Synchronization in interacting scale-free networks

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Introduction. – In the last decades the study of complex networks has been growing strongly due to the large number of systems that exhibit this type of structures. A complex network is a set of nodes that are connected by internal links and the most fundamental property that characterizes its topology is the degree distribution \( P(k) \), which represents the probability that a node has \( k \) neighbors or connectivity \( k \). It has been found that many real systems such as social, communication and biological networks present a degree distribution given by \( P(k) \sim k^{-\lambda} \), where \( \lambda \) is the exponent of the power law and \( k_{\text{min}} \leq k \leq k_{\text{max}} \), where \( k_{\text{min}} \) and \( k_{\text{max}} \) are the minimum and maximum degree of the network. These kind of networks are called Scale Free (SF) and one of its most important features is that in general very heterogeneous, i.e. most nodes of the network have a low connectivity while only a few have a high connectivity (hubs). In recent years the study of synchronization processes in isolate complex networks has been increasing because of its importance in neurobiology [1–5] and population dynamics [6,7]. A common theoretical approach to study synchronization in complex networks is to map this process onto an interface growth model by assigning to each node a scalar field \( h_i \), with \( i = 1, \ldots, N \), where \( N \) is the size of the network. This scalar field could represent, for example, the amount of load on a node in the problem of distributed parallel computing on processors. Without loss of generality we will relate the scalar field to a set of heights on the interface. The most relevant magnitude that characterizes the interface is \( W(t) \equiv W \), which represents the fluctuations of the scalar field around its mean value on the network, given by

\[
W = \left\{ \frac{1}{N} \sum_{i=1}^{N} (h_i - \langle h \rangle)^2 \right\}^{1/2},
\]

where \( \langle h \rangle = \frac{1}{N} \sum_{i=1}^{N} h_i \) is the average of the scalar field over the nodes at time \( t \) and \( \{ \} \) is the average over different network realizations. The roughness evolves in time until it saturates at the steady state. In the saturation regime \( W_s \) is a constant that depends only on \( \lambda \). In complex networks the synchronization of the system is related to the roughness in the saturation regime [8–16].

One of the most simple and used models to study synchronization in complex networks is the Surface Relaxation Model (SRM) [8–12]. In this model, at each time step, a node \( i \) is randomly chosen and the node with the lowest height between the chosen node and all its neighbors evolves increasing its height. It has been found that for isolated SF networks, with \( \lambda < 3 \), \( W_s \sim \ln N \) [8,10,17]
until a critical value $N = N^*$, after which $W_s$ becomes independent of $N$, which means that the system becomes scalable [9,17]. Although it was an interesting result, many real systems are not isolated but interacting with other systems instead. This means that a process that develops in one network can be affected by a process developing in another and vice versa [18–34]. Such is the case of epidemic models where the interaction between networks make it very harmful for the healthy populations because the interaction increases the theoretical risk of infection compared with the same process in isolated networks [35–40].

These interacting systems can be modeled as networks that interact through external links that connect nodes that belongs to different networks. Now the question is, does synchronization in interacting networks performs worse or better than synchronization in isolated networks? In order to answer this question, in this letter we study the synchronization of two SF networks with the same size $N$ that interact through a fraction $q$ of nodes connected, one by one, between them. The model used is the SRM model which we adapt to interacting networks and study the effect of the interaction parameter $q$ on the fluctuations in both networks.

**Model**. – In our model, two uncorrelated SF networks, called $A$ and $B$, with the same size $N$ and exponent $\lambda_A$ and $\lambda_B$, respectively, are built using the Molloy-Reed algorithm [41] disallowing self-loops and multiple connections. As we need a single interface on each network, in order to ensure that we have a single component we use $k_{\text{min}} = 2$ [42]. Each node $i \in \alpha$, with $\alpha = A, B$, has a connectivity $k_i^\alpha$ and we denote the set of its neighbors by $v_i^\alpha$. In order to build the external connections between the networks we connect by simplicity, the first $qN$ nodes in $A$ one by one with the first $qN$ in $B$, where $q$ is the interaction parameter with $0 < q \leq 1$. Notice that if both networks are uncorrelated this procedure is the same as connecting a fraction $q$ of nodes at random. We define the vector $M$, where $M_i$, with $i = 1, \ldots, N$, is equal to 1 if the node $i$ in $A$ has an external connection with $i$ in $B$, and $M_i = 0$ otherwise. In order to simplify the growth rules we choose random initial conditions for the scalar field in the interval $[0,1]$, hence, we avoid the cases in which different nodes have equal heights, as we are only interested in the saturation regime on both networks where the initial condition plays no role.

The evolution rules of the interface growth are given as follows:

1) A network $\alpha$ (with $\alpha = A, B$) is chosen with probability $1/2$ and then a “particle”, which represents the load, is dropped in a node $i$ selected randomly in $\alpha$.

2) The particle diffuses to the node $\epsilon$ that is the node with the lowest height between the node $i$ and its neighbors $v_i^\alpha$.

3) If $M_i = 0$ or $M_\epsilon = 1$ and $h_\epsilon \in \alpha < h_\epsilon \in \beta$ (with $\beta \neq \alpha$) the particle is deposited in $\epsilon \in \alpha$. Otherwise the particle diffuses to the network $\beta$ and is deposited in the node with the lowest height between $\epsilon$ and its neighbors $v_\epsilon^\beta$.

Thus if we denote $\ell \in \alpha$ as the node where the particle is finally deposited, then $h_\ell^\alpha = h_\ell^\beta + 1$. At each Monte Carlo step the time is increased by $1/2N$. In fig. 1 we show a schematic of the rules of the process for the case of a one-dimensional lattice.

**Results and discussions**. – We are interested in the behavior of the fluctuations in the steady state of both networks with the system size above $N^*$ [17], value for which the system is scalable. For isolated SF networks this regime for $\lambda < 3$ is close to $N^* \approx 2 \times 10^5$ [17]. We check that the nature of this regime is due to the distribution of internal connectivities and that it is almost not affected by the interaction parameter $q$. Thus in our research we use $N = N_A = N_B = 3 \times 10^5$ in order to ensure that we are in the scalable regime. We will show our results only for $\lambda_A = 2.6$ and $\lambda_B = 3$, because all the other combinations of the exponents $\lambda$ in $2.5 < \lambda \leq 3$ give qualitatively the same results. We compute the fluctuations in the saturation regime of both networks $W_s^\alpha = \frac{1}{N} \sum_{i=1}^{N} (h_i^\alpha - \langle h_i^\alpha \rangle)^2$, with $\alpha = A, B$ and in fig. 2 we show $W_s^\beta$ as a function of $q$. It is clear that
as the interaction parameter $q$ increases, $W_0^\alpha$ decreases in both networks, which implies that the synchronization improves. From the plot we can also observe that as $q$ increases, the difference between $W_0^A$ and $W_0^B$ becomes smaller, which means that the synchronization in each network becomes mainly controlled by $q$ and not by the internal degree distributions. In the inset of fig. 2 we show $W_0^S/W_0^S(q = 0)$ as a function of $q$. It can be seen how the rate of improvement in the synchronization with the increment of $q$. For example, for $q = 0.3$ the synchronization enhances approximately 20% and for $q = 1$ around 40%. It is worth pointing out that this rate decreases as $q$ increases, and this means that the effect of the optimization gets less significant as the networks have more interconnections between them.

To understand the effect of the interaction parameter $q$ on the optimization of the process we compute the difference between the average height of nodes with degree $k$, denoted by $h_k^\alpha$ and the mean value of the height $\langle h^\alpha \rangle$ as a function of $k$. In fig. 3 we show $h_k^\alpha - \langle h^\alpha \rangle$ as a function of $k$ for different values of $q$. We can see that, for $q = 0$ the heights of low-connectivity nodes, which are the majority in SF networks, are closer to the average height of the network than the heights of high-degree nodes, which are above $\langle h^\alpha \rangle$. This means that hubs are usually overloaded because all their neighbors send them their excess of load, affecting negatively the synchronization of the system. However, as the factor $q$ increases the height of the hubs decreases, approaching to $\langle h^\alpha \rangle$ and becoming independent of $k$ for $q = 1$. In the inset of fig. 3, we can see an amplification of the behavior of $h_k^\alpha$ for the nodes with the lowest connectivity. We can see that as $q$ increases their heights also approach to the average value. These results imply that as we increase the factor $q$ hubs are no longer overloaded and therefore their excess of load is now absorbed by low-connectivity nodes.

In order to understand the diffusion process between networks we want to know how frequently the load spreads from one network to another and how it gets distributed when the rate $\nu_\alpha\beta$ at which a particle spreads from the network $\alpha$ to the network $\beta$ is a function of $q$. With $\nu_{AA}$, $\nu_{AB}$, $\nu_{BB}$ and $\nu_{BA}$, it is clear that $\nu_{AA} + \nu_{AB} = 1$ and $\nu_{AB} + \nu_{BB} = 1$.
of heterogeneity of the deposition process, for di
see that the probabilities
on the exponents of the original degree distributions.

tables 1: Dispersion $\sigma_{\alpha\beta}$ of the distributions $P_{\alpha\beta}$ for different
values of $q$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\lambda_A = 2.6 \lambda_B = 3.0$</th>
<th>$\sigma_{AA}$</th>
<th>$\sigma_{BB}$</th>
<th>$\sigma_{AB}$</th>
<th>$\sigma_{BA}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22.1</td>
<td>7.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>22.4</td>
<td>7.7</td>
<td>2.2</td>
<td>3.3</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>23.9</td>
<td>8.2</td>
<td>2.7</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>25.7</td>
<td>8.6</td>
<td>3.0</td>
<td>5.7</td>
<td></td>
</tr>
</tbody>
</table>

the same, make that the rates have almost no dependence
on the exponents of the original degree distributions.

How the load gets distributed after the diffusion process?
In order to answer this question we compute the probability
$P_{\alpha\beta}(k)$, defined as the probability that a particle
dropped in the network $\alpha$ gets deposited in a node
with degree $k$ in the network $\beta$. In fig. 5 we plot $P_{\alpha\beta}(k)$
with $\alpha, \beta = A, B$ for $q = 0.5$. Also in table 1 we report the
dispersion $\sigma_{\alpha\beta}$ of these distributions, which quantifies the
heterogeneity of the deposition process, for different values
of $q$ and for the isolated networks. From the plot we can see
that the probabilities $P_{\alpha\alpha}(k)$ for $\alpha = A, B$ are very hetero-
geneous and have a similar dispersion to the dispersion
in the isolated networks. This means that when a particle
stays on the network where it was originally dropped
finds an environment with a similar heterogeneity than
in the isolate SF network. Moreover the distributions $P_{\alpha\beta}(k)$
for $\alpha \neq \beta$ are more homogeneous than in the case $\alpha = \beta$ and thus have a lower dispersion. This means that when a particle
crosses from one network to another it finds a more homogeneous neighborhood than in the case in which
stays in the same network, and as we will show below low-
degree nodes are filled more often than high-degree nodes.
Also we can see that the dispersion, for all cases, slightly
grows with $q$, and this is due to the fact that hubs are less
overloaded and can participate more often in the diffusion
process.

![Fig. 5](image_url)

Fig. 5: (Colour on-line) Degree distributions $P_{\alpha\beta}(k)$ of the
particles which get deposited on a node with connectivity $k$
on the network $\beta$ after being dropped on the network $\alpha$ for
$q = 0.5$. With $P_{AA}$ (○), $P_{AB}$ (□), $P_{BA}$ (×) and $P_{BB}$ ($\triangle$).

![Fig. 6](image_url)

Fig. 6: (Colour on-line) Rates $\nu^\alpha_{\alpha\beta}$ and $\nu^\alpha_{\beta\alpha}$ at which a particle
spreads from $\alpha$ to $\beta$ network and gets directly deposited or
deposited on a neighboring node, respectively. With $\nu^\alpha_{AB}$ (○),
$\nu^\alpha_{BA}$ (□), $\nu^\beta_{AA}$ (×) and $\nu^\beta_{AB}$ ($\triangle$).

![Fig. 7](image_url)

Fig. 7: (Colour on-line) Probabilities $P^d_{\alpha\beta}(k)$ and $P^n_{\alpha\beta}(k)$, that
after an external diffusion, the particle gets deposited directly
or gets deposited on a neighboring node with degree $k$, respect-
ly, for $q = 0.5$. With $P^d_{AB}$ (○), $P^n_{AB}$ (□), $P^d_{BA}$ (×) and $P^n_{BA}$ ($\triangle$).

We want to understand the reason that makes the load
gerather deposited on low-connectivity nodes when it
crosses to a different network. In order to explain this ef-
effect we study the diffusion process when the load crosses
to the other network. When a particle spreads from
one network to another, it can be directly deposited on
a node connected by the external connection or it can
be deposited in one of its neighbors. To understand
which of these two scenarios is more probable, we com-
pare $\nu^\alpha_{\alpha\beta}$ and $\nu^\alpha_{\beta\alpha}$, which are the rates at which a particle
that spreads from one network to another gets deposited
directly (d) or in a neighboring (n) node, respectively. No-
tice that $\nu^\alpha_{\alpha\beta} + \nu^\alpha_{\beta\alpha} = 1$. In fig. 6 we plot these rates as a
function of $q$ and we can see that $\nu^\alpha_{\alpha\beta}$ is more important
than $\nu^\alpha_{\beta\alpha}$, which means that for any $q$, most of the times
the particles that cross from one network to another get
directly deposited.

In order to explain the last observation, we define
$P^d_{\alpha\beta}(k)$ and $P^n_{\alpha\beta}(k)$ as the probabilities that, after
an external diffusion, the particle gets directly deposited
or gets deposited on a neighboring node with degree
Then when majority in SF networks. We observe that in average these deposited in nodes with low connectivity, which are the tion is due mainly to the diﬀusion of neighboring deposition, have a wider spectrum, which agrees with the fact that in SF networks there are a few nodes with high degrees that receive the load by diﬀusion from their neighbors. This mechanism reduces the ﬂuctuations, due to a matching of the heights of low-degree and high-degree nodes that is more eﬃcient as q increases. Finally the system is optimally synchronized for q = 1.

Conclusions. – We study the synchronization in two SF networks where the dynamic of growth is ruled by a modiﬁed SRM model. We study the ﬂuctuations in the steady state of each network as a function of the interacting parameter q and we ﬁnd that the synchronization of each network improves when the interconnection between them increases. This improvement in both networks is about a 40% better than in isolated networks when q = 1, which is an important value regarding the decrease of the ﬂuctuations. However, we show that the rate of this improvement decrease with the number of interconnections and this can be an useful result in future research to determine if a larger interconnection between networks apart from q = 1, is worth it.

On the other hand, the improvement in the synchronization is due mainly to the diﬀusion through external connections. We also found that the majority of the particles that travel through the external connections are directly deposited in nodes with low connectivity, which are the majority in SF networks. We observe that in average these nodes usually have the lowest heights in a SF network. Then when q increases, the height of the nodes with low connectivity increases compared to the mean value and the difference with the height of the hubs decreases. We also found various distinctive characteristics of the model, such as the fact that the synchronization and the heights of the hubs become almost independent of the internal degree distribution as q increase, or the fact that the percentage of particles that diﬀuse between networks only depends on q and not on the internal degree distribution of the networks. This result indicates that for high values of q the behavior of the system is ruled by the interconnection between networks and not by the topology of the system.

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