Fractal Boundaries of Complex Networks

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Motivations

- The structural properties of boundaries are important in spread of disease on networks.
- Little attention has been paid to the boundaries of networks.

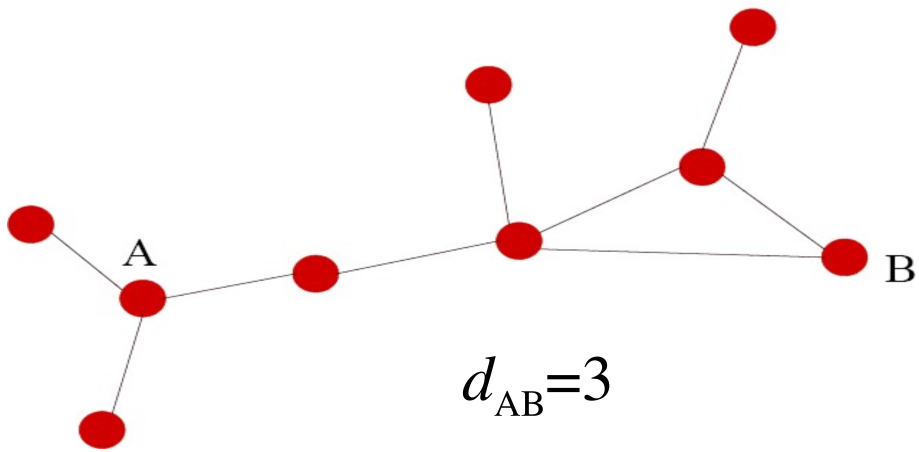
Questions

1. *How do we define the boundaries of networks?*
2. *What are the structural properties of network boundaries?*
3. *How do we apply network boundaries to the study of disease spread?*

Part I

Networks: definition and properties

- Node (vertex) & link (edge)



$d_{AB}=3$

- Distance d between two given nodes.
- Average distance $\bar{d}$ (diameter) of a network.

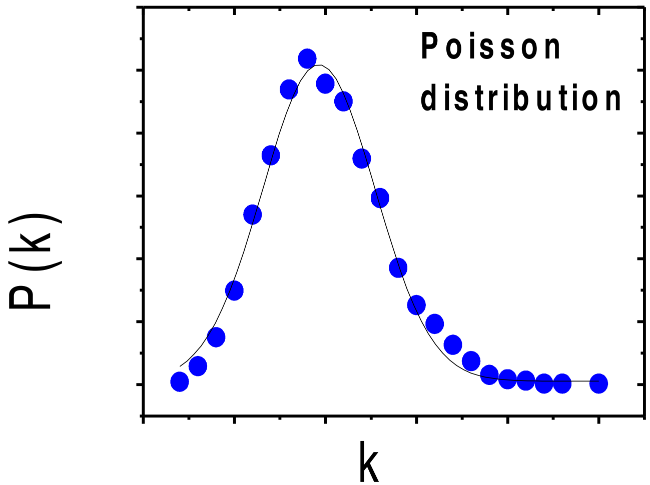
Small-world networks $\rightarrow \bar{d} \approx \log N$

N=10^6 ⟶ $\bar{d} \approx 6$ "six-degree separation"

Network: definition and properties

- Degree: number of links a node has k
- Degree distribution $P(k)$

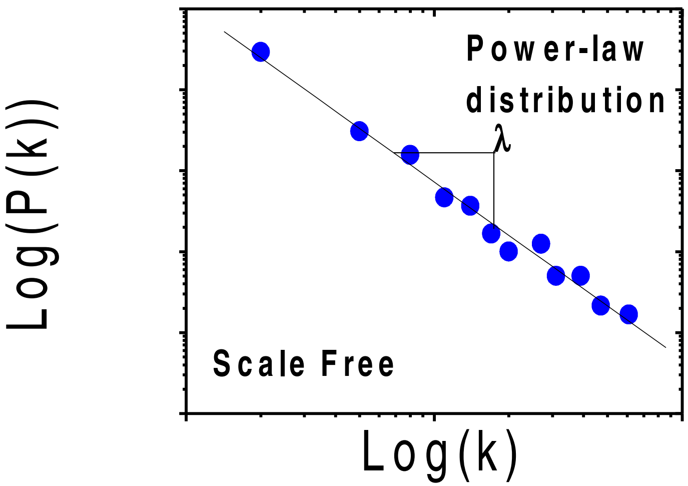
Poisson distribution



$$P(k) \sim \exp(-t)\frac{t^k}{k!}$$

Erdős-Rényi (**ER**)

Power-law distribution



$$P(k) \sim k^{-\lambda}$$

scale-free (**SF**)

Part I

Answer to Q1: *How to define the boundary of network?*

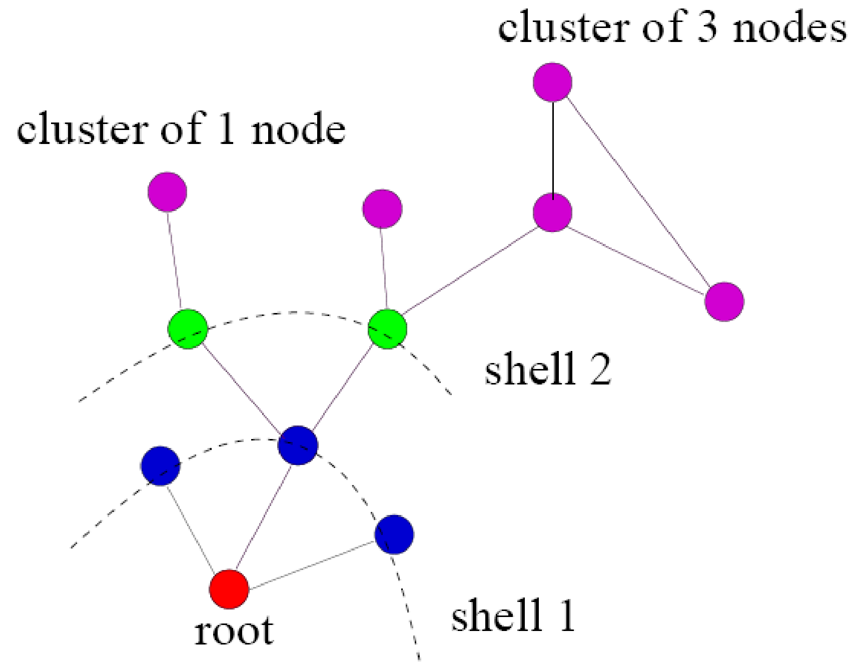
1. Define nodes, which at distance ℓ from root, as shell ℓ.

Number of nodes ***on*** *shell:*

$\ell = 1$, $B_1=3$,

$\ell = 2$, $B_2=2$.

2. Define all nodes **outside** shell ℓ where $\ell > \bar{d}$, as the boundaries.



If we choose boundaries as nodes outside shell ***2***

a) The number of clusters in boundaries: $M_2=3$

b) The sizes of cluster in boundaries: $S_2=3, 1, 1$

Why do we define boundaries in this way?

Because, epidemic starts from random node, spreads along shortest path.

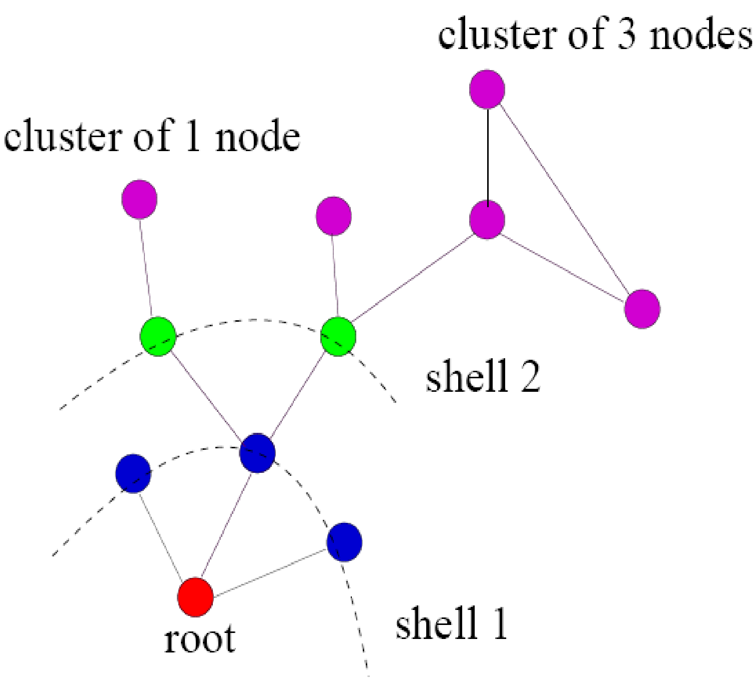
Answer to Q2: *What are the structures of network boundaries?*

1. average degree $< k_\ell >$ of nodes on shell ℓ.
2. $P(B_\ell)$ of number of nodes B_ℓ on shell ℓ, where $\ell > \bar{d}$.

- $P(M_\ell)$ of number of clusters M_ℓ formed by the boundary nodes.

4. Number of clusters $n(S_\ell)$ of size S_ℓ in the boundary.

- S_ℓ as function of the diameter $\bar{d}_\ell$ of the cluster



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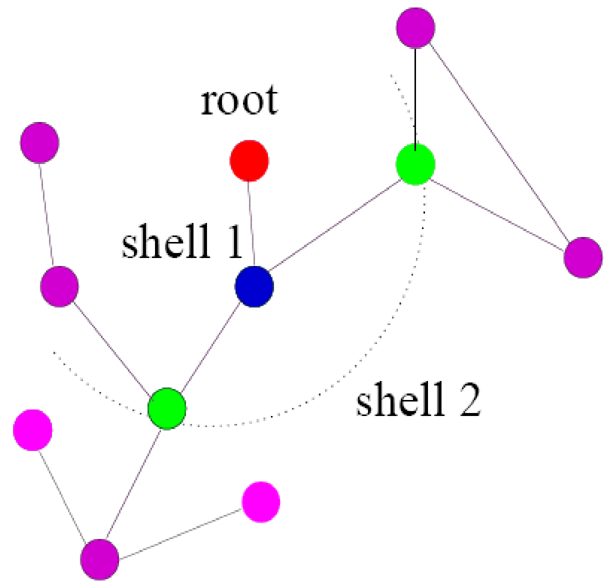

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Part II

Structure of boundaries: $< k_\ell >$

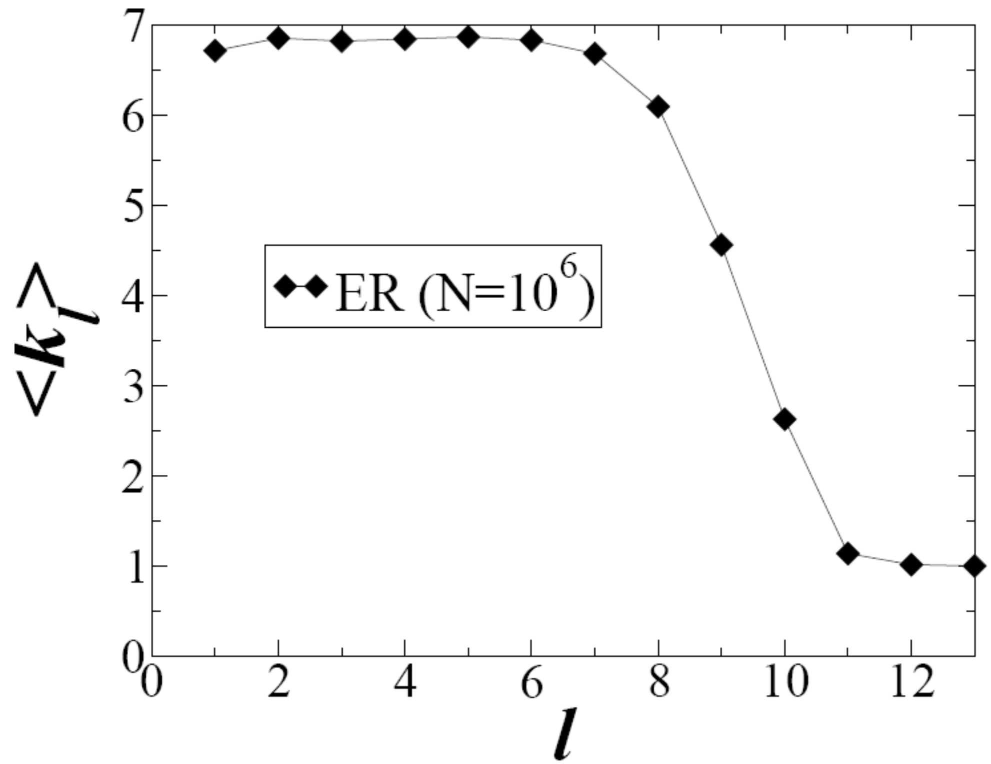
$<k_\ell>$: average degree of nodes **on** shell ℓ

On the boundaries, when $\ell > \bar{d}$

$$< k_\ell > \to k_{\min}$$

Boundary nodes have *smaller* degrees.

ER network <k>=6



Note: $\bar{d} \approx 8$ *for ER with <k>=6*

Do the diluted degrees yield different structural properties from the bulk of the network?

Part II

Structure of boundaries: $P_{cum}(B_\ell)$

B_ℓ: the number of nodes **on** ℓ shell

Probability distribution function:

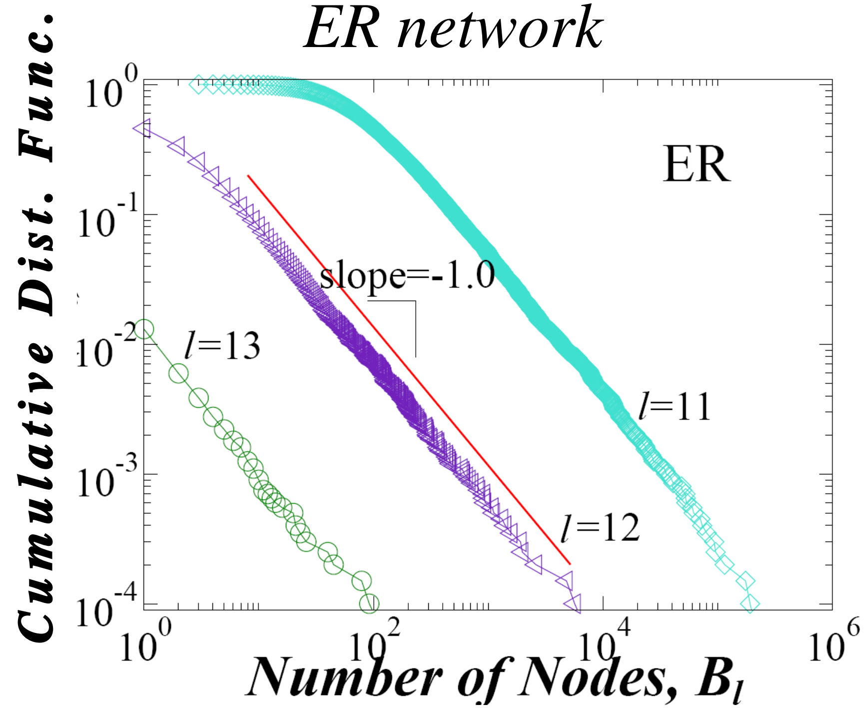
$P(B_\ell)$ probability of having B_ℓ nodes on shell ℓ

Cumulative distribution function:

$P_{cum}(B_\ell)$ probability of having more than B_ℓ nodes

$$P_{cum}(B_\ell) \equiv \int_{B_\ell}^{\infty} P(B)dB$$

ER network



Note: $\bar{d} \approx 8$ *for ER with <k>=6*

$$P_{cum}(B_\ell) \sim B_\ell^{-1.0} \;\; \& \;\; P(B_\ell) \sim B_\ell^{-2.0}$$

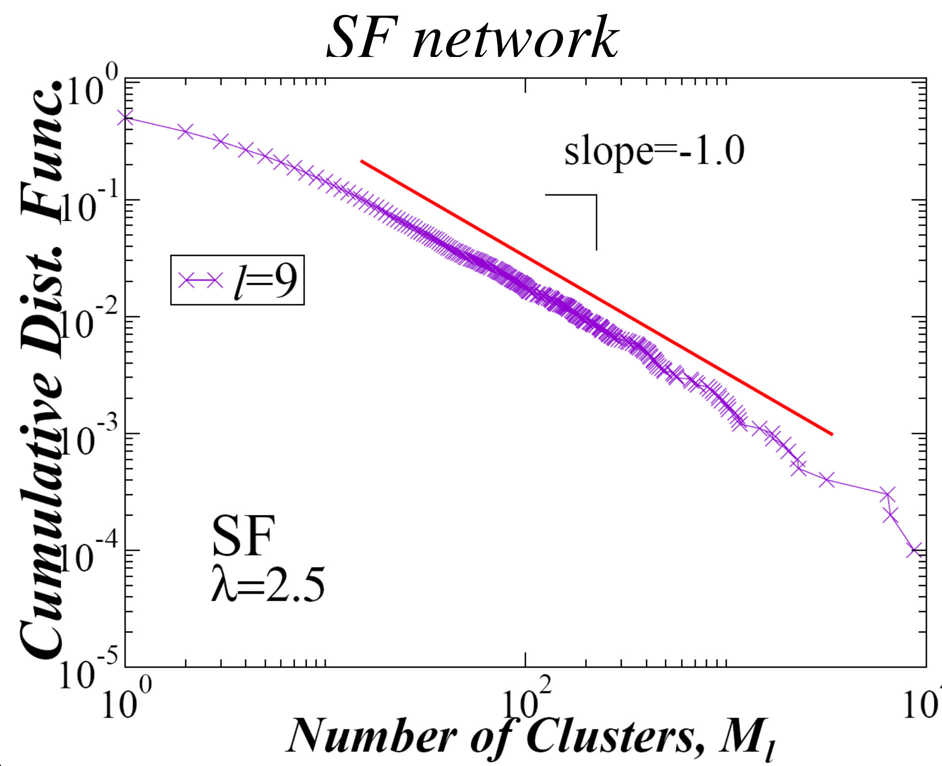
Universal power-law holds for ER with different $<k>$, SF with different λ, and different real networks

→ *Do different networks have similar boundary properties?*

Part II

Structure of boundaries: $P(M_\ell)$

M_ℓ: number of clusters in the boundaries $(\ell > \bar{d})$.



On the boundaries:

$$P_{cum}(M_\ell) \sim M_\ell^{-1.0}$$

Note: $\bar{d} \approx 5$ for SF with λ=2.5

holds for ER with different $\langle k \rangle$, SF with different λ, and different real networks.

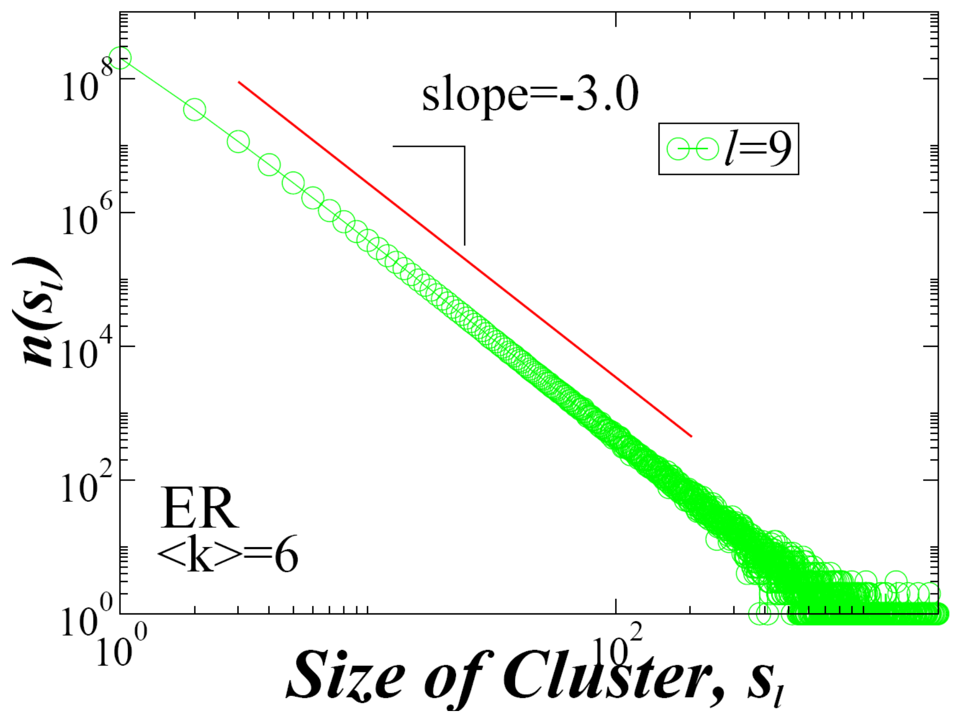
Structure of boundaries: $n(S_\ell)$

S_ℓ : size of clusters in the boundaries ($\ell > \bar{d}$).

$n(S_\ell)$: number of clusters of size S_ℓ.

$$n(S_\ell) \sim S_\ell^{-3.0}$$

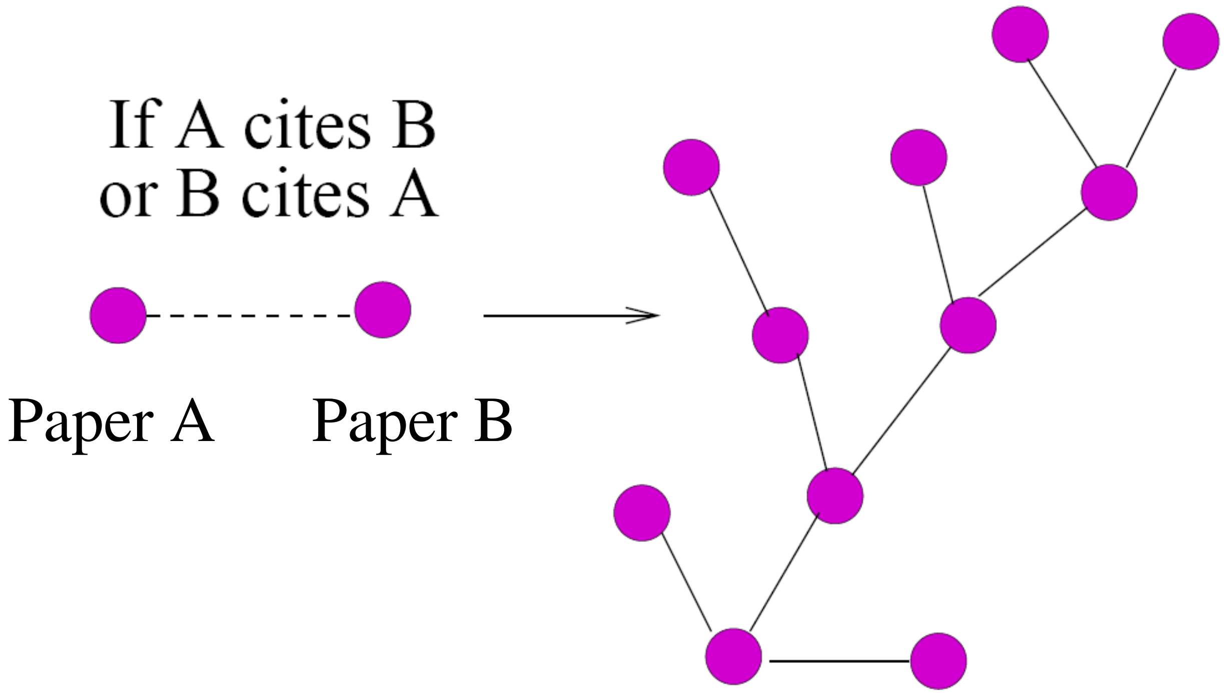
ER network



Note: $\bar{d} \approx 8$ *for ER with <k>=6*

holds for ER with different <*k*>, SF with different λ, and different real networks.

HEP: High Energy Physics citations network



34,401 papers (nodes) and 420,784 citations (links)

Structure of boundaries: S_ℓ vs $\bar{d}_\ell$

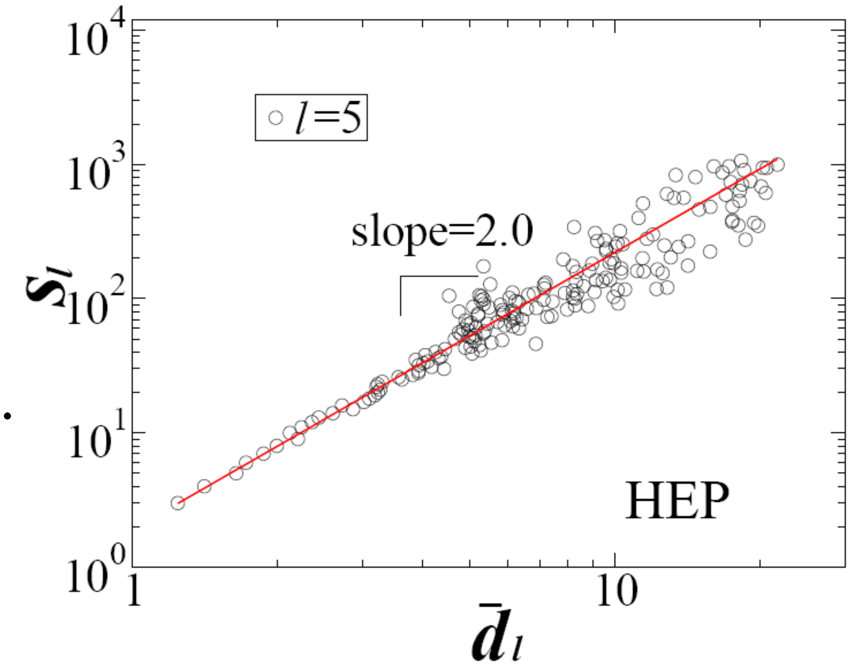
$\bar{d}_\ell$: the diameter of cluster, formed by nodes **outside** shell ℓ.

$$S_\ell \sim \bar{d}_\ell^{\,2} \quad (1)$$

Clusters follow Eq.(1) are fractals [1].

The clusters in the boundaries are fractal clusters.

HEP: High Energy Physics citations network



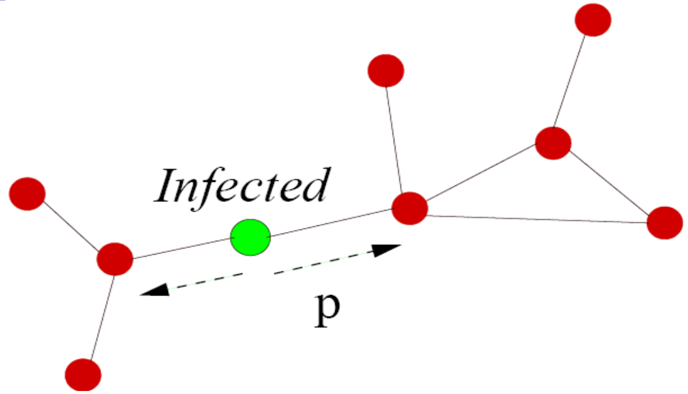
Note: $\bar{d} \approx 4.2$ *for HEP.*

Despite the difference of the networks, their boundary structures are similar!

[1] A. Bunde and S. Havlin, *Fractals and disordered system* (Springer, 1996).

Answer to Q3: *How to apply boundaries to the study of disease spread?*

SIR (**S**usceptible - **I**nfected - **R**ecovered) model



- Disease spread starts from one random chosen node in the network.
- For each time step, an infected node has probability *p* to infect each of its uninfected neighboring nodes.
- After time step **T**, infected nodes are recovered which are no longer infective and cannot be infected again.

Part III

Application of boundaries:

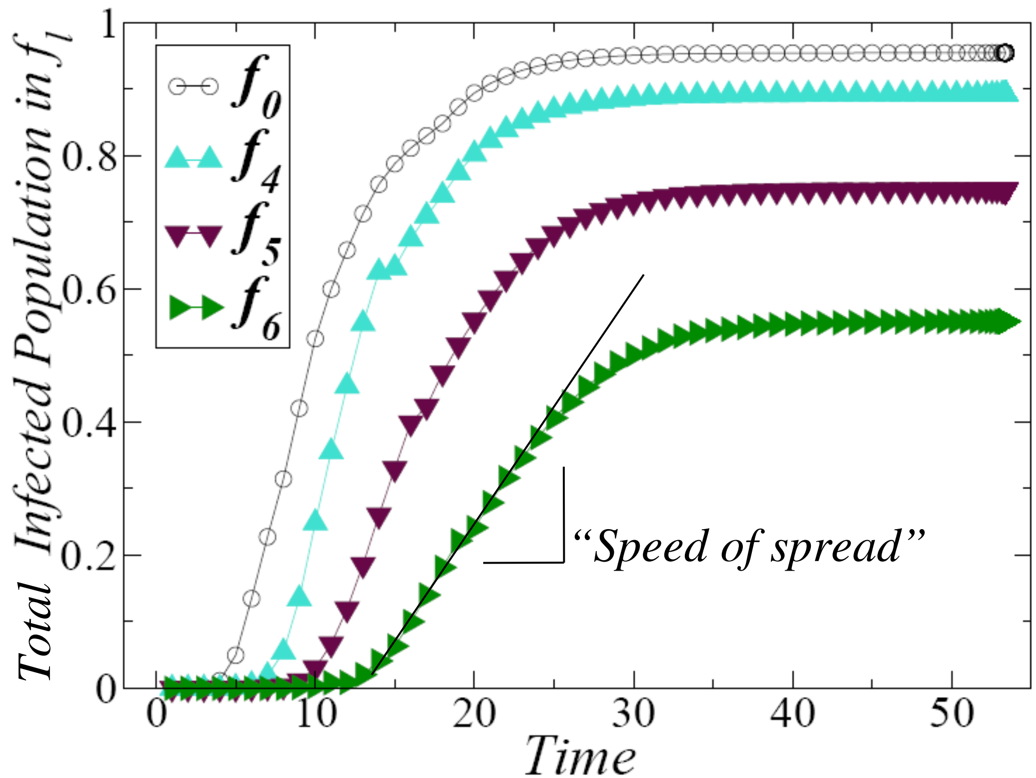
disease spread in network

f_ℓ: fraction of nodes **outside** shell ℓ

$$f_\ell = \sum_{m=\ell+1}^{m_{\max}} \frac{B_m}{N}$$

When disease reaches the boundaries of network, not only the *total infected population*, but also the *spread speed* decreases.

HEP: High Energy Physics citations network (p=0.1,τ=10)



Summary

- We find a power law for $P_{cum}(B_l)$ with a universal exponent "-1" for many types of networks (-2 for probability distribution function).
- Boundaries have interesting structural properties.
- Boundaries have important applications in disease epidemic on networks.

Thank you!