Robustness of a network formed of spatially embedded networks

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We present analytic and numeric results for percolation in a network formed of interdependent spatially embedded networks. We show results for a treelike and a random regular network of networks each with (i) unconstrained dependency links and (ii) dependency links restricted to a maximum Euclidean length r. Analytic results are given for each network of networks with spatially unconstrained dependency links and compared to simulations. For the case of two fully interdependent spatially embedded networks it was found [Li *et al.*, Phys. Rev. Lett. **108**, 228702 (2012)] that the system undergoes a first-order phase transition only for $r > r_c \approx 8$. We find here that for treelike networks of networks (composed of n networks) r_c significantly decreases as n increases and rapidly ($n \geqslant 11$) reaches its limiting value of 1. For cases where the dependencies form loops, such as in random regular networks, we show analytically and confirm through simulations that there is a certain fraction of dependent nodes, $q_{\max}$, above which the entire network structure collapses even if a single node is removed. The value of $q_{\max}$ decreases quickly with m, the degree of the random regular network of networks. Our results show the extreme sensitivity of coupled spatial networks and emphasize the susceptibility of these networks to sudden collapse. The theory proposed here requires only numerical knowledge about the percolation behavior of a single network and therefore can be used to find the robustness of any network of networks where the profile of percolation of a singe network is known numerically.

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I. INTRODUCTION

As network science expanded researchers became aware of the fact that systems often consist of multiple interdependent networks [1–34]. Examples of such systems are power grids that depend on communication networks, individuals who participate in multiple social circles, and metabolic networks that depend on other biological functions. Previous research on networks of networks provided a mathematical framework for understanding the stability of these systems [3,5,10,16]. They found that under many circumstances, these systems undergo a first-order percolation transition rather than the second-order transition which occurs for single networks. Recent work expanded the idea of interdependent networks to a pair of spatially embedded networks [1,35,36]. This represents an important step because many interdependent systems are spatially embedded [10,23,37–46]. Here we generalize the study of percolation of a pair of interdependent spatial networks to percolation of a network of interdependent spatial networks.

In our model, for each pair of interdependent networks a fraction q_{ij} of nodes in network i are assigned a dependent node in network j. The dependencies either follow the "no feedback condition," where if node a in network i depends on node b in network j, then b depends on a as well, or the "feedback condition," where such a constraint is not enforced [31]. For our simulations and theory we applied the "no feedback condition" and note that the "feedback condition" yields an even more vulnerable system [31].

When $n > 2$ the network of networks can assume various topologies. We show some examples of possible configurations in Fig. 1(a). To study the robustness of the network of networks we remove a random fraction $1 - p$ of the nodes from a subset of the networks. This leads to a dynamic cascade where removed nodes cause dependent nodes to be removed in the other networks. The fraction of surviving nodes at the end of the cascade is defined as x. We then find $P_\infty(x)$, the mutual giant connected component where all remaining nodes are in their networks' respective giant component and where the dependencies of all nodes remaining are also still functional.

In order to simplify the study, we follow the method of previous studies [1,35] where square lattices were used for developing the theoretical framework for spatially embedded networks since any other two-dimensional spatial network with finite characteristic link length belongs to the same universality class [47–49]. In the case of spatially embedded networks the dependencies are often restricted such that dependent nodes are within some distance, r, of one another [35]. This quantity r is called the dependency length and forces two dependent nodes in two networks, i and j, with positions (x_i, y_i) and (x_j, y_j) to obey $|x_i - x_j| \leqslant r$ and $|y_i - y_j| \leqslant r$, where x and y represent the positions of the nodes [see Fig. 1(b)].

It has been shown that for a pair of coupled lattices with no restrictions on the length of dependency links ($r = \infty$) there is a first-order percolation transition for any $q > 0$ [1]. If r is finite, then for each value of q there is a critical dependency length, r_c, above which the percolation transition shifts from second order to first order. For two fully interdependent networks ($q = 1$) it was found $r_c \approx 8$ [35] and for lower values of q there is a higher value of r_c [50]. Since lower r values are expected for physical systems, it is of interest whether a higher number of interdependent systems could decrease r_c.

Here we show simulation results for a network of spatially embedded networks with treelike dependencies and for a random regular network of spatial networks, each with (i) no restrictions on the length of dependency links and (ii) dependency links of a maximum finite length, r. The differences between the dependencies assumed in treelike networks of networks and in looplike networks of networks (such as random regular networks) are illustrated in Fig. 2. We also rederive the theory from Gao *et al.* [4,16,31] yet we

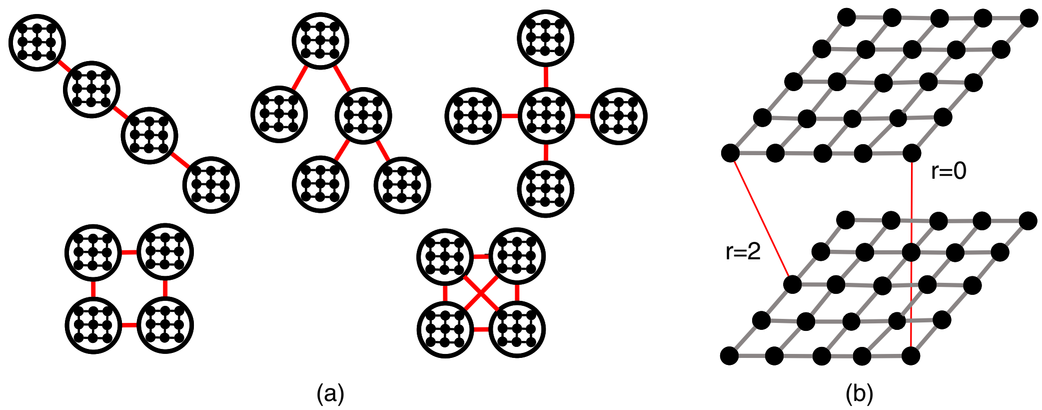

FIG. 1. (Color online) (a) Several examples of possible structures of the network of networks. Within the networks we have connectivity links and between the networks we have dependency links. Examples include a line (top left), a tree (top center), a star (top right), a random regular network of networks where each network has $m = 2$ dependencies (bottom left), and a random regular network of networks with $m = 3$ (bottom right). (b) A graphical representation of r, the length of the dependency links.

generalize the equations to use the percolation profile of a single lattice, $P_\infty(x)$, rather than $g(x)$ which is derived from the generating function. In our case this profile is obtained from simulations of percolation on a $N = 4000 \times 4000$ square lattice averaged over 100 realizations. Our theory is accurate for interdependent networks with no restrictions on the length of the dependency links. While we apply these equations to a lattice, we note that they can be used for any system composed of identical networks if $P_\infty(x)$ is known for the percolation of a single network.

II. TREELIKE NETWORK OF INTERDEPENDENT SPATIALLY EMBEDDED NETWORKS

A. Dynamics of cascading failures

We begin by examining the cascading dynamics for a treelike network of networks [in Fig. 1(a) the top structures] where the length of dependency links is unconstrained. Li *et al.* [31] derived P_∞ of the cascading failure of two interdependent networks as a function of iteration count. If a fraction $1 - p$ of nodes is removed from each network, then p_i, the fraction of survived nodes at the ith iteration, is $p_i = p^2 \frac{P_\infty(p_{i-1})}{p_{i-1}}$. For n networks in a treelike configuration a node is in the mutual giant component if it and the $n - 1$ nodes it depends on are all in their respective networks' giant components. Thus $g(p_i) = P_\infty(p_{i-1})/p_{i-1}$ is the probability for a node to be in the giant component after $1 - p_i$ fraction of nodes are removed. Now $g(p_i)$ must be raised to the $n - 1$ power since each node has $n - 1$ dependencies. This gives

$$p_i = p^n \left(\frac{P_\infty(p_{i-1})}{p_{i-1}} \right)^{n-1}. \tag{1}$$

Each iteration represents reducing all networks to their giant components and removing nodes which have dependencies outside the giant component. The next iteration then factors in the nodes removed due to having dependencies outside the previous giant component and again reduces each network to its giant component. The process repeats until a steady state is reached [see Fig. 3(a)]. In the limiting case of only a single network, $n = 1$, we get $p_i = p$ and there is no cascading effect. Further, if $n = 2$, we obtain the same result as in Ref. [35].

An alternate method of counting involves observing how the failures propogate across the links in the network of networks [21]. The initial attack on each network occurs at $t = 1$. A node which depends on a failed node then fails at $t = 2$ and in general for a node that failed at $t = t_n$, its dependent nodes fail at $t = t_n + 1$ [see Fig. 3(b)].

B. Size of the giant component after the cascade

We examine what happens to p_i when $i \to \infty$ and the system reaches steady state. We define $x \equiv p_{i\to\infty}$ and note that x represents the total fraction of nodes removed after the cascade including those removed due to interdependencies. For a given $1 - p$ fraction of nodes removed from the network of networks, $1 - x$ is the fraction that would have to be removed from a single network to obtain an equivalent giant component. Solving for x we get

$$x = p\sqrt[n]{P_\infty(x)^{n-1}}. \tag{2}$$

In Fig. 4(a) we observe that the theory of Eq. (2) shows excellent agreement with the simulations for all values of p. We also see there that close to p_c, the percolation threshold, the system collapses through a long plateau (see Fig. 3). The mechanism behind this cascade, identified by Zhou *et al.* [51], is a second-order percolation transition occurring simultaneously with the first-order transition during the plateau. The number of iterations at p_c and the value of p_c both increase as the number of networks increases [see Fig. 4(b)].

To calculate p_c we must find where the two sides of Eq. (2) are tangent at their intersection. We take the derivatives of both sides and get

$$\frac{n}{n-1} P_\infty(x_c) = x_c P'_\infty(x_c), \tag{3}$$

$$p_c = \frac{x_c}{P_\infty(x_c)^{(n-1)/n}}, \tag{4}$$

where x_c is the x value which corresponds to p_c based on Eq. (2) and $P'_\infty(x_c)$ is the derivative of $P_\infty(x)$ at x_c.

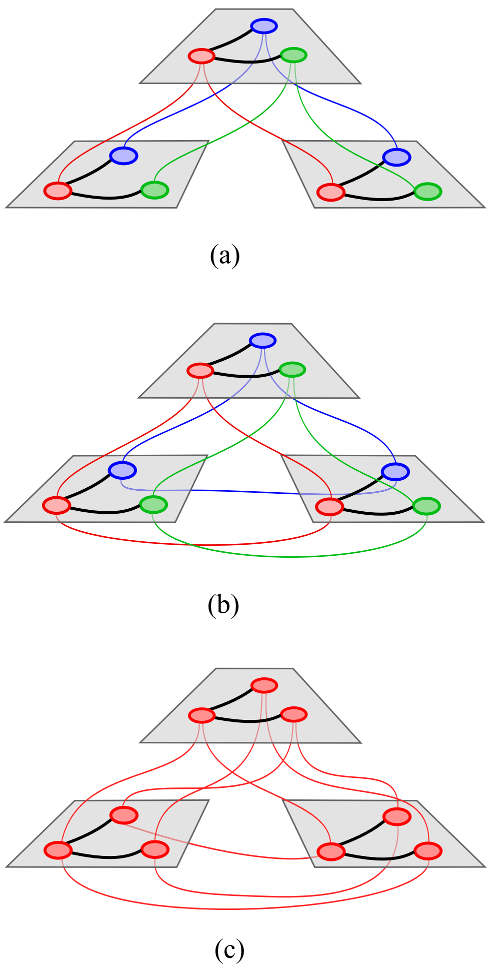

FIG. 2. (Color online) Illustration of treelike network of networks and looplike network of networks. (a) In this treelike network of networks the mutually interdependent nodes are distinguished by color and the treelike topology guarantees that the size of a mutually interdependent set be exactly n (assuming full interdependency, $q = 1$, as in this example). (b) In this network of networks with loops the dependency behavior is identical to (a) because the added dependency links are redundant and do not change the partition to sets of mutually interdependent nodes. Thus, with respect to dependency and cascading failures, (b) can be regarded as a treelike network of networks. (c) In contrast, if the loops are not closed, a situation can emerge in which all of the nodes are dependent upon one another, i.e., the size of the set of mutually interdependent nodes can be up to $N \times n$. Cases (a) and (b) are described in Sec. II and case (c) is described in Sec. III.

If a fraction $1 - p$ is removed from only a single network, then the fraction remaining after the initial removal is p instead of p^n, therefore p^n in Eqs. (2) to (4) must be replaced with p. This gives

$$x = \sqrt[n]{p P_\infty(x)^{n-1}}. \tag{5}$$

$$p_c = \frac{x_c^n}{P_\infty(x_c)^{(n-1)}}, \tag{6}$$

and Eq. (3) remains the same after the substitution. Results for n networks according to Eqs. (3) and (6) are shown in Fig. 5 as the top curve ($q = 1.0$).

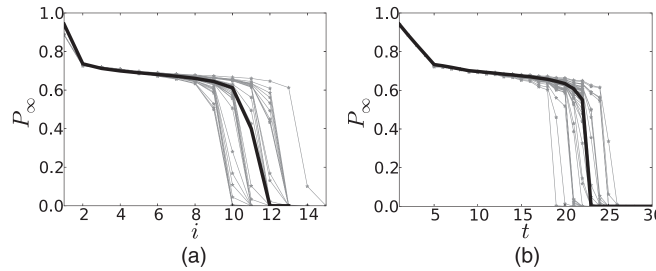

FIG. 3. Theory for $P_\infty(x)$, the size of the giant component, after a number of iterations is shown as the thick black curve and 40 simulated realizations on lattices of size $N = 500 \times 500$ are shown as the lighter curves at $p = 0.942 < p_c = 0.944$. (a) P_∞ as a function of the number of iterations, i, is shown to fit well with the theory of Eq. (1). These results are for five networks in a line, yet for this method the shape of the tree has no effect on the number of iterations. (b) P_∞ as a function of t iterations according to the method in Gao *et al.* [21] for five networks in a line fits well with the theory.

C. Starlike network of spatially embedded networks with $q < 1$

If we examine a starlike network of spatial networks with dependency links of unrestricted lengths [see Fig. 1(a)] we can also present an analytic solution for percolation for any value of the coupling q. Rederiving the equations in Ref. [16] for $P_\infty(x)$ instead of $g(x)$ we find that the size of the giant component after a fraction $1 - p$ of nodes is removed from

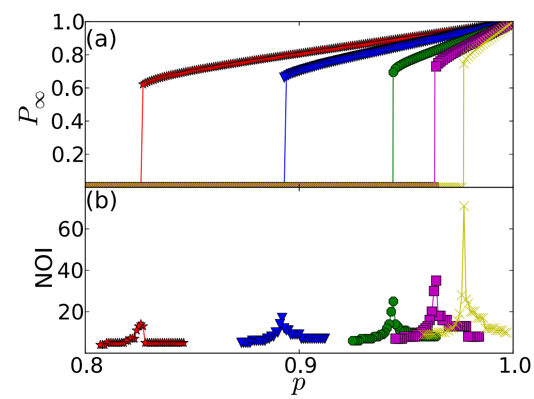

FIG. 4. (Color online) (a) Theory (lines) and simulations (symbols) of the size of the giant component, $P_\infty(x)$, as a function of the fraction of surviving nodes p, for networks of interdependent lattices of size $N = 250 \times 250$ with treelike structure are shown. These results are again for a network of networks in a line yet the results are the same for any other tree formation. Results are shown for $n = 2$ (red stars), $n = 3$ (blue triangles), $n = 5$ (green circles), $n = 7$ (purple squares), and $n = 10$ (yellow-green x's). As shown, all of the transitions are first order and the simulations fit well with the theory. Further, increasing the number of networks is seen to quickly increase p_c, indicating that the system becomes more vulnerable as n increases. (b) Here we observe that the number of iterations (NOI) it takes for the system to arrive at steady-state diverges at the critical threshold p_c. The number of iterations at p_c increases both with the number of networks, n, and the size of the networks, N [51].

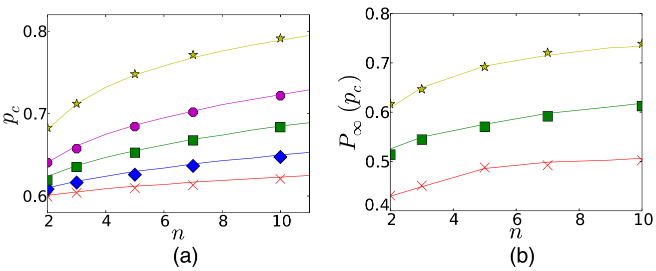

FIG. 5. (Color online) (a) The critical threshold p_c as a function of the number of networks, n, in a star formation is plotted for $q = 0.5$ (red x's), $q = 0.6$ (blue diamonds), $q = 0.7$ (green squares), $q = 0.8$ (purple circles), and $q = 1.0$ (yellow-green stars). A fraction $1 - p$ of nodes are removed only from the central network. Simulations on lattices with $N = 250 \times 250$ ($n > 2$) or $N = 500 \times 500$ ($n = 2$) show excellent agreement with the theory. (b) $P_\infty(p_c)$, the size of the giant component at criticality, is shown as a function of the number of networks (symbols are as before). Simulations on lattices of size $N = 500 \times 500$ fit well with the theory.

the central network can be described by

$$x_1 = p\left[q\frac{P_{\infty_2}(x_2)}{x_2} - q + 1\right]^{n-1},$$
$$x_2 = pq\frac{P_{\infty_1}(x_1)}{x_1}\left[q\frac{P_{\infty_2}(x_2)}{x_2} - q + 1\right]^{n-2} - q + 1, \tag{7}$$

where the subscript 1 refers to the central network and the subscript 2 refers to all the other networks. Results of theory and simulations for p_c and $P_\infty(p_c)$ can be seen in Fig. 5. Note that when $q = 1$ Eqs. (7) reduce to Eq. (5).

D. Dependency links of finite length r

We now examine a network of spatial networks with treelike dependencies but now with a finite maximum dependency length, i.e., $r < \infty$. Due to the finite r, the dependency-induced damage is not distributed uniformly and the analytic theory is not valid and we are limited to simulations. We randomly remove a fraction $1 - p$ of the nodes from each network and note that the results can be converted to a case where nodes are removed from just a single network using $p^n \to p$. Previous research on a pair of interdependent lattices found that when r is small, p_c increases monotonically until r reaches r_c. At this point p_c is at a maximum and the percolation transition changes from a second-order transition to a first-order transition. As r increases further, p_c decreases and approaches its limiting value at $r = \infty$ [35,50].

First, we analyze the giant component as a function of p for different numbers of interdependent networks in a tree. In Fig. 6(a) we observe that the system now undergoes a first-order transition even for $r = 2 (< 8)$ if there are a sufficient number of networks. Our simulations reveal that the results are the same (for $q = 1$) regardless of whether the network of networks is in a star or a line, i.e., the results are independent of the shape of the tree. The critical dependency length above which the collapse becomes first order, r_c, is also where p_c is at a maximum, thus by looking at the graph of p_c vs r we can find r_c. In the inset of Fig. 6(b) we observe that this critical

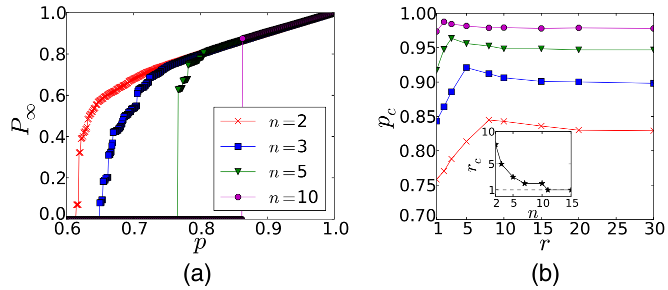

FIG. 6. (Color online) (a) Simulations of P_∞, the fractional size of the giant component as a function of the fraction of surviving nodes p for n fully interdependent ($q = 1$) spatial networks of size $N = 100 \times 100$ in a tree formation with $r = 2$ are shown. As the number of networks increases the transition becomes first-order (here at $n = 5$). (b) The percolation threshold p_c is plotted as a function of r (number of lattice units) for lattices with $N = 250 \times 250$ with symbols as defined in Fig. 6(a). The change from a second-order to a first-order transition occurs at r_c, when p_c reaches a maximum. The inset shows how this critical value, r_c, varies with the number of networks n. The critical dependency length can reach as low as $r_c = 1$ if there is a sufficient number of interdependent networks (here $n = 11$). Note that already at $r = 30$ the p_c becomes very close to the theory of Eqs. (3) and (4) for $r = \infty$. In this case since a fraction $1 - p$ was removed from all networks, we must take $p_c^{1/n}$ in order to get results that agree with Eqs. (3) and (6) and Fig. 5(a). The lines in this figure are a guide to the eye.

dependency length decreases significantly as we increase the number of networks in the tree. When $n = 11$, we get $r_c = 1$ which is its limiting value, i.e., for any $n > 11$, $r_c = 1$.

III. RANDOM REGULAR NETWORK OF INTERDEPENDENT SPATIALLY EMBEDDED NETWORKS

A. Unconstrained dependency links

Our previous results (Sec. II) were only valid when the network of networks configuration does not contain loops (treelike structures). This is not always realistic and thus we now derive results for a random regular (RR) network of spatial networks (which include loops). These results are also accurate for any network of networks with fixed degree such as a lattice of networks. Gao *et al.* [4,31] showed that for such a dependency configuration the actual number of networks, n, is irrelevant, and, instead, the results depend only on m, the number of networks each network depends on. In this case if all nodes are interdependent ($q = 1$) a failure of a single node can propogate from network to network unhindered due to the the loops and eventually lead the entire system to collapse. If the dependency path forms a closed loop (i.e. node a depends on b which depends on c which depends on a), then the fact that the network of networks topology has loops does not affect the dependency behavior and the behavior is the same as for treelike networks above (see Fig. 2). We therefore study the case $q < 1$ and remove a fraction $1 - p$ of the nodes from all the networks. Gao *et al.* [4,31] obtained

$$x = p[qyg(x) - q + 1]^m$$
$$y = p[qyg(x) - q + 1]^{m-1}, \tag{8}$$

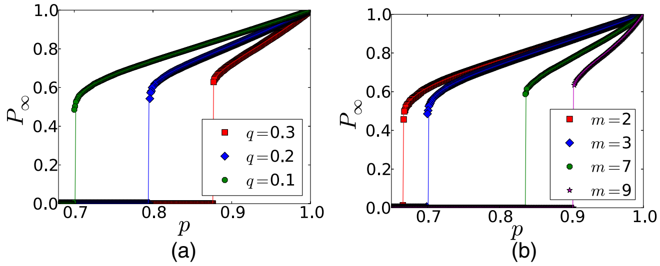

FIG. 7. (Color online) The fraction of nodes in the giant component, $P_\infty(x)$ as a function of p according to both theory (lines) and simulations (symbols) for interdependent lattice networks of size $N = 250 \times 250$ with random regular dependencies where $r = \infty$ are shown. As seen all of the transitions are first order and the simulations fit well with the theory. We use $n = m + 1$ networks in the simulations, but the results depend only on m, the number of dependencies, each network has and not on n, the total number of networks in the system. Results for (a) different values of q and (b) different values of m are shown.

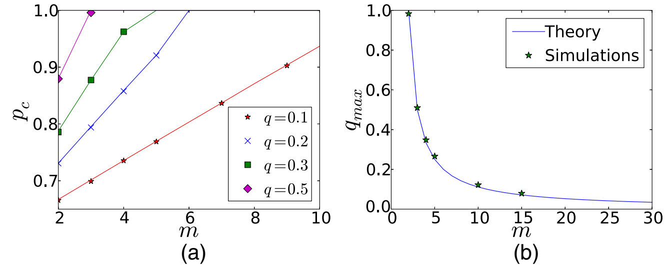

FIG. 8. (Color online) (a) The critical threshold p_c for an RR network of spatial networks is plotted as a function of m, the number of dependencies each network has for several fractions of interdependent nodes q. The lines represent the theory according to Eq. (10) and the symbols represent simulations. It is worth noting that once the number of dependencies reaches a certain value, $p_c \to 1$ for a given q value. This q value represents $q_{\max}$, the maximum dependency the system can sustain. (b) The maximum coupling fraction between networks, $q_{\max}$, above which $p_c \to 1$ is plotted as a function of m. As seen, $q_{\max}$ decreases rapidly with m, indicating that as each network has more dependencies less coupling is required for the network to fail after even the smallest attack. Simulations (symbols) on lattices of size $N = 250 \times 250$ are shown to fit well with the theory of Eqs. (13) and (14) (lines).

where y represents the percolation damage from the other networks. The system in Eq. (8) can be solved by eliminating y from the second equation and obtaining a single equation for x. After substituting $g(x) = P_\infty(x)/x$ we obtain

$$P_\infty(x)p^{2/m}q = x^{2/m} + (xp)^{1/m}(q-1), \tag{9}$$

which can be solved self-consistently for x to obtain $P_\infty(x)$. Simulations and theory according to Eq. (9), based on the numerical form of $P_\infty(x)$ for a single lattice, are shown in Fig. 7. Note that when $m = 1$ and $q = 1$ Eq. (9) reduces to the equation for a pair of interdependent lattices.

From Eq. (9) we can get p_c by rearranging Eq. (9) to

$$p_c = \left[\frac{x_c^{1/m}}{2q P_\infty(x_c)}\left(q - 1 \pm \sqrt{(q-1)^2 + 4q P_\infty(x_c)}\right)\right]^m \tag{10}$$

by noting that Eq. (9) is quadratic in $p^{1/m}$. To calculate x_c we take derivatives of both sides of Eq. (9) and obtain

$$m x_c q P'_\infty(x_c) p_c^{2/m} = 2x_c^{2/m} + p_c^{1/m}(q-1)x_c^{1/m}. \tag{11}$$

We then substitute Eq. (10) into Eq. (11) to arrive at

$$\begin{aligned} m x_c P'_\infty(x_c)&\left(q - 1 \pm \sqrt{(q-1)^2 + 4q P_\infty(x_c)}\right)^2 \\ &= 8q P_\infty(x_c)^2 + 2(q-1)P_\infty(x_c) \\ &\quad\times \left(q - 1 \pm \sqrt{(q-1)^2 + 4q P_\infty(x_c)}\right), \end{aligned} \tag{12}$$

which can be solved numerically for x_c. Simulations and theory according to Eq. (10) are shown in Fig. 8(a).

B. Maximum coupling, $q_{\max}$

From the curves of p_c in Fig. 8(a) it is clear that for each value of m there is some maximum coupling $q_{\max}$ above which $p_c = 1$ and removing even a single node will lead the entire network to collapse. We can solve for $q_{\max}$ by using Eq. (9) and setting $p = 1$. This gives

$$q_{\max} = \frac{x_{\max}^{2/m} - x_{\max}^{1/m}}{P_\infty(x_{\max}) - x_{\max}^{1/m}}. \tag{13}$$

We can then solve for $x_{\max}$ using Eq. (12). Explicitly,

$$\begin{aligned} m x_{\max} P'_\infty(x_{\max})&\left(x_{\max}^{2/m} - x_{\max}^{1/m}\right) \\ &= -x_{\max}^{3/m} + P_\infty(x_{\max})\left(2x_{\max}^{2/m} - x_{\max}^{1/m}\right). \end{aligned} \tag{14}$$

After we have $x_{\max}$ we substitute it into Eq. (13) and obtain $q_{\max}$, which is plotted in Fig. 8(b). If $m = 1$ we obtain a value of $q_{\max} > 1$ which is unphysical for our system implying that for this case there is no $q_{\max}$ that will lead to a collapse.

C. Dependency links of finite length r

We now analyze random regular networks of networks where dependency links are of a finite maximum length, r. We observe in Fig. 9 that the shape of the p_c-r curve is similar to that of trees and the transition switches from second order to first order above r_c which is the value of r when p_c is at a maximum. As the number of neighbors m increases, the

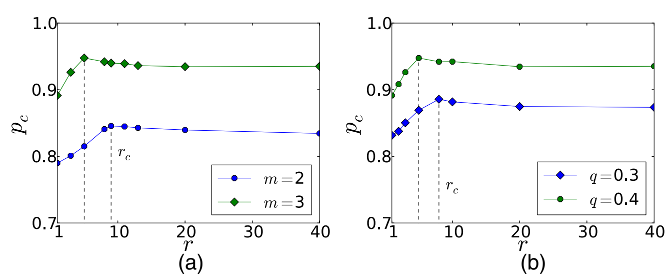

FIG. 9. (Color online) The shift from a second-order to first-order transition at r_c (number of lattice units) occurs where the critical threshold p_c reaches a maximum. This maximum is seen to occur for smaller r as (a) the number of dependent networks increases (with $q = 0.4$) and (b) the interdependent fraction, q, between the networks increases (with $m = 3$). Simulations are performed on lattices of size $N = 250 \times 250$. The lines are a guide to the eye.

critical dependency length r_c decreases significantly for any value of q. Further, for high values of q the transition is first order, even for $m = 2$ and $r = 1$. Thus a system with as few as three networks, fully interdependent ($m = 2$), experiences a first-order percolation transition for high q. In Fig. 9 we observe that r_c decreases as m increases and also as q increases.

Analogous to the results for $r = \infty$ we find that for finite r there also exists a $q^r_{\max}$ above which the system collapses for even a single failure. For small m values $q^r_{\max} > q^\infty_{\max}$ but $q^r_{\max}$ converges quickly to $q^\infty_{\max}$ as m increases, and for $m \gtrsim 15$ they are essentially equivalent. Thus networks of spatial networks with many dependencies are extremely vulnerable, even for small q and small r.

IV. DISCUSSION

In summary, we have applied the percolation framework of a network of networks [5,16,28,31] to the case of networks formed of n spatially embedded networks [1,35]. We provide analytic results for a treelike network of networks and a random regular network of networks where no restrictions are applied on the length of the dependency links. Further we provide simulations for these two cases and find excellent agreement with the theory. We also study simulations for each case when dependency links are of a finite length, r. For n networks in a tree configuration we find that the critical dependency length, r_c, above which a first-order transition occurs decreases significantly as more networks are added and with enough interdependent networks the system undergoes a first-order transition even for $r = 1$, i.e., with nearest-neighbor dependencies.

For both finite and infinite r, if the dependencies contain loops we find that there exists a critical value of coupling $q_{\max}$ above which the system will collapse even if a single node is removed. This $q_{\max}$ decreases significantly as m increases and for high values of m this $q_{\max}$ is extremely low, regardless of the length of the dependency links.

Our framework for a general network of networks does not require analytic knowledge of $P_\infty(x)$ for a single network; instead, numerical results are sufficient. Therefore our method is useful for empirical systems where the single network topology does not conform to a simple mathematical rule.

These results emphasize the vulnerability of interdependent spatially embedded networks and show that many colocalized interacting systems can collapse suddenly. Our model here can help explain sudden failures seen in many real-world systems such as power grids.

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