Chapter 20
Spreading of Failures in Interdependent Networks

Louis M. Shekhtman, Michael M. Danziger and Shlomo Havlin

20.1 Introduction

Most studies of spreading focus on how some physical objects move in space, yet spreading can also involve other phenomenon. Recent research has explored the spreading of *failures* in various complex systems like power grids, communications networks, financial networks and others. In these systems, when one failure occurs it can trigger a cascade wherein that failure spreads to other parts of the system. Failure spreading can have dramatic results leading to blackouts, economic collapses, and other catastrophic events.

In order to combat this problem, it is often useful to model and understand the physical mechanisms of failure spreading. While it would be ideal if failures could be prevented entirely, this is unlikely since every system will experience failure at one time or another. Rather, the approach and models reviewed in this chapter focus on the mechanisms of how initial failures spread and how this information can be used to mitigate the spreading. It is also noteworthy that many of the same models used here in the context of failures in infrastructure networks can also be used to identify influential individuals who are capable of spreading a message to a large audience in social networks.

A specific focus here will be on the mechanism involved in the realistic situation where two systems are interdependent such that components of one system cannot function without components of the other. This could be for example, the case in the context of a communication tower that needs to receive power from a

L. M. Shekhtman (✉) · M. M. Danziger · S. Havlin
Department of Physics, Bar Ilan University, Ramat Gan, Israel
e-mail: lsheks@gmail.com

M. M. Danziger
e-mail: mmdanziger@gmail.com

S. Havlin
e-mail: havlins@gmail.com

A. Bunde et al. (eds.), *Diffusive Spreading in Nature, Technology and Society*,
https://doi.org/10.1007/978-3-319-67798-9_20

nearby power station. If the power station fails, that failure immediately spreads to the communication tower. The failure of the communication tower can then lead to additional failures that also impact the power grid (either directly or indirectly) and so on. Understanding how these failures spread in both time and space is critical in order to ensure that large-scale complex systems remain functional.

A further challenge examined here involves the question of how to optimally repair and recover a system after it has experienced some failures. While it may seem simple to repair failed components of a system, it will be ineffective if the spreading of the failures has not first been contained. Making repairs is useless if the failures quickly spread to the repaired components once again. Instead, repairs must be made in a clear and purposeful way in order to restore the system to a functional state.

Many complex systems such as power grids, communication systems, the internet, and biological systems have recently been modeled as *complex networks*. This representation of the system involves defining sets of nodes that are connected to one another through links. Precisely what constitutes a node and/or link will depend on the exact system being analyzed. For example, in power grids the nodes are typically defined as the power stations and the links are powerlines that connect the power stations. In communication networks, nodes could be antenna towers and towers that are in range of one another are linked. Many different systems can be modeled in such a manner and researchers have discovered that while different systems have unique properties, many global network properties remain true across multiple systems.

Most of the properties discovered in complex networks relate to the structure of the connections between the nodes. The number of connections of a particular node is known as its degree, k. Networks where connections are assigned purely randomly (Erdős-Rényi networks) have a degree distribution that is Poisson. The most important feature of Poisson distributions, at least in the context of complex networks, is that they have a typical mean degree, $< k >$ and it is highly unlikely for any node to have a degree that is substantially larger or smaller than $< k >$. Explicitly the likelihood of a node to have degree k, is given by $P(k) = < k >^k e^{-k}/k!$. (Note also, that in the text we will simply refer to the mean degree of an Erdős-Rényi network as k rather than $< k >$ and that it is common to do so in the literature). Early research found that the distribution of the degree in real networks often takes the form of a power law. This means that the likelihood of a node to have degree k is proportional to $k^{-\gamma}$, i.e. $P(k) \sim k^{-\gamma}$. Notably, if $\gamma < 3$, then some nodes end up with far more connections than others and the variance tends to infinity. Networks with this property are known as scale-free networks [1]. Another unique feature present in many networks is the existence of tightly connected communities that have many links to other nodes in the same community (module), but few links to nodes outside of the community [2]. This property is often referred to as modularity and it is highly ubiquitous in many networks. Lastly, many networks, like power grids, are embedded in physical space (spatial networks) and the expense of creating long-range links forces most links to be of short length [3]. There are many other significant structures that exist in networks which are more fully reviewed in one of the recent books on the subject [4, 5]. More information on diffusion in complex networks can be found in [6] or in [7, 8].

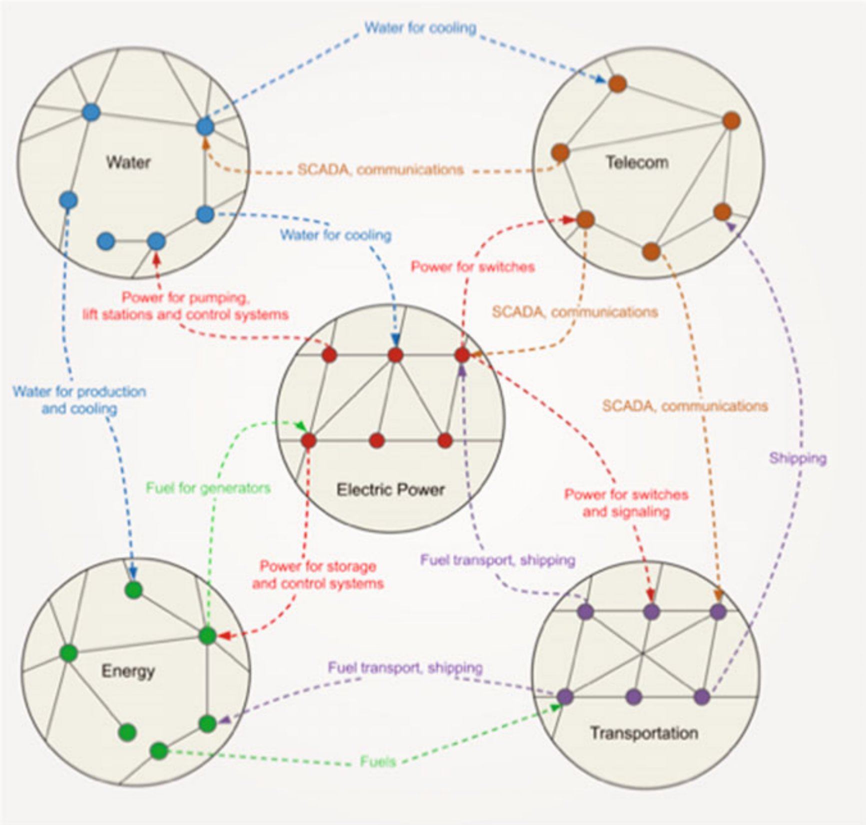

Fig. 20.1 Interdependence in modern infrastructure causes failures to spread between systems. This is the result of multiple systems needing power for switches, supervisory control and data acquisition (SCADA), fuel transport, and other resources. After [20]

Further research has led to the recognition that many networks do not exist in isolation, but rather a network is often only one of several interdependent networks [9–14]. This situation refers to the case where a node in one network, say a communication antenna tower, depends on a node in another network, say a power station. This relationship can be described through the existence of a new type of link known as a *dependency* link [15–17]. Whereas connectivity links represent the idea that some sort of flow occurs between the two connected nodes (e.g. flow of electricity in power grids, flow of information in communication networks), dependency links mean that if the node being depended upon fails then the dependent node also fails. Such situations are especially common in infrastructure, but they can also arise in biological systems [18] and financial networks [19]. An example detailing the interdependence between different infrastructure networks can be found in Fig. 20.1.

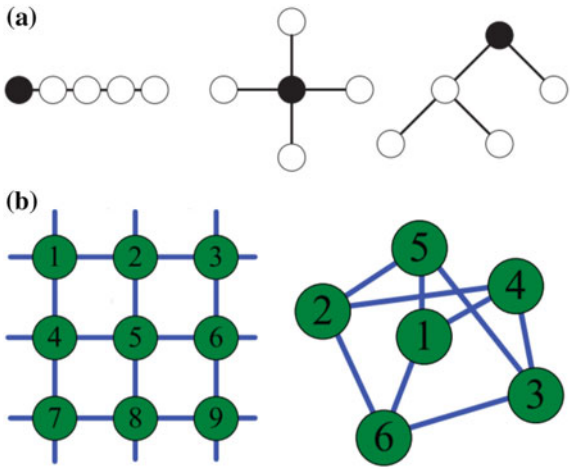

Fig. 20.2 Different types of interdependent networks are shown in the figure. In this case, each node shown above actually represents an entire network and the links in the figure represent the existence of dependency links between two networks. The path through which failure spreading in interdependent networks occurs is determined according to which networks have dependency links between them. (**a**) The top structures are various treelike networks of networks (NON) structures and (**b**) the bottom structures are NONs with loops. On the left is a lattice and on the right is a random-regular NON structure. After [21, 22]

As seen in Fig. 20.1, interdependencies can take complex forms. This has led researchers to refer to interdependent networks as 'networks of networks' (NON). Dependency links exist between specific pairs of networks and the structure of the network of networks is defined according to which pairs of networks have dependency links. A few examples of networks of networks are shown in Fig. 20.2. Simple examples include cases where the NON dependencies form a tree, a single loop, and a random-regular configuration where all networks depend on the same number of networks.

One of the most important properties of many networks and systems in general is their robustness to failures. As discussed briefly above, power stations can become overloaded or fail for other reasons and communication antenna towers can have problems due to bad weather or other issues. While efforts are always made to minimize the frequency of these failures, they are bound to occur. When analyzing the system as a whole, it is desirable to optimize the network such that it can still continue functioning even if some of the nodes fail. In many cases the functioning of the system can be quantified by asking how many nodes remain connected after some nodes fail. For example, for communications networks it is often most relevant to ask, "How many nodes can communicate after some others fail?" or in the context of power grids, "How many power stations are still linked to the grid after some stations fail?" The failure of one node can cause other nodes to become disconnected from the network as a whole and fail as well, thus the initial failures are magnified and can spread throughout the network.

The question of what fraction of a system remains connected after some set of failures can be answered through percolation theory from physics. Percolation theory essentially determines clusters of nodes that are connected to one another such that flow can occur between them. The largest cluster, which contains the largest number of nodes, is referred to as the *giant connected component* and is described by P_∞ [4, 5, 23, 24]. Explicitly, P_∞ is defined as the fraction of nodes remaining in the largest connected component (or equivalently, the likelihood of a node to be in the largest component) at some point in a percolation process. For our purposes, only nodes that are part of this largest cluster are considered functional whereas all other nodes are considered to have failed. The goal in designing resilient systems is to maximize the size of the giant component for any case of failures.

Percolation theory was able to discover that scale-free networks (i.e. those whose degree distribution follows a power-law) are far more resilient to random failures than random networks. In other words, if the same number of failures occur in both random and scale-free networks, a larger fraction of a scale-free network will remain connected. More precisely, in contrast to random networks where only a finite fraction of nodes must be removed, for scale-free networks only if nearly all of the nodes are randomly removed, will the network become totally fragmented [25]. In any case, for both isolated random and isolated scale-free networks, slightly increasing the number of initial failures only slightly increases the number of total failures. In other words, the transition from a functioning to non-functioning state is continuous.

Failures in interdependent networks occur and spread through two different mechanisms. The first is the same as in single networks, i.e. failures of nodes lead further nodes to become disconnected from the giant component. The second mechanism is through failures spreading due to the dependency links. As mentioned previously, a node at one end of a dependency link relies on the node at the other end of the link to function. If a node on one end of a dependency link fails, the node on the other end of the link also fails. As we will see in the next section, such mechanisms can lead to abrupt collapse.

20.2 Robustness of Interdependent Networks

Percolation methods were widely applied to solve problems in single networks [4, 5, 25–27] and recent research has expanded these methods to interdependent networks [15, 16, 21, 28, 29]. In interdependent networks, when some nodes fail, they cause other dependent nodes to fail [15]. The failure of these dependent nodes then disconnects other nodes from the giant component and leads to the failure of more dependent nodes. In this manner, failures spread through the system until a steady state is reached. It is noteworthy that because of the cascade, removing a single additional node can cause the system to collapse entirely, i.e. the transition is abrupt and first-order [15, 16, 28, 30]. This is significantly different from isolated networks where the transition is continuous.

In the initial work on interdependent networks, Buldyrev et al. [15] calculated the final fraction of functional nodes after the cascade analytically. They also carried out numerical simulations to verify their results. In explaining the results from [15], it is important to note the result for percolation of a single Erdős-Rényi network, namely $P_\infty = p(1 - e^{-kP_\infty})$, where p is the fraction of the network that survives the initial failures and k is the average degree of the network [31–33]. It is also noteworthy that since P_∞ appears on both sides of the equation and no additional simplification is possible, the equation is transcendental and can only be solved numerically. If p nodes survive the initial failures in a system of two interdependent Erdős-Rényi networks, the size of the giant component is described by [15, 28],

$$P_\infty = p(1 - e^{-kP_\infty})^2. \tag{20.1}$$

While the difference between the formulas for P_∞ for single and interdependent networks may seem small, namely a power of 2 instead of 1 on the right side of the equation, this small change has dramatic consequences leading to the long cascades and abrupt failures described throughout this review. Essentially, the power of 2 can be understood by recognizing that nodes now must be both in the giant component of their own network and have their dependent node be in the giant component of the second network. Gao et al. [21, 22, 28, 34] later solved several cases that involved more than two networks. In the case of n interdependent Erdős-Rényinetworks with full dependency such that they form a tree (Fig. 20.2a), the size of the giant component is given by [28]

$$P_\infty = p(1 - e^{-kP_\infty})^n. \tag{20.2}$$

Gao et al. [21, 28] also solved several other simple structures of networks of networks analytically.

Other papers by Bianconi et al. [35, 36], Baxter et al. [30, 37], Hu et al. [38], Kim et al. [39], Lee et al. [40] and Cellai et al. [41] have also obtained further analytic results for interdependent networks.

20.3 Interdependent Networks with Realistic Features

In this section we will provide a brief review on more realistic models of failure spreading in interdependent networks. The internet and power-grids, as well as other networks, are not purely random and instead contain non-random structure. This structure influences the spreading of failures.

One common feature is a degree distribution that is scale-free. It was found for interdependent scale-free networks, that a broader degree distribution makes the networks *more* vulnerable to the spreading of failures [15, 42].

Another realistic feature that has recently been found to influence failure spreading in single and interdependent networks is modularity [43, 44]. Both of those studies [43, 44] examined the case of attacks on interconnected nodes, i.e. nodes that

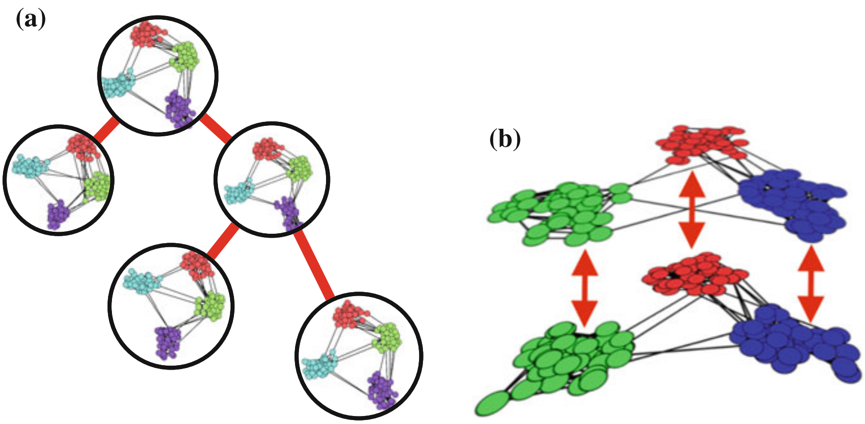

Fig. 20.3 Here we show examples of modular structure in interdependent networks. In this case, each of the interdependent networks has a modular structure, i.e. they are segregated into distinct tightly connected communities. This is seen by the fact that inside each black circle in (**a**) there is a network with modular structure. Further, each of the communities is highlighted with a different color. The specific structure shown in (**a**) is a treelike network of modular networks. The dependency links are restricted such that a node in a particular module in one network will depend on a node that is in the same module in the second network (i.e. dependency links are between nodes of the same color). This is illustrated clearly in (**b**). After [43]

connect between two communities. Another study using a similar model, showed that attacks on interconnected nodes lead to very fast spreading of failures, especially in the Western U.S. power grid [45]. Shekhtman et al. [43] solved analytically the case where there are several networks each of which has the same number of modules of the same size. Dependency links were also restricted to be between corresponding modules in different layers. An example of where this model is realistic is the case of infrastructure within and between cities. Each city has its own infrastructure and the interdependence occurs within the city. At the same time, different infrastructure networks will connect both within and across several cities. Consider the example of a coupled system of a power grid and a communications system. Most likely, a power station and a communication tower that depend on one another will be in the same city. This is true even though both the communication tower and the power station have connections to other cities. The model is visualized in Fig. 20.3. Failures can lead to collapse that occurs in one or two stages, i.e. there can be two transitions. When two transitions occur, the first is the result of modules separating but continuing to function independently. After additional failures, the modules themselves also collapse.

Failure spreading is also influenced by spatial features. It is well accepted that many infrastructures are embedded in space, including power grids and many communication networks [3, 46]. This embeddedness has significant influence on how failures spread. To simplify studies of spatial networks, 2D lattices are often used as models and it is noted that any other embedded network is in the same universality class [24, 47]. An early study on spatially embedded networks found that they

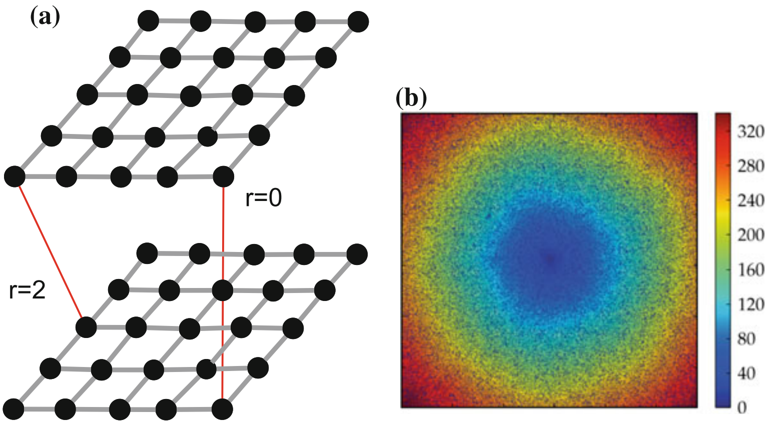

Fig. 20.4 **a** Dependency links can be restricted such that only pairs of nodes within some distance r are allowed to be interdependent. In the figure the cases of $r = 0$ and $r = 2$, where the pairs are zero and two lattice spaces apart are shown. After [51]. **b** The radial spreading at criticality is shown. The redder regions end up failing at later times (as shown in the colorbar on the right) in comparison to the central regions. After [50]

are extremely vulnerable in the sense that two interdependent networks can collapse abruptly even if only a few of the nodes in each network are interdependent [48].

Another study examined a case where the dependency links are restricted to a maximal length, r (demonstrated in Fig. 20.4a). This accounts for the fact that dependencies are most likely to be short range and that it is highly unlikely for example that a communication tower in the Eastern United States is dependent on a power grid in the Western United States. For short range dependency lengths, i.e. low r, the percolation transition is continuous, but for larger r, the transition is abrupt. The shift between the behaviors occurs when r reaches a critical value, $r_c \approx 8$ [49]. Above this critical dependency length, the percolation transition occurs in such a way that failures spread radially outward from an initial damage site until they end up finally consuming the entire network [50], see Fig. 20.4b.

Later works incorporated additional spatial features in order to move towards even more realistic models of interdependent spatial infrastructure including considering the case of NON formed of more than just two networks [50–53].

The cascade in interdependent networks can be mapped to other cascades like blackouts in power grids [54, 55]. Most blackouts and other failures spread in a predictable manner. Understanding the spatio-temporal spreading is crucial in order to understand and contain such failure spreading. Specifically, it has been found that the spatio-temporal dynamics of the cascade can be used to identify a specific dependency correlation distance that determines how failures spread [56]. This dependency correlation distance defines how far the failures are likely to spread. In that work, Zhao et al. [56] studied the case of overload failures in spatially embedded networks and examined how failures propagate in space and time. The authors defined load according to the well-known *betweenness* centrality, which measures how many

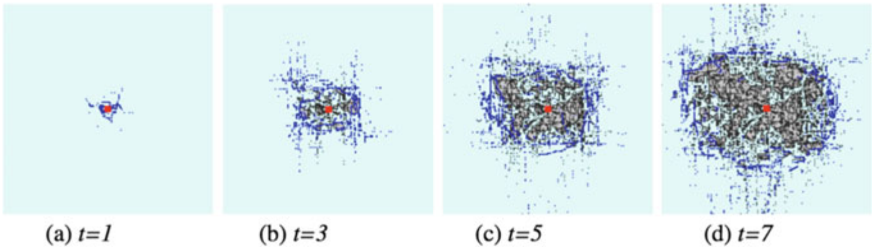

Fig. 20.5 The propagation of failures in a synthetic overloaded system are shown. The red nodes in the center represent the initial failures. At each time step, additional nodes that fail due to overloads are shown in blue. Nodes that have already failed are shown in black. As seen, the spread occurs almost radially outward from the location of the initial failures. After [56]

shortest paths go through a particular link [54]. The initial load depends on the structure of the network and nodes that have more shortest paths going through them have a higher load. After an initial set of localized failures, the paths between nodes change and load becomes redistributed, especially around the failed nodes. However, due to this redistribution, some other nodes will also become overloaded and fail. This process will continue either until the load manages to rebalance or until the entire network collapses. The dependency correlation distance describes how far the direct effects of the initial redistribution are felt. Zhao et al. [56] studied how the failures spread as a function of the tolerance, α. The quantity α is defined such that $1+\alpha$ times the original load is the maximal load above which the node becomes overloaded and fails. They found that for all values of α the spreading of failures occurs radially from the initial failures and spreads at approximately constant velocity. As α increases, the velocity of the spreading of failures decreases. This is intuitive as it means that the system is able to accept a higher increased load without failing. An example of the spreading in a synthetic power grid can be seen in Fig. 20.5. These results support the model of interdependent spatial networks studied in previous works [49, 57, 58] where now the velocities can be mapped to the length of the dependency links [56]. As explained earlier, the length of the dependency links represents the distance between two nodes that rely on one another. The velocity of the failure spreading in the model from [56] has the similar meaning of how quickly failures from a node in one location reach a node in another location. The specific procedure for mapping between these two quantities is described in [56].

Other aspects relating to more realistic models of interdependent networks have also been analyzed in many further works which consider many types of network structures and conditions on dependency links [38, 41, 59–72].

20.4 Localized Attacks on Interdependent Networks

Another realistic feature that has recently been incorporated into understanding the resilience of both single and interdependent networks is localized attack [57, 73, 74]. For localized attack on a pair of spatially embedded networks it was found that

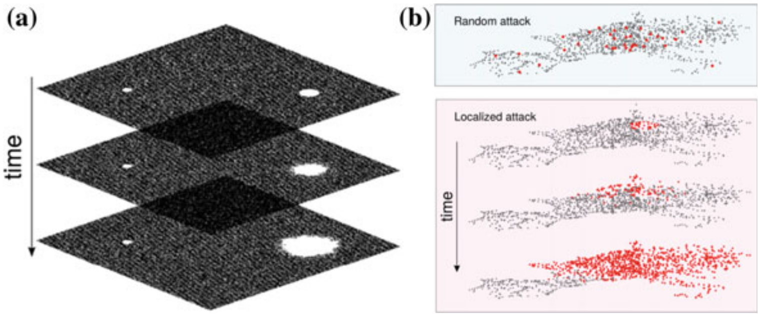

Fig. 20.6 **a** Depending on the initial size of the hole, it may either spread through a system of interdependent networks (the hole on the right) or not (the hole on the left). Whether the hole spreads, depends only on the degree of the networks and not on the number of total nodes in the system. **b** Here a localized attack is shown to spread on an interdependent system with a layout according to the European Power grid, whereas a random attack does not spread. After [57]

a 'hole' above a certain critical size must be made in one of the networks in order for the failures to spread throughout the entire system (see Fig. 20.6). The researchers in [57] found that even though a network may be robust to random attacks, it can be vulnerable to localized attacks. In addition, the critical size of the 'hole,' denoted r_h^c, that must be made to collapse the system is independent of the size of the system and instead depends only on the degree. This behavior is vastly different from the case of a single spatially embedded network where the size of the hole necessary for total system failure scales with the size of the system. Localized attacks are particularly relevant in the realistic case of an Electromagnetic Pulse (EMP) detonation, i.e. a short burst of electromagnetic energy that damages all electronic devices within some radius.

20.5 Recovery in Single Networks and Interdependent Networks

In order to understand the spreading of failures, it is also important to consider how to repair failures as they spread throughout a system. This question is of course highly relevant since while the goal is always to prevent failures from occurring, all systems will experience failure at some point. To address this question, researchers have begun studying how to optimally repair and recover a system like a power grid or the internet. It was found in [75] that when node recoveries are introduced in a simple dynamic cascade model [76], the system can spontaneously recover. The model contains three key parameters: one describes the fraction of internally failed nodes (p^*), a second governs the time for recovery to occur (τ), and the third describes the probability of failure due to lack of support from external nodes (r). For the case of small networks, r and p^* in the system, due to stochasticity will not be fixed, but will instead wander in phase space near their average values. This exploration of phase

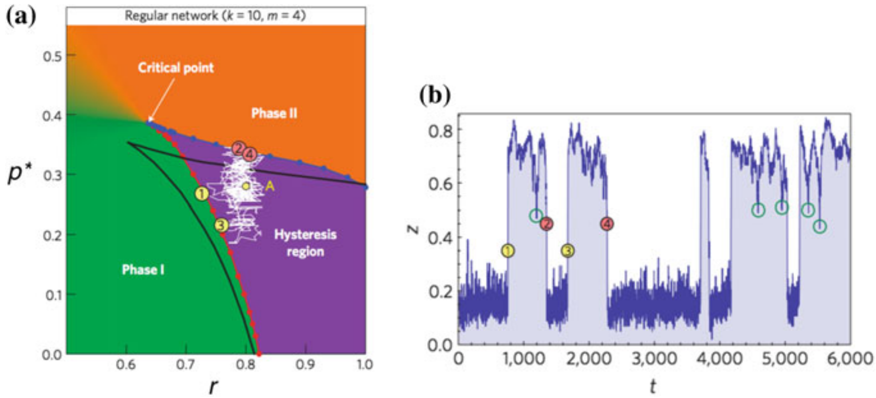

Fig. 20.7 **a** Here we show the phase space describing the state of the system according to the number of internally failed nodes (p^*) and the likelihood of external failure (r). The white line shows the trajectory of the system as it diffuses in phase space between failed (Phase II) and functional states (Phase I). When the system crosses the red line (points 1 and 3) it transits to a recovered state and when it crosses the blue line (points 2 and 4) it moves to a failed state. **b** Here we show the time evolution of the system in accordance with the diffusion in phase space according to the trajectory shown in (**a**) where the y-axis, z represents the fraction of active nodes at a given time t. As seen in the figure there are two clear states, either a failed or recovered state. The system changes between these two states, but never exists in an intermediate state. After [75]

space causes the system to dynamically recover or fail over time. In Fig. 20.7a, when the system crosses the blue line it reaches a failed state and when it crosses the red line it recovers. The crossing of these points can also be observed in Fig. 20.7b according to the corresponding numbered transitions. When the goal is to repair the system, a global planner will make repairs such that they reduce the likelihood of external failure, r, and help the system to pass the red line which represents the transition to a repaired state.

One example of a real system where this model was applied is stock-market networks. In such networks, each stock represents a node and it is connected to other stocks based on correlations between their stock prices. Stocks that are going up can be considered to be in a functional state and stocks that are falling can be considered in a failed state. Each stock (company) has an internal probability of failure (p^*) which could occur due to internal problems that are inherent to the company. Next there is a time (τ) it takes the company to fix the problems that caused the stock price drop, i.e. to recover. Lastly, because stocks are connected to one another they require support from one another and thus there is a probability (r) for a stock to drop if other stocks in the same or related sectors are falling. Naturally, the probability of a stock to fail, p^*, and the probability for failure due to the collapse of other stocks, r, will change based on overall market conditions, recent shocks to the markets, and for other reasons as well. The explicit application of this model to stock-market networks and a comparison to real data for the S&P500 can be found in Majdandzic et al. [75].

It was recently shown that a similar but much richer phenomenon occurs in interdependent networks [77]. The study in [77] found an optimal repairing strategy, which describes how many repairs should be made in each network in order to move the system towards a functional state.

In addition there have been several other studies on restoration of interdependent networks [78, 79].

20.6 Conclusions

Modern systems are becoming more and more interdependent especially through the use of SMART technologies, which require information from both their own system and from other systems. This information is then used to optimize the performance of each system based on the functioning of the other systems. Examples are SMART grids, SMART cities, and the internet of things (IOT). Understanding how failures spread both within and between the different systems that form SMART cities is crucial in order to ensure the stability of these highly interdependent systems. Methods from diffusion, percolation and physics in general can serve as useful tools to contain and predict the spreading of failures in these systems. Furthermore, models of interdependent networks have also explained the spreading of failures in other areas like finance [19, 80]. Continuing to study how failures spread in real-world systems is a crucial area of research and will likely provide many additional interesting results.

Acknowledgements We acknowledge the Israel-Italian collaborative project NECST, Israel Science Foundation, ONR, Japan Science Foundation, BSF-NSF, and DTRA (Grant no. HDTRA-1-10-1- 0014) for financial support.

References

1. A.L. Barabási, R. Albert, Science **286**, 5439 (1999)
2. M. Girvan, M. Newman, PNAS **99**, 12 (2002)
3. M. Barthélemy, Phys. Rep. **499**, 1 (2011)
4. M. Newman, *Networks: An Introduction* (OUP Oxford, 2010)
5. R. Cohen, S. Havlin, *Complex Networks: Structure, Robustness and Function* (Cambridge University Press, 2010)
6. M. Newman, A.L. Barabási, D.J. Watts, *The Structure and Dynamics of Networks* (Princeton University Press, 2011)
7. S. Havlin, E. López, S.V. Buldyrev, H.E. Stanley, Diffus. Fundam. **2**, 4 (2005)
8. S. Havlin, E. López, S.V. Buldyrev, H.E. Stanley, in *Leipzig, Einstein, Diffusion*, ed. by J. Kärger (Leipziger Universitätsverlag, 2007), p. 144
9. M. Kivelä, A. Arenas, M. Barthélémy, J.P. Gleeson, Y. Moreno, M.A. Porter, J. Complex Netw. **2**, 3 (2014)
10. G. Bianconi, Phys. Rev. E **87**, 062806 (2013)
11. S. Boccaletti, G. Bianconi, R. Criado, C.I. Del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendina-Nadal, Z. Wang, M. Zanin, Phys. Rep. **424**, 4 (2014)

12. A. Vespignani, Nature **464**, 7291 (2010)
13. S.M. Rinaldi, J.P. Peerenboom, T.K. Kelly, Control systems. IEEE **21**, 11 (2001)
14. E.A. Leicht, R.M. D'Souza, arXiv e-prints (2009)
15. S.V. Buldyrev, R. Parshani, G. Paul, H.E. Stanley, S. Havlin, Nature **464**, 08932 (2010)
16. R. Parshani, S.V. Buldyrev, S. Havlin, Phys. Rev. Lett. **105**(4), 048701 (2010)
17. S.W. Son, G. Bizhani, C. Christensen, P. Grassberger, M. Paczuski, EPL (Europhys. Lett.) **97**, 1 (2012)
18. A. Bashan, R.P. Bartsch, J.W. Kantelhardt, S. Havlin, P.C. Ivanov, Nat. Commun. **3**, 702 (2012)
19. D.Y. Kenett, S. Havlin, Mind Soc. **1** (2015)
20. J. Gao, D. Li, S. Havlin, Natl. Sci. Rev. **1**, 346 (2014)
21. J. Gao, S.V. Buldyrev, H.E. Stanley, S. Havlin, Nat. Phys. **8**, 40–48 (2012)
22. J. Gao, S.V. Buldyrev, H.E. Stanley, X. Xu, S. Havlin, Phys. Rev. E **88**, 062816 (2013)
23. D. Stauffer, A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, 1994)
24. A. Bunde, S. Havlin, *Fractals and Disordered Systems* (Springer, New York, 1991)
25. R. Cohen, K. Erez, D.B. Avraham, S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000)
26. D.S. Callaway, M. Newman, S.H. Strogatz, D.J. Watts, Phys. Rev. Lett. **85**, 5468 (2000)
27. M. Newman, I. Jensen, R.M. Ziff, Phys. Rev. E **65**, 021904 (2002)
28. J. Gao, S.V. Buldyrev, S. Havlin, H.E. Stanley, Phys. Rev. Lett. **107**, 195701 (2011)
29. F. Radicchi, Nat. Phys. **11**, 7 (2015)
30. G.J. Baxter, S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, Phys. Rev. Lett. **109**, 248701 (2012)
31. B. Bollobás, *Modern Graph Theory*, vol. 184 (Springer, 1998)
32. P. Erdős, A. Rényi, Publ. Math. Debrecen **6**, 290 (1959)
33. P. Erdős, A. Rényi, Acta Math. Hung. **12**, 1 (1961)
34. J. Gao, S.V. Buldyrev, S. Havlin, H.E. Stanley, Phys. Rev. E **85**, 066134 (2012)
35. G. Bianconi, S.N. Dorogovtsev, J. Mendes, Phys. Rev. E **91**, 012804 (2015)
36. G. Bianconi, S.N. Dorogovtsev, Phys. Rev. E **89**, 062814 (2014)
37. G.J. Baxter, S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, in *Networks of Networks: The Last Frontier of Complexity, Understanding Complex Systems*, ed. by G. D'Agostino, A. Scala (Springer, 2014)
38. Y. Hu, D. Zhou, R. Zhang, Z. Han, C. Rozenblat, S. Havlin, Phys. Rev. E **88**, 052805 (2013)
39. J.Y. Kim, K.-I. Goh, Phys. Rev. Lett. **111**, 058702 (2013)
40. K.M. Lee, J.Y. Kim, W.K. Cho, K.-I. Goh, I.-M. Kim, New J. Phys. **14**, 033027 (2012)
41. D. Cellai, E. López, J. Zhou, J.P. Gleeson, G. Bianconi, Phys. Rev. E **88**, 052811 (2013)
42. D. Zhou, J. Gao, H.E. Stanley, S. Havlin, Phys. Rev. E **87**, 052812 (2013)
43. L.M. Shekhtman, S. Shai, S. Havlin, New J. Phys. **17**, 123007 (2015)
44. S. Shai, D.Y. Kenett, Y.N. Kenett, M. Faust, S. Dobson, S. Havlin, Phys. Rev. E **92**, 062805 (2015)
45. B.R. da Cunha, J.C. González-Avella, S. Gonçalves, PloS One **10**, 11 (2015)
46. P. Hines, S. Blumsack, E. Cotilla Sanchez, C. Barrows, in *System Sciences (HICSS)* (2010)
47. L. Daqing, K. Kosmidis, A. Bunde, S. Havlin, Nat. Phys. **7**, 481–484 (2011)
48. A. Bashan, Y. Berezin, S.V. Buldyrev, S. Havlin, Nat. Phys. **9**, 667 (2013)
49. W. Li, A. Bashan, S.V. Buldyrev, H.E. Stanley, S. Havlin, Phys. Rev. Lett. **108**, 228702 (2012)
50. M.M. Danziger, A. Bashan, Y. Berezin, S. Havlin, J. Complex Netw. **2**, 460 (2014)
51. L.M. Shekhtman, Y. Berezin, M.M. Danziger, S. Havlin, Phys. Rev. E **90**, 012809 (2014)
52. M.M. Danziger, A. Bashan, Y. Berezin, S. Havlin, in *Signal-Image Technology Internet-Based Systems (SITIS)* (2013)
53. M.M. Danziger, L.M. Shekhtman, Y. Berezin, S. Havlin, EPL (Europhys. Lett.) **115**, 36002 (2016)
54. A.E. Motter, Y.C. Lai, Phys. Rev. E **66**, 065102 (2002)
55. I. Dobson, B.A. Carreras, V.E. Lynch, D.E. Newman, Chaos: an interdisciplinary. J. Nonlinear Sci. **17**, 026103 (2007)
56. J. Zhao, D. Li, H. Sanhedrai, R. Cohen, S. Havlin, Nat. Commun. **7**, 10094 (2016)
57. Y. Berezin, A. Bashan, M.M. Danziger, D. Li, S. Havlin, Sci. Rep. **5**, 8934 (2015)

58. M.M. Danziger, A. Bashan, S. Havlin, New J. Phys. **17**, 043046 (2015)
59. S. Shao, X. Huang, H.E. Stanley, S. Havlin, Phys. Rev. E **89**, 032812 (2014)
60. X. Huang, S. Shao, H. Wang, S.V. Buldyrev, H.E. Stanley, S. Havlin, EPL (Europhys. Lett.) **101**, 18002 (2013)
61. D. Zhou, H.E. Stanley, G. D'Agostino, A. Scala, Phys. Rev. E **86**, 066103 (2012)
62. C.M. Schneider, N. Yazdani, N.A.M. Araújo, S. Havlin, H.J. Herrmann, Sci. Rep. **3**, 1969 (2013)
63. C. Buono, L.G. Alvarez-Zuzek, P.A. Macri, L.A. Braunstein, PloS One **9**, 3 (2014)
64. S. Watanabe, Y. Kabashima, Phys. Rev. E **89**, 012808 (2014)
65. R. Parshani, C. Rozenblat, D. Ietri, C. Ducruet, S. Havlin, EPL (Europhys. Lett.) **92**, 68002 (2010)
66. M. Li, R.R. Liu, C.X. Jia, B.-H. Wang, New J. Phys. **15**, 093013 (2013)
67. S.V. Buldyrev, N.W. Shere, G.A. Cwilich, Phys. Rev. E **83**, 016112 (2011)
68. Y. Kornbluth, S. Lowinger, G. Cwilich, S.V. Buldyrev, Phys. Rev. E **89**, 032808 (2014)
69. L.D. Valdez, P.A. Macri, H.E. Stanley, L.A. Braunstein, Phys. Rev. E **88**, 050803 (2013)
70. L.D. Valdez, P.A. Macri, L.A. Braunstein, J. Phys. A: Math. Theoret. **47**, 055002 (2014)
71. X. Huang, J. Gao, S.V. Buldyrev, S. Havlin, H.E. Stanley, Phys. Rev. E **83**, 065101 (2011)
72. G. Dong, J. Gao, R. Du, L. Tian, H.E. Stanley, S. Havlin, Phys. Rev. E **87**, 052804 (2013)
73. S. Shao, X. Huang, H.E. Stanley, S. Havlin, New J. Phys. **17**, 023049 (2015)
74. X. Yuan, S. Shao, H.E. Stanley, S. Havlin, Phys. Rev. E **92**, 032122 (2015)
75. A. Majdandzic, B. Podobnik, S.V. Buldyrev, D.Y. Kenett, S. Havlin, H.E. Stanley, Nat. Phys. **10**, 34 (2013)
76. D.J. Watts, PNAS **99**, 9 (2002)
77. A. Majdandzic, L.A. Braunstein, C. Curme, I. Vodenska, S. Levy-Carciente, H.E. Stanley, S. Havlin, Nat. Commun. **7**, 10850 (2015)
78. M.A. Di Muro, C.E. La Rocca, H.E. Stanley, S. Havlin, L.A. Braunstein, Sci. Rep. **6**, 22834 (2016)
79. M. Stippinger, J. Kertész, Phys. A: Stat. Mech. Appl. **416**, 481–487 (2014)
80. L. Bargigli, G. di Iasio, L. Infante, F. Lillo, F. Pierobon, Quant. Finan. **15**, 4 (2015)