Percolation of hierarchical networks and networks of networks

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Much work has been devoted to studying percolation of networks and interdependent networks under varying levels of failures. Researchers have considered many different realistic network structures such as scale-free networks, spatial networks, and more. However, thus far no study has analyzed a system of hierarchical community structure of many networks. For example, infrastructure across cities is likely distributed with nodes tightly connected within small neighborhoods, somewhat less connected across the whole city, and even fewer connections between cities. Furthermore, while previous work identified interconnected nodes, those nodes with links outside their neighborhood, to be more likely to be attacked or to fail, in a hierarchical structure nodes can be interconnected in different layers (between neighborhoods, between cities, etc.). We consider here the case where the nodes with interconnections at the highest level of the hierarchy are most likely to fail, followed by those with interconnections at the next level, etc. This is because nodes at higher levels of the hierarchy have the longest links as well as having more flow passing through them. We develop an analytic solution for percolation of both single and interdependent networks of this structure and verify our theory through simulations. We find that, depending on the number of levels in the hierarchy, there may be multiple transitions in the giant component (fraction of connected nodes), as the network separates at the various levels. Our results show that these multiple jumps are a feature of hierarchical networks and can affect the vulnerability of infrastructure networks.

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I. INTRODUCTION

The robustness of infrastructure systems can be understood through the frameworks of complex networks, percolation, and interdependent networks [1–12]. The initial research on network robustness was later expanded to include various network structures such as different degree distributions [13–15], clustering [16–18], spatial embedding [19–21], and, quite recently, community structure [22,23]. Additional research has considered various types of attacks on these networks such as degree-based attacks [24,25], localized attacks [26,27], and attacks based on nodes linking across communities [22,23,28].

Despite these advances, there remain network structures that are likely relevant for robustness that have not yet been studied. Among these is a hierarchical structure, which we study here, where communities connect loosely with one another to form larger communities, which then connect to one another, and so on [29–32] (see Fig. 1). In the context of infrastructure robustness these hierarchical modules likely describe real neighborhoods overlapping to form cities, which then overlap to form states, etc., which are then interconnected among themselves.

Furthermore, in this model, the nodes at the highest level of the hierarchy (e.g., between states) are likely more vulnerable to failure or attack than those at the next highest level, which are in turn more vulnerable than those at an even lower level, etc. This is because the nodes at higher levels have longer distance links between them which are more likely to fail or be attacked [33] and also have higher betweenness [22], which yields an additional load on them [34,35]. Moreover, recent work by da Cunha *et al.* [28] showed that attacks on these 'interconnected nodes' are an optimal form of attack on the US power grid, an infrastructure system of critical interest.

II. FAILURE AND ATTACK IN COMPLEX NETWORKS

We now provide a short review of the analytic methods for studying the effects of various attacks on complex networks. In the next sections we make use of these methods to find an analytic solution describing the fractional size of the giant component in our model under attack. We recall the definitions from Callaway *et al.* [25] for the generating function of a variable x,

$$G(x) = \sum_{k=0}^{\infty} P_i(k)x^k, \tag{1}$$

where k is the number of links and $P_i(k)$ is the likelihood that a node has exactly k links.

For targeted attack, they also define

$$F_0(x) = \sum_{k=0}^{\infty} r_k P_i(k)x^k, \tag{2}$$

where the symbols are as before, except that r_k represents the likelihood that a node with exactly k links fails. Next, the generating function of the branching process, $F_1(x)$, is given by

$$F_1(x) = F_0'(x)/G'(1), \tag{3}$$

where $F_0'(x)$ means the derivative of $F_0(x)$ with respect to x, and likewise for $G'(1)$.

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Given the above, the distribution of sizes of clusters of connected nodes reached by following a randomly chosen edge is given by

$$H_1(x) = 1 - F_1(x) + xF_1(H_1(x)). \tag{4}$$

Similarly, the distribution of sizes of clusters from a randomly chosen node is given by

$$H_0(x) = 1 - F_0(1) + xF_0(H_1(x)). \tag{5}$$

It was noted that above the percolation threshold, this refers to the sizes of clusters that are not in the giant component and thus $H_0(1)$ gives the fraction of nodes that are not in the giant component [25]. The fraction of nodes in the giant component, P_∞, can thus be found by

$$P_\infty(x) = 1 - H_0(1) = F_0(1) - F_0(u), \tag{6}$$

where u is given by

$$u = 1 - F_1(1) + F_1(u). \tag{7}$$

When a $1 - p$ fraction of nodes is removed randomly from an Erdős-Rényi network (Poisson distribution of links) with average degree $\langle k\rangle$ per node, it is found that $G(x) = e^{\langle k\rangle(x-1)}$ [36] and $F_0(x) = pG(x)$, where p is the fraction of surviving nodes. Continuing the derivation leads to the classical result from Erdős-Rényi, $P_\infty = p(1 - e^{-\langle k\rangle P_\infty})$.

For the case of modular networks with Poisson-distributed inter- and intraconnections, where the average degree of interconnections is k_{inter}, the average degree of intraconnections is k_{intra}, and r is the fraction of interconnected nodes that survive, the generating functions are [22]

$$G(x) = e^{(k_{\text{intra}}+k_{\text{inter}})(x-1)}, \tag{8}$$

$$F_0(x) = e^{k_{\text{intra}}(x-1)-k_{\text{inter}}}(1 - r) + rG(x), \tag{9}$$

$$F_1(x) = F_0'(x)/G'(1). \tag{10}$$

Following the above derivation leads to the formula for P_∞ in modular networks [22]

$$P_\infty = \begin{cases} e^{-k_{\text{inter}}}(1-r)(1-e^{-k_{\text{intra}}P_\infty}) + r(1-e^{-(k_{\text{intra}}+k_{\text{inter}})P_\infty}), & 0 < r < 1, \\ \frac{p}{m}(1-e^{-mk_{\text{intra}}P_\infty}), & r = 0, \end{cases} \tag{11}$$

where k_{inter} is the average degree of interconnections, k_{intra} is the average degree of intraconnections, and r is the fraction of interconnected nodes that remain. Once $r = 0$, all interconnected nodes are removed and the model continues with removal of the remaining nodes randomly.

Note that for $r = 0$, the value of P_∞ is divided by m since at this point the modules are separated and thus the fraction of nodes in the giant component is scaled by 1 over the number of modules. We note that the earlier study [22] found that the network may segregate into separate modules before collapsing or it may collapse all together as a single network. It has also been found that if nodes are targeted entirely randomly (i.e., no preference for attacking interconnected nodes), then P_∞ is the same as for an Erdős-Rényi network with degree $k_{\text{inter}} + k_{\text{intra}}$. [22]

More details on the above derivations can be found in Refs. [22,23,25].

III. MODEL

We now develop and analyze a stochastic block model [37–40] with overlap among the various blocks (modules). We first define a vector $\vec{m}$ describing the number of distinct modules or communities (blocks) in each layer of the hierarchy. In the first layer we always consider the entire network as a single community, thus $m_1 = 1$. The next layer, m_2, counts how many modules are in the second layer. Next is the total number of modules in the third layer, followed by the number of modules in the fourth layer, etc. We also assume, for simplicity, that all of the m_j modules in layer j are broken down into the same fixed number of m_{j+1} modules. For example, for the network shown in Fig. 1, we define $\vec{m} = [1, 3, 12, 36]$; since the top layer is a connected graph, in the next layer we have 3 modules, then a total of 12 modules (i.e., each of the 3 is broken down into 4 smaller modules), and, finally, 36, since each of the 12 modules is broken down into 3 additional ones.

We next define the vector $\vec{k}$, which describes the average degree between nodes connected in each layer of the network. Thus, if in the highest layer the nodes have an average of 0.1 link to nodes in other modules, this will be the first entry, k_1, in $\vec{k}$. If the average degree in the next layer is 0.3, then that

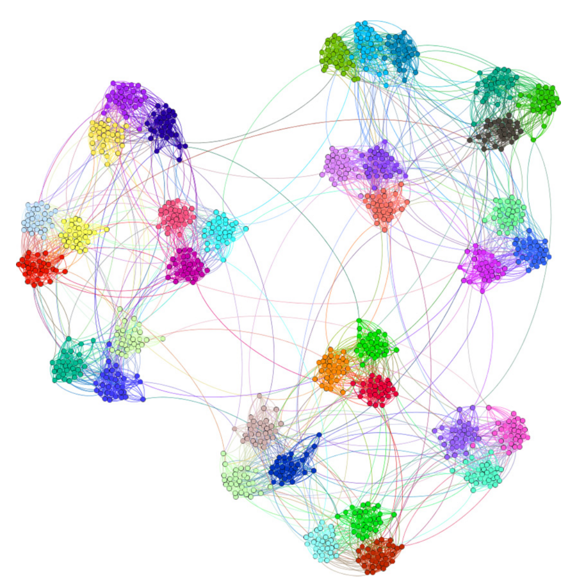

FIG. 1. Model illustration. For this realization, the model has four hierarchical layers. In the top layer there are three modules, each of which is broken down into four modules, each of which is then broken down into three modules, which are not broken down further. We can describe this configuration of hierarchical modules by the vector $\vec{m} = [1, 3, 12, 36]$.

will be the second entry, k_2, etc. We assume that the entries in $\vec{k}$ should be strictly increasing since we expect there to be more links within communities in a lower layer than in a higher layer (e.g., neighborhoods are more tightly connected than cities).

In our model, we carry out a targeted attack on the nodes of the network, assuming that nodes that are interconnected in the top level are most likely to fail, followed by those connected in the second level, etc. To do so, we must determine how many nodes are connected in each level and convert from the survival likelihood of nodes in a given layer, r_i, to the overall survival likelihood, p. First, we must estimate the fraction of nodes that are connected in level i. We note that the distribution of links in each layer is Poisson, with the likelihood of a node's having k links being given by $P(k) = k_i^k e^{-k_i}/k!$, where k_i is the average degree in layer i. We can then find the likelihood of not having any links as $P(0) = e^{-k_i}$ and thus the likelihood of having at least one link (being interconnected) is $1 - e^{-k_i}$. We define r_i as the survival probability of interconnected nodes in layer i. For the top layer of interconnections we can find how the $1 - p$ overall fraction of nodes removed from the network corresponds to the $1 - r_1$ fraction of interconnected nodes removed using [22]

$$p = r_1(1 - e^{-k_1}) + e^{-k_1}. \tag{12}$$

This equation can be understood by recognizing that the likelihood of a node's *not* having an interlink (i.e., not being interconnected) in layer 1 is given by e^{-k_1} and therefore the likelihood of being interconnected is $1 - e^{-k_1}$. Since r_1 is the fraction of interconnected nodes that survive, the overall survival probability is given by multiplying r_1 by the fraction of nodes that are interconnected and adding the fraction of nodes that are not interconnected (since all noninterconnected nodes survive the attack).

After we have removed all interconnected nodes in the top level, we then begin removing interconnected nodes in the next level of the hierarchy. In order to convert from the survival probability of nodes in this next level, r_2, to the overall survival probability, p, we must take into account that some of the nodes removed from the previous layer were likely interconnected in this layer too, but as they have already been removed we must remove them from our calculation.

We do so by first finding the value of p for removing all interconnected nodes in each respective layer. For the first layer, this is when $r_1 = 0$ and thus the value of p at which all interconnected nodes in layer 1 are removed is $p_{\mathrm{co}_1} = e^{-k_1}$, where we have defined p_{co_1} as the *cutoff* value for layer 1. For the next layer the cutoff at which all interconnected nodes are removed is given by recognizing that the fraction of interconnected nodes in this next layer is $1 - e^{-k_2}$, but we have already removed a $1 - p_{\mathrm{co}_1}$ fraction of nodes. We must recognize that some nodes are also likely to be interconnected in *both* layers and we must make sure not to double-count them. We can thus find p_{co_2} using the inclusion-exclusion principle as

$$\begin{aligned} p_{\mathrm{co}_2} &= 1 - (1 - e^{-k_2} + 1 - e^{-k_1} - (1 - e^{-k_2})(1 - e^{-k_1})) \\ &= e^{-k_2 - k_1}, \end{aligned} \tag{13}$$

where in the top line $(1 - e^{-k_1})$ is the fraction of interconnected nodes in layer 1 and $(1 - e^{-k_2})$ is the fraction of interconnected nodes in layer 2, and we subtract $(1 - e^{-k_2})(1 - e^{-k_1})$, which is the fraction of interconnected nodes in both layers that were double-counted. To get the cutoff at which all interconnected nodes in either layer 1 or layer 2 are removed, we take 1 minus the fraction of nodes that are interconnected in either of our two layers. Simplifying terms gives the bottom line in Eq. (13).

Another, simpler way to arrive at Eq. (13) is to note that we now remove all nodes with an interlink in either the first layer or the second layer. The total degree in these layers is just the sum $k_1 + k_2$. Given, as we can immediately recognize based on the Poisson distribution for $k_1 + k_2$, that $e^{-(k_1+k_2)}$ is the fraction of nodes that will not be connected in either layer, if we remove all nodes that are interconnected in either of these layers, we will be left with exactly a fraction $e^{-(k_1+k_2)}$ of nodes.

We can continue along the above lines to recognize that as we move farther down the layers, we must include an additional k_i for each layer i. Thus, for a given layer i, the cutoff value of p for which all interconnected nodes in that layer are removed is

$$p_{\mathrm{co}_i} = e^{-\sum_{j=1}^{i} k_j}. \tag{14}$$

Having solved the case where all nodes in a given layer are removed, we can now consider the values of p for which only some fraction, $0 < r_i < 1$, of nodes in layer i survives. We can then convert from r_i, the survival probability in layer i (after all nodes in higher layers have been removed), to p using

$$p = r_i\left(p_{\mathrm{co}_{i-1}} - p_{\mathrm{co}_i}\right) + p_{\mathrm{co}_i}, \tag{15}$$

which can be understood by noting that the p_{co_i} fraction of nodes will always survive since they are not interconnected in layer i (or any higher layer) and the $r_i(p_{\mathrm{co}_{i-1}} - p_{\mathrm{co}_i})$ fraction of interconnected nodes in layer i survives.

IV. ANALYTIC THEORY FOR A SINGLE HIERARCHICAL NETWORK

We now present a theory for hierarchical networks, generalizing the results in Eq. (11). For the top layer, we can make use of the previous results on modular networks by setting our average interconnected degree to the degree in the layer we are attacking, namely, k_1. To determine the intra degree, we note that all connections below the layer we are attacking are randomly distributed and thus the average intra degree will be replaced with the sum of the degrees below the layer we are attacking, $\sum_{i=2}^{l} k_i$, where l is the number of layers. We thus obtain

$$\begin{aligned} P_\infty &= e^{-k_1}(1 - r_1)\left(1 - e^{-(\sum_{i=2}^{l} k_i)P_\infty}\right) \\ &\quad + r_1\left(1 - e^{-(\sum_{i=1}^{l} k_i)P_\infty}\right), \quad p_{\mathrm{co}_0} < p. \end{aligned} \tag{16}$$

Note that the result in Eq. (16) is only accurate until we have removed all the nodes that are interconnected in the top layer, i.e., as long as $p_{\mathrm{co}_0} < p$. Also note that Eq. (16) is the same as Eq. (11) when $r > 0$, with $k_{\mathrm{inter}} = k_1$ and $k_{\mathrm{intra}} = \sum_{i=2}^{l} k_i$.

Once we have removed all interconnected nodes in the first layer, we then move on to removing nodes that are interconnected in the second layer. In this case, the average degree of interconnections is now k_2 and the average degree of intraconnections is $\sum_{i=3}^{l} k_i$. Furthermore, we must recall that the survival probability has already dropped to p_{co_1}, which is distributed randomly from the perspective of these lower layers. We also must note that at this point the network is already split into m_2 separate modules and nodes that are interconnected in layer 2 survive with a probability of only r_2. Accounting for this gives

$$m_2 P_\infty = p_{\text{co}_1}\big[e^{-k_2}(1-r_2)\big(1-e^{-(\sum_{i=3}^{l} k_i)m_2 P_\infty}\big) + r_2\big(1-e^{-(\sum_{i=2}^{l} k_i)m_2 P_\infty}\big)\big], \quad p_{\text{co}_2} < p < p_{\text{co}_1}. \tag{17}$$

Note that this equation is only accurate for values of p for which all interconnected nodes in layer 1 are removed, but not all interconnected nodes in layer 2 are removed, i.e., $p_{\text{co}_2} < p < p_{\text{co}_1}$.

We can generalize the above results for all values of p to find

$$m_j P_\infty = p_{\text{co}_{j-1}}\big[e^{-k_j}(1-r_j)\big(1-e^{-(\sum_{i=j+1}^{l} k_i)m_j P_\infty}\big) + r_j\big(1-e^{-(\sum_{i=j}^{l} k_i)m_j P_\infty}\big)\big], \quad p_{\text{co}_j} < p < p_{\text{co}_{j-1}}. \tag{18}$$

To apply Eq. (18) for a given p one must first examine the p_{co_j} values to determine between which two cutoffs p is and then convert p to a corresponding r_j value. Finally, one plugs r_j, $p_{\text{co}_{j-1}}$, and the other parameter values into Eq. (18).

We compare in Fig. 2 the theory of Eq. (18) and simulations of a corresponding network, observing excellent agreement between them. The figure shows multiple discontinuities in P_∞ as a function of p. Such multiple transitions have previously been observed in a few models that consider bootstrap percolation or percolation on interdependent networks [41–43], however, to the best of our knowledge, multiple (more than two) transitions have not been observed under targeted attack with ordinary percolation as in our model. As all interconnected nodes in a particular layer are removed, the system experiences a discontinuous jump. We note that the number of layers minus 1 serves as an upper bound on the number of potential jumps, as there may not be more jumps than that, however, there may be *fewer* jumps, depending on the parameters. This generalizes the results in Ref. [22], where only for certain sets of parameter values did the network separate into distinct communities before collapsing entirely, while for other parameter values there was only a single collapse where all the communities failed in tandem. In the next section we assess the number of discontinuous jumps for a given set of parameters. While one could of course examine this by plotting the results of Eq. (18) and then observing how many jumps take place, we provide more intuition into the underlying process by considering the number of jumps explicitly via analytical considerations.

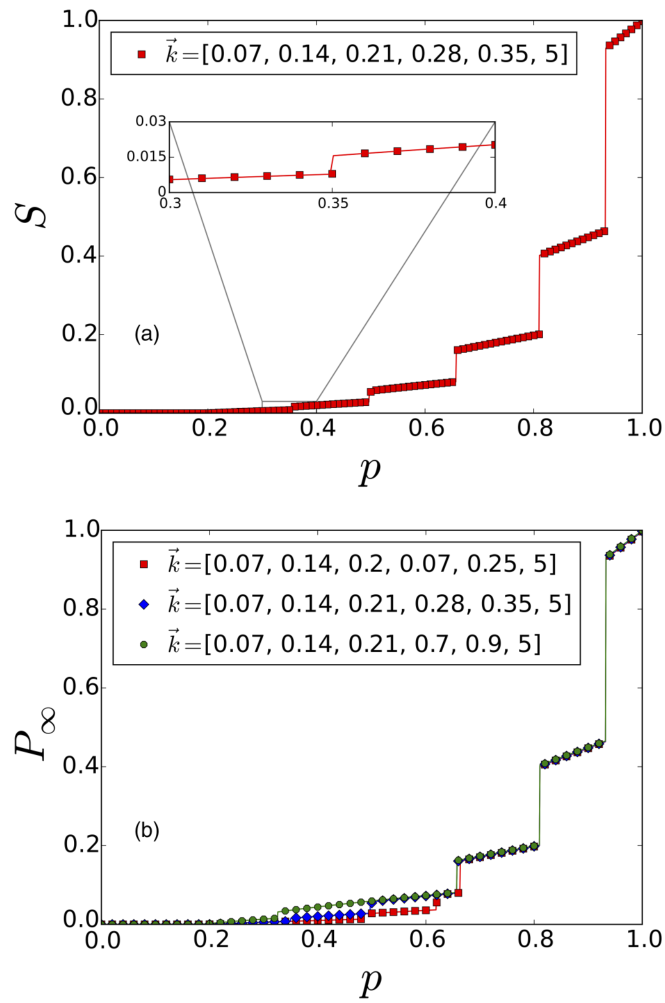

FIG. 2. Comparison between simulation results and theory. In both (a) and (b) there are six layers, with each layer splitting a module into two other modules; thus there are $\vec{m} = [1, 2, 4, 8, 16, 32]$ modules in each layer, with the average degrees between nodes in each layer given by the vector $\vec{k}$ in the legend. We have different values for the degree in each layer as given in the legends to (a) and (b). Lines represent the theory of Eq. (18) and points are simulations averaged over 10 runs on networks of $N = 10^6$ nodes.

A. Number of abrupt jumps

Here we evaluate the expected number of jumps that will take place using our analytic theory from above. The key insight is to note that jumps will occur when all interconnected nodes in a particular layer are removed, yet there remain enough total surviving nodes that the network in the next lower layer remains connected. For the limiting case where all nodes are connected in a particular layer, then the network will collapse before lower layers are reached since all nodes will already have been removed.

This condition can be expressed mathematically by recognizing that we need the value of p_c, the critical threshold of the remaining intralinks, to be lower than the value of p for which all interconnected nodes in a given layer i are removed. The classical result of Erdős-Rényi informs us that there will remain a giant component as long as $p > \frac{1}{\langle k \rangle}$, where $\langle k \rangle$ is the average degree, which for our case is $\langle k \rangle = \sum_{j=i+1}^{l} k_j$ or the sum of the degrees at all lower levels. The point at which all interconnected nodes in layer i are removed is given by the cutoff value defined in Eq. (14). Overall, the condition that

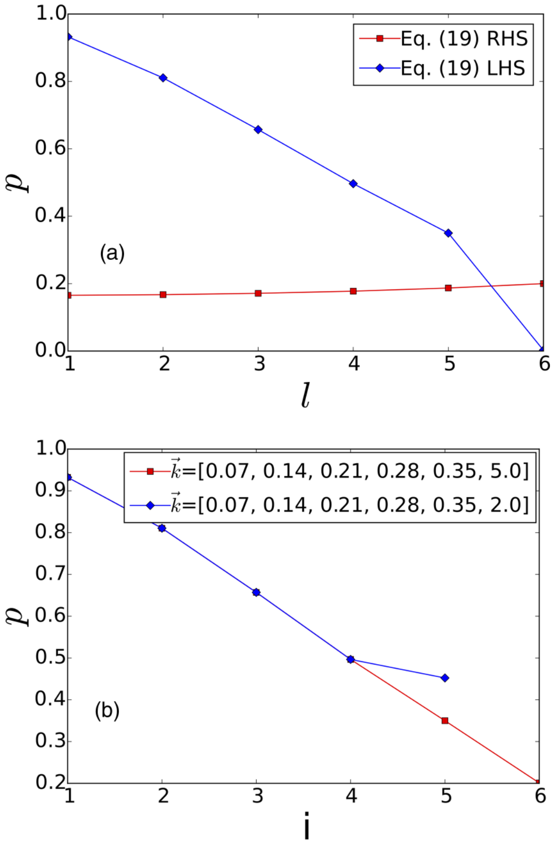

FIG. 3. (a) RHS and LHS of Eq. (19) for the network in Fig. 2(a). We observe that for the first $l \leqslant 5$ layers the value of p for which all interconnected nodes are removed [LHS of Eq. (19)] is greater than the value at which network intraconnectivity breaks down [RHS of Eq. (19)]. However, for the sixth layer this is no longer true and we observe the continuous percolation transition of a random network. (b) Values of the ith critical points for both the network described in Fig. 2(a) and another network which has a smaller degree in its bottom layer. We note how changing the degree in the bottom layer affects the number of jumps since, for the network with a lowest layer average degree of 2, before removing all interconnected nodes in the fifth layer, the network already breaks apart.

there will be a jump once all interconnected nodes in layer i are removed is

$$e^{-\sum_{j=1}^{i} k_j} \geqslant \frac{1}{\sum_{j=i+1}^{l} k_j}. \tag{19}$$

We note that, assuming all $k_i > 0$, then $e^{-\sum_{j=1}^{i} k_j}$ is strictly decreasing as i increases. Furthermore, the above assumption also implies that $1/\sum_{j=i+1}^{l} k_j$ is strictly increasing as i increases (since the denominator must decrease, as there are fewer k_j terms). Therefore, once the condition of Eq. (19) is first violated for a particular layer, we know that it will continue to break down for lower layers and thus we can be sure that our number of jumps is the number of layers for which Eq. (19) is valid.

We plot the two sides of Eq. (19) in Fig. 3(a), where for $l \leqslant 5$ we see that the left-hand side (LHS) of the equation is larger than the right-hand side (RHS). Compared to the number of jumps in Fig. 2(a) we see that the network indeed experiences five abrupt jumps as expected (see inset for the fifth jump).

B. The *p* values of the jumps

Having predicted above the number of jumps that the network will undergo, we can now analyze the multiple values of p_c, the critical thresholds at which the transitions occur. We first note that as long as the LHS of Eq. (19) is greater than the RHS of the same equation, there will be a transition at the point of the LHS of the equation. After these i transitions, there will be one final $i+1$st transition, which will be continuous as opposed to abrupt. We can find the point at which this final transition occurs by generalizing Eq. (21) from [23], which gives a form for the critical transition point of a modular network that has experienced both targeted attack of the type proposed here and random failures on all nodes. The formula found there was

$$\begin{aligned} &r_c^2\left(p_{\text{rand}}^2 k_{\text{intra}} k_{\text{inter}} e^{-k_{\text{inter}}}\right) + r_c\big(p_{\text{rand}} k_{\text{inter}} + p_{\text{rand}} k_{\text{intra}} \\ &- p_{\text{rand}} k_{\text{intra}} e^{-k_{\text{inter}}} - p_{\text{rand}}^2 k_{\text{intra}} k_{\text{inter}} e^{-k_{\text{inter}}}\big) \\ &+ \left(p_{\text{rand}} k_{\text{intra}} e^{-k_{\text{inter}}} - 1\right) = 0, \end{aligned} \tag{20}$$

where p_{rand} represents the survival probability due to the random failures, r_c represents the critical threshold, and k_{intra} (k_{inter}) represents the degrees of intraconnected (interconnected) nodes, respectively.

For our case of hierarchical networks, the random failures are represented by the attacks on nodes that were interconnected in higher layers. Overall, the probability of surviving the random attacks is p_{co_i}, thus p_{rand} is replaced by p_{co_i}. Furthermore, the inter degree k_{inter} is now given by k_{i+1} and the intra degree k_{intra} is $\sum_{j=i+2}^{l} k_j$. Thus, we can find the point of transition, which we call r_{i+1}, by solving Eq. (21), which is a quadratic formula for r_{i+1}:

$$\begin{aligned} &r_{i+1}^2\left(p_{\text{co}_i}^2\left(\sum_{j=i+2}^{l} k_j\right) k_{i+1} e^{-k_{i+1}}\right) + r_{i+1}\left(p_{\text{co}_i} k_{i+1}\right. \\ &+ p_{\text{co}_i}\left(\sum_{j=i+2}^{l} k_j\right) - p_{\text{co}_i}\left(\sum_{j=i+2}^{l} k_j\right) e^{-k_{i+1}} \\ &\left.- p_{\text{co}_i}^2\left(\sum_{j=i+2}^{l} k_j\right) k_{i+1} e^{-k_{i+1}}\right) \\ &+ \left(p_{\text{co}_i}\left(\sum_{j=i+2}^{l} k_j\right) e^{-k_{i+1}} - 1\right) = 0. \end{aligned} \tag{21}$$

After finding r_{i+1} we can convert it to a value of p using Eq. (15). We note a slight subtlety in this system, in that even for the case where the hierarchical network is completely isolated in the lowest level, we do not precisely recover the critical threshold of a random network with $k = k_{i+1}$. This is because we are targeting only those nodes which have at least one link. This leads to a slight correction where we obtain $r_{i+1} = 1/k_{i+1}$ (and then convert this to a value for the last p_c), rather than obtaining the usual $p_c = 1/k_{i+1}$. In most cases this correction will be quite small, as for any reasonable value of k at the lowest level, there will be very few nodes that do not have even a single link. For example, for the case of the network represented by the top line of the legend in Fig. 3(b), the transition for the sixth layer takes place at $p_c \approx 0.201$ as opposed to $1/k_6 = 0.2$. Nonetheless, it is worth noting this discrepancy.

V. INTERDEPENDENT NETWORKS

Much recent research has also explored the resilience of interdependent networks where the nodes of one network depend on nodes in another network [5,6,12,36,44–49]. One example is that of a communication network that is interdependent with a power grid, yet more complex interdependencies are also possible [50,51]. Many of these interdependent networks will likely possess the hierarchical structure described above. Therefore we now extend our theory from a single hierarchical network to the case of networks of interdependent networks (NONs).

We assume that each network in the interdependent system is formed of the same hierarchical structure, i.e., there is the same number of modules in each level. Again, this is realistic since the numbers of cities, neighborhoods, etc., that exist for the power grid are likely the same as those for a communications network as well as for other infrastructures. Further, we assume that nodes are dependent on other nodes within the same module in the lowest level. This corresponds to the assumption that nodes are most likely dependent on resources from nodes in their own neighborhood, i.e., a power station depends on a communication tower in the same neighborhood, and vice versa.

In the case of interdependent networks formed of n networks with $n > 2$, the structure of the dependencies can take various shapes. Among these are both treelike structures, where networks depend on one another such that their dependencies form a tree, or looplike structures, where the dependencies form loops. Here we consider (i) treelike structures and (ii) a random-regular (RR) network of networks where each network depends on exactly z other networks. Furthermore, one can allow for differing levels of interdependence where only some fraction q of nodes between two networks is interdependent, whereas the $1 - q$ fraction of nodes is autonomous, with no dependency. This could be the case, for example, if some communications towers have their own generators for power supply [52].

For the case of interdependent networks, it was noted [23] that the framework described above for failure and attack on complex networks can be extended by noting that each node now has an additional random probability p_{dep} of failure due to the presence of the dependency links. The precise expression for p_{dep} will depend on the number of dependent networks, the amount of the dependencies (q), and the structure of the dependencies (treelike, looplike, etc.). Equations (6) and (7) can be rewritten generally for any p_{dep} as [23]

$$P_\infty(x) = p_{\text{dep}}(F_0(1) - F_0(u)) \tag{22}$$

and

$$1 = p_{\text{dep}} \frac{F_1(u) - F_1(1)}{1 - u}. \tag{23}$$

A. Treelike network formed of hierarchical networks

Here we introduce the theory for a NON formed of n interdependent networks such that they form a tree. We also note that Eqs. (15) and (16) remain valid as for the single-network case, as we still target the nodes using the same procedure.

We assume that all nodes between pairs of interdependent networks have dependency links ($q = 1$). Further, we assume the no-feedback condition, meaning that if node a in network n_1 depends on node b in network n_2, then node b also depends on node a. We restrict the dependencies to be within the smallest community (i.e., the lowest layer of the hierarchy) for each set of interdependent networks. Finally, we attack the nodes of only one of the networks and let the attack propagate to the other networks as well as back to the original one. We note that the results are not dependent on the structure of the tree or the network in the tree from which the nodes are originally removed [36].

To include the effects of the interdependencies, we must add an additional likelihood of failure based on the interdependence. For a treelike network of n interdependent networks with the dependencies within the neighborhoods, the likelihood that all of a node's interdependent nodes will survive is $p_{\text{dep}} = (1 - e^{-(\sum_{i=j}^{l} k_i) m_j P_\infty})^n$ [23,36]. For a node to survive, it must survive and its dependent nodes must survive, thus p_{dep} is multiplied by our previous result from Eq. (18). This gives us the following solution for the treelike NON:

$$\begin{aligned} m_j P_\infty = {} & p_{\text{co}_{j-1}} \big[e^{-k_j} (1 - r_j) \big(1 - e^{-(\sum_{i=j+1}^{l} k_i) m_j P_\infty} \big) \\ & + r_j \big(1 - e^{-(\sum_{i=j}^{l} k_i) m_j P_\infty} \big) \big] \big(1 - e^{-(\sum_{i=j}^{l} k_i) m_j P_\infty} \big)^{n-1}, \\ & p_{\text{co}_j} < p < p_{\text{co}_{j-1}}. \end{aligned} \tag{24}$$

We note that numerical simulations show excellent agreement with the theory of Eq. (24) (Fig. 4). We also note that in the case of interdependent networks, the final transition is now also abrupt [5,6,36]. The abruptness of this transition is caused by a long cascade process that takes place in interdependent networks and that has been previously found for different models [53].

B. Random-regular network formed of hierarchical networks

Finally, we consider the case of an RR NON where each network depends on exactly z other networks. We assume that for each pair of interdependent networks only a fraction q of the nodes is interdependent and we allow feedback (in contrast to what was done for the treelike NON). However, we still restrict the dependencies such that they must be within the same community in the lowest level of the hierarchy. We also carry out the attack on all the networks, as opposed to attacking only one of them in the treelike case.

The effects of the dependencies now imply that the likelihood of a node's surviving all interdependencies is $p_{\text{dep}} = (1 - q + q m_j P_\infty)^z$ [36,54]. Again, a node must survive in its

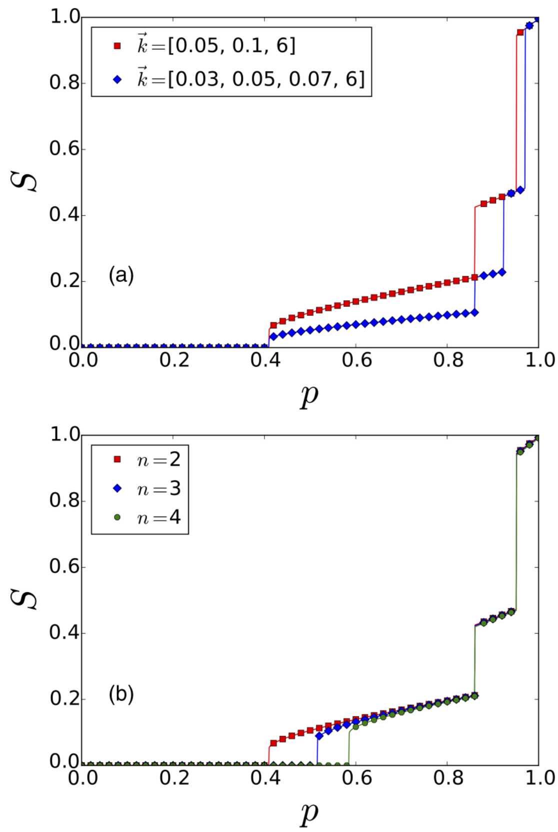

FIG. 4. (a) The case of two interdependent networks with the values of the degree in each layer as given in the legend. Points are simulations averaged over 10 realizations of networks with $N = 10^6$ nodes and lines are theory from Eq. (24). (b) Variation of the number of networks, with the degree vector fixed to $\vec{k} = (0.05, 0.1, 6)$.

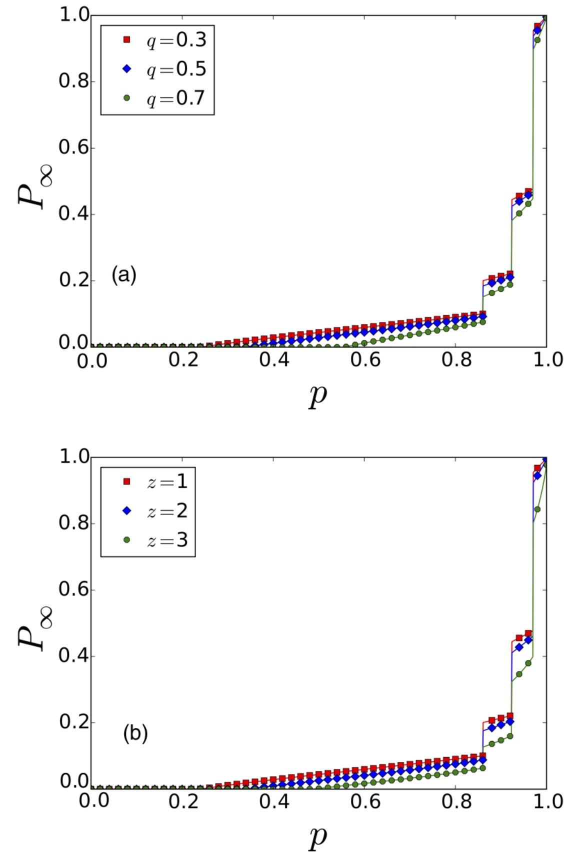

FIG. 5. Random-regular network of networks where each network depends on z other networks such that they form loops. We vary both (a) q, the level of interdependence between the networks (with $z = 1$), and (b) z, the number of networks each network depends on (with $q = 0.3$). Symbols are simulations averaged over 10 realizations on networks with $N = 10^6$ nodes, and lines are theory from Eq. (25).

own network as well, so combining this with Eq. (18) yields

$$P_\infty = p_{\text{co}_{j-1}}\left[\frac{1}{m_j}e^{-k_j}(1-r_j)\big(1-e^{-(\sum_{i=j+1}^{n}k_i)m_jP_\infty}\big) + r_j\big(1-e^{-(\sum_{i=j}^{l}k_i)m_jP_\infty}\big)\right](1-q+qm_jP_\infty)^z,$$
$$p_{\text{co}_j} < p < p_{\text{co}_{j-1}}. \quad (25)$$

For the RR NON the last transition will be continuous (for low values of q), as for an RR NON formed of Erdős-Rényi networks the transition may be continuous [54]. We observe excellent agreement between the theory of Eq. (25) and the simulations in Fig. 5.

VI. REALISTIC EXTENSIONS

Here we consider two basic extensions of the framework developed above. One type of extension involves considering degree distributions that are not Poisson and uncorrelated in each layer, while the second considers trade-offs between adding links in different layers of the hierarchy. We also discuss several other possible extensions but leave them for future work.

A. Varying the degree distributions

The first extension we consider is a degree distribution that is not Poisson in each layer of the hierarchy. For this purpose we use a power-law distribution, with the likelihood of a node having k links given by $P(k) \sim k^{-\lambda}$. This type of degree distribution is common to many networks such as social networks, biological networks, and others [55].

We consider the case where the lowest layer of our hierarchy has a power-law distribution, while the links in higher layers have a Poisson distribution as in the above versions of our model. In Fig. 6(a) we present results for two different hierarchical networks with different degree distributions in higher layers and a scale-free distribution in the lowest layer. In both cases we still observe the characteristic multiple jumps as in our earlier models. In fact, in the case of scale-free networks, since $p_c \to 0$ for an isolated scale-free network with $\lambda < 3$, we expect that the number of jumps will nearly always approach its upper bound of 1 less than the number of layers $(l-1)$, which will be followed by a final transition at $p_c \to 0$. Relating back to Eq. (19) a scale-free distribution with $\lambda < 3$ would imply that the RHS of the equation is always near 0 and thus for every layer the LHS will be greater.

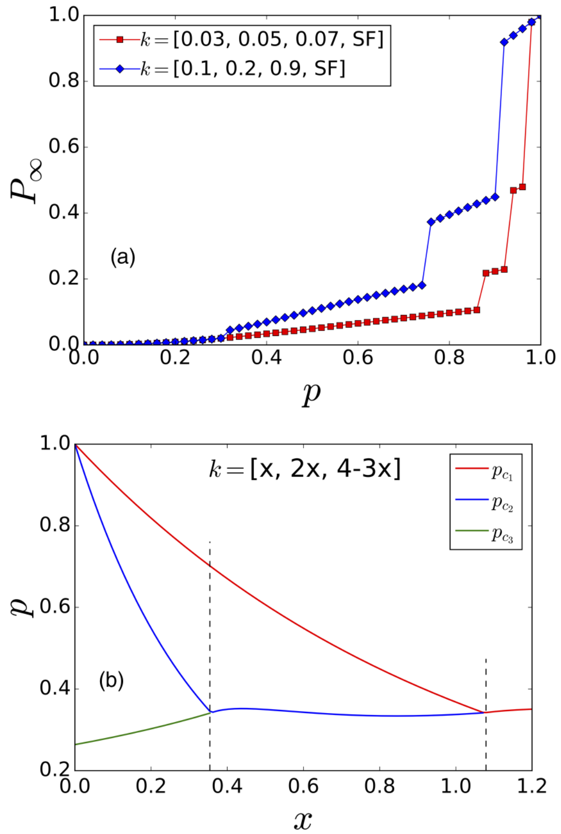

FIG. 6. (a) Two distinct hierarchical networks ($n = 1$) with a scale-free distribution in the lowest layer of the network. In the higher layers, the degree distribution is Poisson, with the average given by the entries in $\vec{k}$. For the scale-free distribution we set $\lambda = 2.5$, $k_{\min} = 2$, and $k_{\max} = 1000$. Results are averaged over 10 realizations with $N = 10^6$, and we note that the lines here are not theory but, rather, only a visual guide, as all results are from simulations. (b) A single network with a hierarchical structure composed of three layers, each with a Poisson distribution. However, we vary the degree in each layer according to a fixed equation and fixed maximum degree. We assume that in the highest layer the degree is given by x, the next layer is given by $2x$, and the final layer is given by $4 - 3x$. We then plot the locations of p_{c_i} vs x. We highlight with dashed vertical lines the two critical values of x where we move from having three transitions to two ($x \approx 0.35$) and from having two transitions to a single one ($x \approx 1.05$). All solid lines are based on the theory from Eq. (18).

Our results in Fig. 6(a) indeed confirm this, as we find that for the case shown of $l = 4$ there are always three jumps.

Furthermore, if we consider the case where the scale-free distribution is not in the lowest layer, we expect that all nodes in the scale-free layer would have to be removed before the network would segregate and thus there would be no jumps past the layer where the scale-free distribution existed. Alternatively, one could restrict *a priori* a specific set of nodes to have interlinks between them in the given layer according to a scale-free distribution. In this case once all such nodes were removed, the network would presumably segregate into distinct communities as long as the chosen set was not overly large.

A second extension involving degree distributions could relate to having the same nodes be interconnected in each layer. In this sense, one would first choose some set of nodes to be interconnected in the next-to-lowest layer and then choose a subset of those nodes to be interconnected in the above layer, followed by a subset of those nodes interconnected in the next higher layer, etc. If all the layers remained Poisson distributed, the main effect of these correlations between interconnections would be that the cutoff values of p in Eq. (14) would change. In this case, rather than having the cutoff be given by the sum of the degrees in all the layers, it would be given by $p_{\text{co}_i} = e^{-k_i}$, as only nodes that are interconnected in the given layer have been removed since those with interlinks in higher layers also must have interlinks in the lower layers (note that we assume that the k_i values are decreasing as one moves up the layers). The remainder of our original derivation would remain virtually unchanged. Interlinks between layers could also simply be correlated rather than the absolute nature suggested above and then the calculation of the cutoff values would be more complicated.

B. Trade-offs between links in different layers

Another extension we consider is trade-offs among adding links in various layers. Ideally, to consider such trade-offs one should specify the costs of failures in connectivity in each layer of the hierarchy and also specify the cost of adding a link in each layer. A framework for considering costs similar to these can be found in a recent work [56].

To give an example of a simpler version, we consider a hierarchical network of three layers with a fixed total degree. Furthermore, we place a condition stating that the number of links in the second layer must be twice (or, more generally, any factor of) the number of links in the first layer. Using Eqs. (19) and (21) we then find the critical point(s) of transition for the given network. In Fig. 6(b) we show how the critical points vary with the average degree in the first layer (defined as x). We find that there exist several possible critical values of x where the system moves from having three transitions to two and where it moves from having two transitions to a single one. These critical values of x represent what could be considered optimal trade-offs in preserving connectivity in specific layers. If we consider, for example, the first critical point in x, x_1, we note that this is the lowest value of x for which the middle layer does not segregate into communities and instead its connectivity is preserved as long as the overall network remains connected. Likewise, the second critical point in x represents the first point at which the network does not separate in the highest layer and instead the entire hierarchical network collapses all at once. This simple analysis shows that indeed trade-offs do exist between adding links in the different layers and provides some basic intuition into the more general case where costs are assigned to links in the different layers.

VII. DISCUSSION

In this work, we have studied the robustness of networks and networks of interdependent networks with a hierarchical structure. This structure is very common for many infrastructure networks, biological networks, and others. We have found analytical solutions and confirmed these solutions through simulations for isolated hierarchical networks and for two different structures of interdependent hierarchical networks. The resilience of the network depends on the number of communities in each layer of the hierarchy, the degree in each layer of the hierarchy, the fraction of nodes removed, and also the parameters governing the interdependence (if present). We have also extended our framework to consider the more realistic case of a scale-free distribution and different trade-offs between adding links in the various layers.

Our results show that hierarchical networks can undergo multiple abrupt transitions depending on the above parameters and that these transitions represent the separation of the network in different layers of the hierarchy. These results have potential applications in optimization of the resilience of networks in infrastructure and other fields.

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