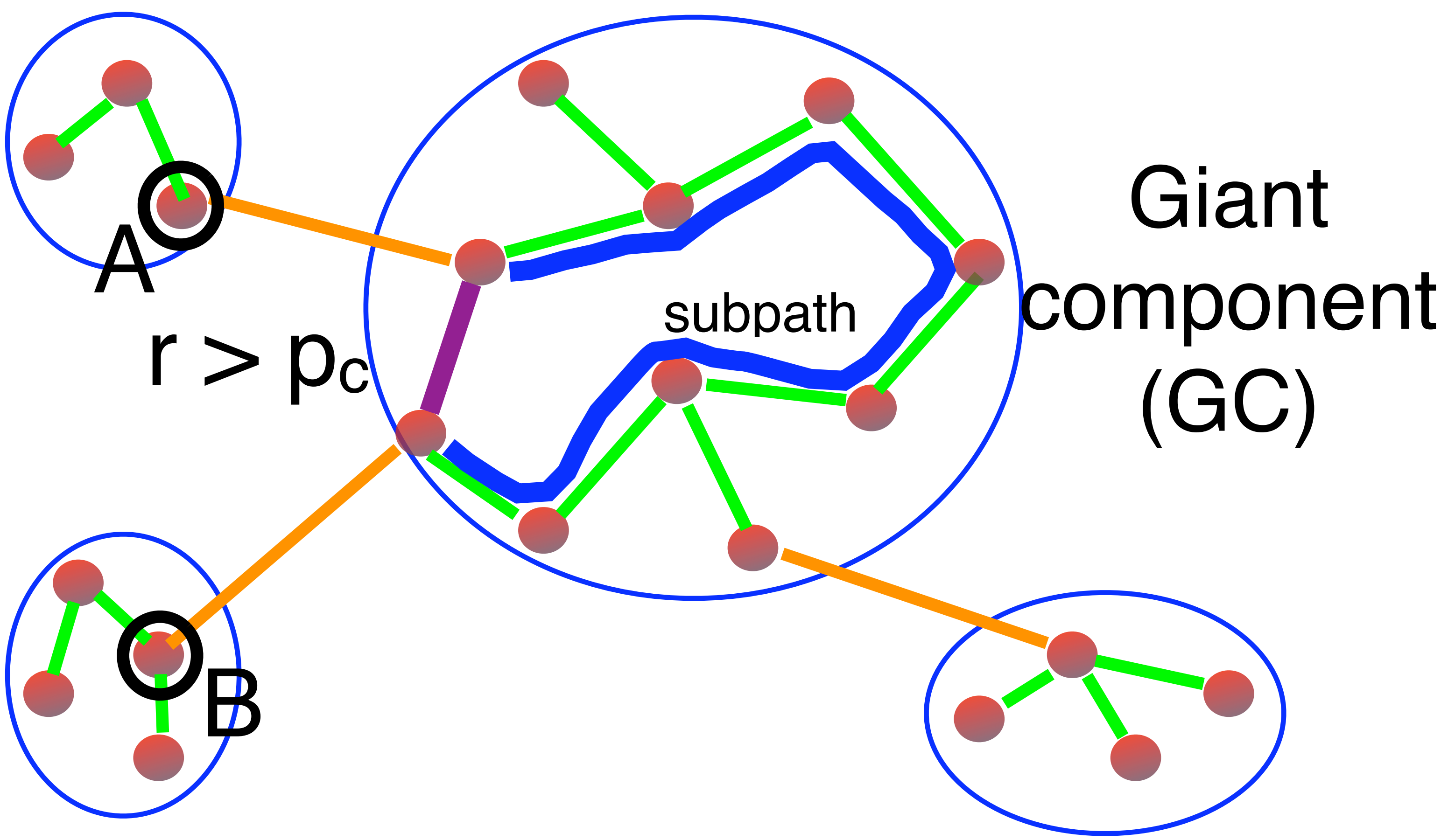
Since giant component size ($O(N^{2/3})$) >> size of finite clusters ($O(\ln N)$), path length within GC provides the dominant contribution ℓ_∞.

$$\therefore\ \ell_\infty \sim S^{1/2} \sim (N^{2/3})^{1/2} = N^{1/3}$$

Deriving $a_x(N)$

$\ell_{opt(a,N)} \sim \ln N$ Weak disorder regime ($a \ll a_x(N)$)

$\ell_{opt\,(a,N)} \sim N^{1/3}$ Strong Disorder regime ($a \gg a_x(N)$)



As disorder strength "a" is decreased,

A "shortcut" link with $r > p_c$ can be used if :

Weight of the added link ≤ Weight of subpath within GC

Crossover to weak disorder

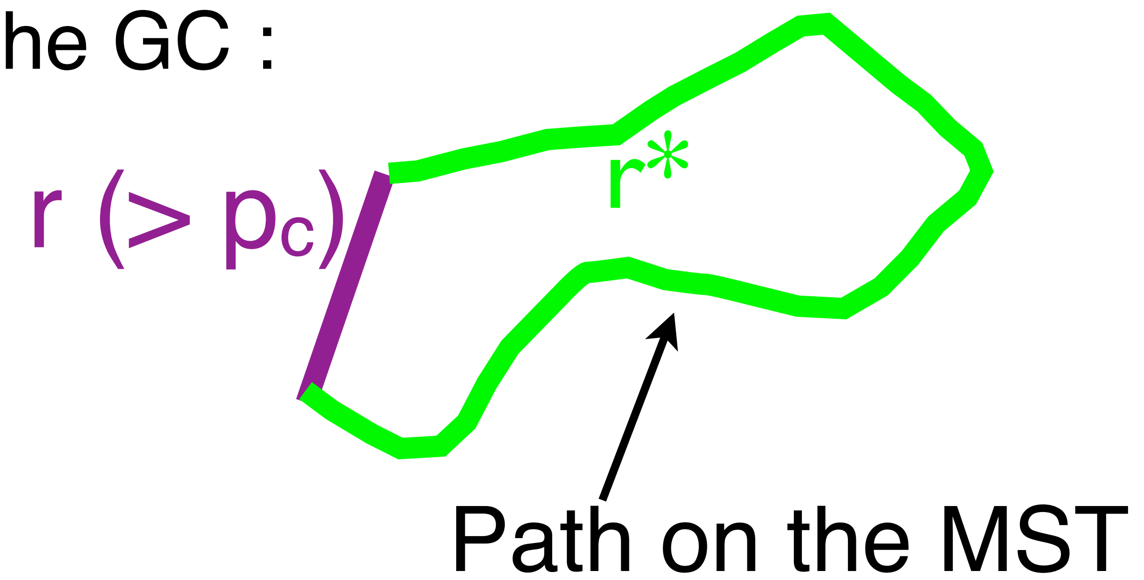
Assume that any path in the GC has $r \in [0,p_c]$, uniform:

Then, for a typical path of length ℓ_∞, the kth largest random number has value (on average): $r_k = kp_c / \ell_\infty$

Average weight of a typical path within the GC :

$$\sum_{k=1}^{\ell_\infty} \exp(akp_c / \ell_\infty) = \exp(a\, r^*)$$

where $r^* \approx p_c + \dfrac{\ln(\ell_\infty / ap_c)}{a}$



For a shortcut to be feasible:

$$r^* = r$$

$\Rightarrow \ell_\infty >> ap_c$

$\Rightarrow$ for $a << a_x = \ell_\infty/p_c$ the optimal path deviates from the MST

Scaling ansatz for ℓ_{opt} (a,N) :

$\therefore$ We expect :

$\ell_{opt(a,N)} \sim \ln N$ for $a << a_x(N) = \ell_\infty(N)/p_c$

$\ell_{opt}(a,N) \sim N^{1/3}$ for $a >> a_x(N)$

We therefore propose a scaling ansatz for $\ell_{opt}(a,N)$:

$$\ell_{opt}(a,N) \sim \ell_\infty(N)\, F\, (a/a_x(N))$$

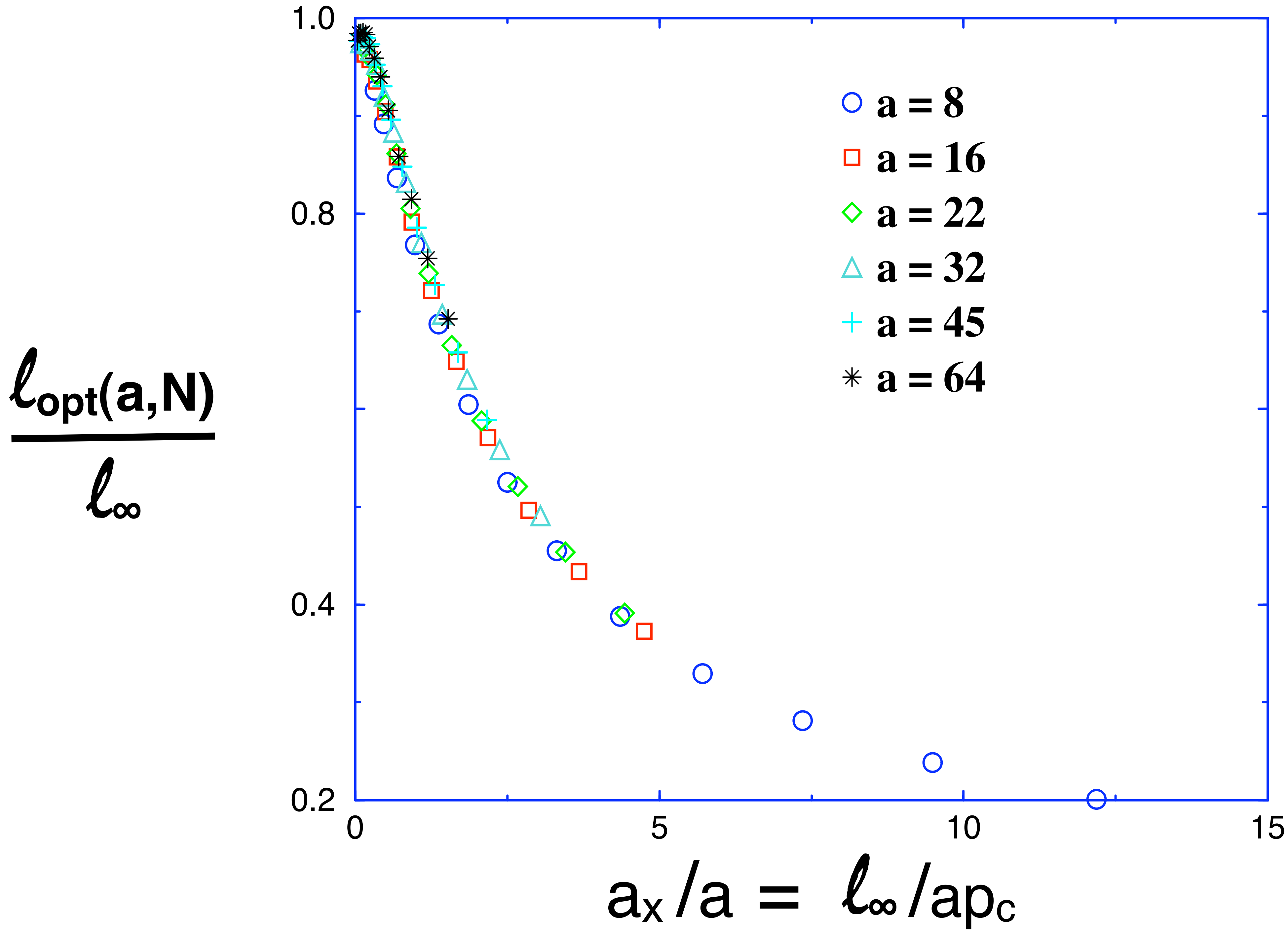
where $F(u) = -u \ln u \quad u >> 1$

$= \text{const} \quad u << 1$

Test of the scaling ansatz

$$\ell_{opt}(a,N) \sim \ell_\infty(N)\, F\,(a/a_x)$$

$$F(u) = -u \ln u \quad u \gg 1$$
$$= \text{const} \quad u \ll 1$$



S. Sreenivasan, T. Kalisky, L.A. Braunstein, S.V. Buldyrev, S. Havlin and H. E. Stanley, PRE 70, 046133 (2004)

Summary

(i) We find the scaling of $\ell_\infty(N)$, $a_x(N)$ and a scaling ansatz for $\ell_{opt}(a,N)$ and test it by simulations.

(ii) We establish a relationship between the MST and clusters at the percolation threshold of the graph.

Further extensions

(i) Results extended to general distributions.

Y. Chen, E. Lopez, S. Havlin and H. E. Stanley , PRL 96 068702 (2006)

P. Van Mieghem and S. Van Langen, PRE 71 056113 (2005)

(ii) Current flow paths on strongly disordered random resistor networks

Z. Wu, E. Lopez, S. V. Buldyrev, L. A. Braunstein, S. Havlin and H. E. Stanley, PRE 045101(R) (2005)

Collaborators:

L. A. Braunstein
S. V. Buldyrev
R. Cohen
S. Havlin
T. Kalisky
E. Lopez
G. Paul
Z. Toroczkai