Interdependent networks with identical degrees of mutually dependent nodes

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We study a problem of failure of two interdependent networks in the case of identical degrees of mutually dependent nodes. We assume that both networks (A and B) have the same number of nodes N connected by the bidirectional dependency links establishing a one-to-one correspondence between the nodes of the two networks in a such a way that the mutually dependent nodes have the same number of connectivity links; i.e., their degrees coincide. This implies that both networks have the same degree distribution $P(k)$. We call such networks correspondently coupled networks (CCNs). We assume that the nodes in each network are randomly connected. We define the mutually connected clusters and the mutual giant component as in earlier works on randomly coupled interdependent networks and assume that only the nodes that belong to the mutual giant component remain functional. We assume that initially a $1-p$ fraction of nodes are randomly removed because of an attack or failure and find analytically, for an arbitrary $P(k)$, the fraction of nodes $\mu(p)$ that belong to the mutual giant component. We find that the system undergoes a percolation transition at a certain fraction $p = p_c$, which is always smaller than p_c for randomly coupled networks with the same $P(k)$. We also find that the system undergoes a first-order transition at $p_c > 0$ if $P(k)$ has a finite second moment. For the case of scale-free networks with $2 < \lambda \leqslant 3$, the transition becomes a second-order transition. Moreover, if $\lambda < 3$, we find $p_c = 0$, as in percolation of a single network. For $\lambda = 3$ we find an exact analytical expression for $p_c > 0$. Finally, we find that the robustness of CCN increases with the broadness of their degree distribution.

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I. INTRODUCTION

The robustness of interdependent networks has been recently studied by Buldyrev *et al.* [1] within the framework of the mutual percolation model. They found that two randomly connected networks with arbitrary degree distributions randomly coupled by bidirectional dependency links completely disintegrate via a cascade of failures if the fraction p of the nodes that survive the initial attack is less than some critical value $p = p_c > 0$. Moreover, the transition at p_c is of the first-order type; i.e., the fraction of the functional nodes $\mu(p)$ that survive after the cascade of failures has a step discontinuity at $p = p_c$ changing from $\mu_c = \mu(p_c) > 0$ for $p = p_c$ to zero for $p < p_c$. This behavior was observed even for scale-free (SF) networks with a power-law degree distribution $P(k) \sim k^{-\lambda}$ with $2 < \lambda \leqslant 3$. The explanation of this behavior is based on the fact that in that model the nodes with large degree (hubs) in one network may depend on the nodes with small degree in another network. The nodes with small degree can be isolated from a giant component in one network by removal of a small fraction of nodes and thus cause the malfunction of the hubs in the other network. In real-world interacting networks, the hubs in one network are more likely to depend on the hubs of another network [2]. This can significantly enhance the robustness of the interdependent networks. In general, the correlations among the degrees of the mutually dependent nodes can be described by a matrix $P(k_1|k_2)$ that specifies the conditional probabilities to find a node with degree k_1 in one network, provided it depends on a node with degree k_2 in another network. This matrix can be quite complex and may depend on many parameters. For each parameter set the model can be readily studied by computer simulations [2], but in order to get a general understanding of the correlation effects, it is desirable to solve the problem analytically at least in some limiting cases.

In this paper we study the mutual percolation problem in the case of the strongest possible correlations; namely, we study the case in which both networks (A and B) have the same number of nodes N connected by bidirectional dependency links establishing a one-to-one correspondence $\mathcal{D}$ between the nodes of the two networks in such a way that mutually dependent nodes have the same number of connectivity links; i.e., their degrees are identical: $P(k_1|k_2) = 1$ for $k_1 = k_2$ and $P(k_1|k_2) = 0$ otherwise. This implies that both networks have the same degree distribution $P(k)$. For brevity we will call such networks correspondently coupled networks (CCNs), while we will refer to the model studied in Ref. [1] as randomly coupled networks (RCNs). Following Ref. [1], we assume that the nodes in each network are randomly connected.

As in Ref. [1] we begin by randomly removing a fraction $1-p$ of the nodes of network A and removing all the A links connected to these removed nodes. Due to the dependence between the networks, all the nodes in network B that depend on the removed A nodes must also be removed. The B links connected to the removed B nodes are then also removed. As nodes and links are sequentially removed, each network begins to fragment into connected components, which we call clusters. The clusters in network A and the clusters in network B are different since each network is connected differently. A set of nodes a in network A and the corresponding set of nodes $b = \mathcal{D}a$ in network B form a mutually connected set, if

(i) Each pair of nodes in a is connected by a path that consists of nodes belonging to a and links of network A, and

(ii) Each pair of nodes in b is connected by a path that consists of nodes belonging to b and links of network B.

We call a mutually connected *set* a mutually connected *cluster* if it cannot be enlarged by adding other nodes and still satisfy the conditions above. Only mutually connected clusters are potentially functional.

As has been shown in Ref. [1] the majority of mutually connected clusters consist of single mutually dependent nodes. The probability of finding a mutually connected cluster consisting of two or more nodes becomes negligible as the number of nodes in the networks goes to infinity. However, aside from finite mutually connected clusters a giant mutually connected cluster that constitutes a nonzero fraction of nodes can exist if p is not too small.

This giant mutual cluster is called the mutual giant component. We assume that only the nodes that belong to the mutual giant component remain functional. The mutual giant component can be found by an iterative algorithm [1] that is equivalent to a physically meaningful process of the cascade of failures.

Here we find analytically the fraction of nodes $\mu(p)$ that belong to the mutual giant component for the case of CCNs. We find that as in Ref. [1], the system undergoes a percolation transition at a certain fraction $p = p_c$, which, however, is always smaller than p_c for RCNs with the same degree distribution with the exception of random regular graphs [3] for which both values coincide. Moreover, we find that the system undergoes a first-order transition at $p_c > 0$ if the degree distribution has a finite second moment. For the practically important case of SF networks [4–8] with $2 < \lambda \leqslant 3$, for which the second moment diverges, the transition becomes a second-order transition. If $\lambda < 3$, we find that $p_c = 0$ as in the percolation of a single network [9], while for $\lambda = 3$ we find an exact analytical expression for $p_c > 0$. The change in transition order has been also observed in interdependent networks with partial coupling [10]. We also investigate how the broadness of the degree distribution affects the robustness of CCNs.

II. GENERATING FUNCTIONS AND THE CASCADE PROCESS

A. First stage

We will describe the stages of the cascade of failures in CCNs in terms of the generating function of their degree distribution [11,12]:

$$G(x) = \sum_{k=0}^{\infty} P(k)x^k, \qquad (1)$$

and the generating function of the associated branching process [13]:

$$H(x) = \frac{G'(x)}{G'(1)} = \frac{1}{\langle k\rangle}\frac{dG(x)}{dx}, \qquad (2)$$

where $\langle k\rangle \equiv G'(1)$ is the average degree. It is known that the degree distribution $\tilde{P}(k,p)$ of a network $\tilde{A}$ that is obtained by random removal of a fraction $1-p$ of nodes from a network A with the original degree distribution $P(k)$ is related to $P(k)$ through a binomial expansion [11]:

$$\tilde{P}(k',p) = \sum_{k\geqslant k'} P(k)p^{k'}(1-p)^{k-k'}C_k^{k'}, \qquad (3)$$

where $C_k^{k'} = k!/[k'!(k-k')!]$ are binomial coefficients. Accordingly [11], the generating function of this distribution is

$$\tilde{G}(x,p) = G(xp+1-p). \qquad (4)$$

The fraction of nodes that do not belong to the giant component of a network is given by [12,14,15]

$$r = G(f), \qquad (5)$$

where f is the smallest nonnegative root of a transcendental equation:

$$f = H(f). \qquad (6)$$

The degree distribution of nodes that do not belong to the giant component is given by [14]

$$P_o(k,f) = P(k)f^k/r. \qquad (7)$$

Accordingly the degree distribution of nodes in the giant component is given by

$$P_i(k,f) = P(k)(1-f^k)/(1-r). \qquad (8)$$

Thus the degree distribution in the giant component of a decimated network after random removal of a $1-p$ fraction of nodes is

$$\tilde{P}_i(k',f,p) = \tilde{P}(k',p)[1-f(p)^{k'}]/[1-r(p)], \qquad (9)$$

where

$$r(p) = \tilde{G}[f(p),p], \qquad (10)$$

and $f(p)$ satisfies the transcendental equation

$$f(p) = \tilde{H}[f(p),p]. \qquad (11)$$

In order to find the original degree distribution in the giant component of network A we must restore the links that lead to the randomly removed nodes. If a node in the decimated network A has a degree k', it might have any degree $k \geqslant k'$ in the original network A with probability $P(k|k')$ given by Bayes' formula:

$$P(k|k') = P(k)C_k^{k'}p^{k'}(1-p)^{k-k'}/\tilde{P}(k',p). \qquad (12)$$

Thus the total probability that a node in the giant component has a degree k is

$$P_1(k) = \sum_{k'\leqslant k} P(k)C_k^{k'}p^{k'}(1-p)^{k-k'}\frac{\tilde{P}_i(k',f,p)}{\tilde{P}(k',p)}, \qquad (13)$$

or using Eq. (9),

$$P_1(k) = \sum_{k'\leqslant k} P(k)C_k^{k'}p^{k'}(1-p)^{k-k'}\frac{1-f(p)^{k'}}{1-r(p)} = P(k)\frac{1-[f(p)p+1-p]^k}{1-r(p)}. \qquad (14)$$

Introducing the new notations $f_1 = f(p)$ and

$$t_1 = f_1p+1-p \qquad (15)$$

and using Eqs. (10) and (4), we obtain the generating function of this degree distribution:

$$G_1(x) = \frac{G(x)-G(xt_1)}{1-G(t_1)}. \qquad (16)$$

The fraction of nodes in the giant component of the decimated network A is $1 - r_1$, where $r_1 = G(t_1)$. Because the decimated network has Np nodes, the size of the giant component A_1 of network A after random removal of $(1 - p)$ nodes is $N_1 = Np(1 - r_1)$.

B. Second stage

We assume that only nodes that belong to A_1 are functional; thus after the first stage of the cascades of failures, only a $p(1 - r_1) < p$ fraction of the nodes in network B remain functional. Thus we expect further disintegration of network B at the second stage of the cascade, and its giant component B_2 will be even smaller than A_1. We define a set of nodes $B_1 = \mathcal{D}(A_1)$ by projecting A_1 onto network B using the one-to-one correspondence $\mathcal{D}$ between the nodes of networks A and B established by dependency links. Since the degree of each node in network B is the same as the degree of its dependent node in network A, the giant component A_1 of network A obtained at the first stage of the cascade has the same degree distribution as the set B_1 in network B. Thus the degree distribution of set B_1 coincides with the degree distribution of set A_1, which is given by Eq. (14). Moreover, from the point of view of network B, the nodes in B_1 are randomly selected and randomly connected. The structure of Eq. (14) implies that the selection process of B_1 can be interpreted as random selection of nodes from the original network B by first removing $1 - p$ fraction of nodes due to the original attack and then removing the nodes that do not belong to the giant component of $\tilde{A}$. From the point of view of network B these nodes are removed at random with probability t_1^k, which depends only on the node degree, k.

Thus, to compute B_2 we can use the same approach used at the first stage, but applied to the new network B_1 with the new degree distribution given by Eq. (14). The only problem is that many of the links outgoing from network B_1 are ending at the nodes that do not belong to network B_1, and thus for computation of B_2 these links must be removed. The probability p_1 of a random link originating in network B_1 to end up in B_1 is equal to the ratio of the number of links originating in network B_1:

$$L_1 = N_1 \sum k P_1(k) = pN\langle k\rangle[1 - G'(t_1)t_1/\langle k\rangle] \quad (17)$$

to the total number of links $N\langle k\rangle$. Therefore,

$$p_1 = \frac{L_1}{N\langle k\rangle} = p(1 - s_1), \quad (18)$$

where

$$s_1 = t_1 G'(t_1)/\langle k\rangle. \quad (19)$$

Accordingly, the degree distribution of links connecting the nodes of network B_1 is

$$\tilde{P}_1(k', p) = \sum_{k\geqslant k'} P_1(k) p_1^{k'} (1 - p_1)^{k-k'} C_k^{k'}, \quad (20)$$

and the generating function of this distribution is

$$\tilde{G}_1(x, p_1) = \frac{G(xp_1 + 1 - p_1) - G[t_1(xp_1 + 1 - p_1)]}{1 - r_1}. \quad (21)$$

Thus the size N_2 of the giant component B_2 is $N_2 = p(1 - r_1)[1 - r(p_1)]N$, where $r(p_1) = \tilde{G}_1(f_2, p_1)$ and $f_2 = \tilde{H}_1(f_2, p_1)$. Introducing a new notation

$$t_2 \equiv f_2 p_1 + 1 - p_1 \quad (22)$$

and taking into account Eq. (21), we see that

$$f_2 = \frac{G'(t_2) - G'(t_1 t_2)t_1}{\langle k\rangle(1 - s_1)} \quad (23)$$

and $N_2 = p(1 - r_1)\{1 - [G(t_2) - G(t_2 t_1)]/(1 - r_1)\}N$. Using that $r_1 = G(t_1)$, we get

$$N_2 = p[1 - G(t_1) - G(t_2) + G(t_1 t_2)]N. \quad (24)$$

We can compute the original degree distribution $P_2(k)$ in B_2 using Bayes' formula in the same way as we obtained the distribution $P_1(k)$:

$$P_2(k) = P(k)\frac{\left(1 - t_1^k\right)\left(1 - t_2^k\right)}{1 - G(t_1) - G(t_2) + G(t_1 t_2)}. \quad (25)$$

C. Third stage

On the third stage of the cascade we will compute the giant component A_3 of network A, which is the result of further disintegration of A_1 because the nodes in A_1 that do not belong to $A_2 = \mathcal{D}(B_2)$ failed at the second stage. We can again apply the same technique because the set of nodes A_2 is a selection of nodes in A_1, which is made independent of its topology. From the point of view of network A, this selection is a random selection that can depend only on the degree of a node.

Because the degrees of the mutually dependent nodes in networks A and B coincide, the degree distribution in the set A_2 is given by Eq. (25). We can rewrite Eq. (14) for the degree distribution of nodes in A_1 as

$$P_1(k) = P(k)\frac{1 - t_1^k}{1 - G(t_1)}. \quad (26)$$

Comparing Eqs. (26) and (25) we see that the only significant difference between them is the factor $1 - t_2^k$ in the numerator of Eq. (25), while the expressions in the denominators are just normalization factors. Thus the distribution $P_2(k)$ is the degree distribution of a set of nodes obtained from the set A_1 by random deletion of some nodes in A_1 with probability t_2^k, which depends on the degree of the node k. Thus from the point of view of the network A the set A_2 can be obtained from A_1 by random deletion of some nodes with probability t_2^k.

The only difference with the situation at the second stage is that A_2 is selected not from the random subset of nodes $\tilde{A}$ but from its connected giant component A_1. Accordingly, we must find a way to replace A_1 by some random selection of nodes out of the original network A. Recall that $\tilde{A}$ is obtained by randomly selecting nodes of the original network A with probability p. In order to obtain A_2 from A_1, we must delete the nodes from A_1 with probability t_2^k. We achieve the same result if we randomly delete nodes from $\tilde{A}$ with the same probability. Let us denote $\tilde{A}_2$ a set obtained from $\tilde{A}$ by random deletion of nodes with probability t_2^k. It is clear that the giant component of $\tilde{A}_2$ coincides with A_3, the giant component of A_2. This is true because it is equivalent to first find all the paths between all pairs of nodes in $\tilde{A}$ and then delete some paths due to deletion of nodes, or to first delete the nodes and find all the paths among the remaining nodes of $\tilde{A}$. The set $\tilde{A}_2$ is the result of first selecting nodes from A with probability p and then selecting the remaining nodes with probability $1 - t_2^k$. This is equivalent to selecting nodes from the network A at random with probability $p(1 - t_2^k)$. Thus A_3 is the giant component of a subset of nodes of the original network A selected at random with probability $p(1 - t_2^k)$. Note also that B_2 obtained on the second stage is the giant component of network B after random selection of nodes with probability $p(1 - t_1^k)$. Thus the third stage in the cascade of failures is equivalent to the second stage with the replacement of t_1 by t_2. Accordingly, from the point of view of the network B, $B_3 = \mathcal{D}A_3$ can be obtained from B_2 by random deletion of nodes with probability t_3^k, which can be obtained from t_2 using the same algorithm by which we obtained t_2 from t_1.

D. Recursive relations

Generalizing, for stage i we arrive at a recursive relation between t_i and t_{i+1}. Namely, once we know t_i we can find t_{i+1}, as well as the size of the giant component at the stage $i + 1$:

$$N_{i+1} = p[1 - G(t_i) - G(t_{i+1}) + G(t_i t_{i+1})]N \qquad (27)$$

and the degree distribution of the nodes inside this giant component:

$$P_{i+1}(k) = P(k)\frac{\left(1 - t_{i+1}^k\right)\left(1 - t_i^k\right)}{1 - G(t_i) - G(t_{i+1}) + G(t_i t_{i+1})}. \qquad (28)$$

In order to find t_{i+1} from t_i we repeat the steps used deriving t_2 from t_1 by first introducing

$$s_i = t_i G'(t_i)/\langle k\rangle \qquad (29)$$

and

$$p_i = p(1 - s_i) \qquad (30)$$

in analogy to Eqs. (19) and (18). Then

$$t_{i+1} \equiv f_{i+1} p_i + 1 - p_i, \qquad (31)$$

where f_{i+1} satisfies a transcendental equation analogous to Eq. (23):

$$f_{i+1} = \frac{G'(t_{i+1}) - G'(t_i t_{i+1}) t_i}{\langle k\rangle(1 - s_i)}. \qquad (32)$$

Excluding f_{i+1} and s_i from Eq. (32) we find that t_{i+1} is given by the smallest non-negative root of the equation:

$$t_{i+1} = (1 - p) + \frac{p}{\langle k\rangle}[G'(t_i)t_i + G'(t_{i+1}) - t_i G'(t_i t_{i+1})]. \qquad (33)$$

To start the iterative process we must take into account the definition of t_1 given in Eqs. (15) and (11), which is equivalent to a transcendental equation

$$t_1 = (1 - p) + \frac{p}{\langle k\rangle} G'(t_1), \qquad (34)$$

which is the same as Eq. (33) if we introduce $t_0 \equiv 0$.

III. THE MUTUAL GIANT COMPONENT AND THE PHASE TRANSITION

The cascade of failures will stop when $t_{i+1} = t_i = t$, and hence the fraction of nodes in the mutual giant component $\mu = \lim_{i\to\infty} N_i/N$ is given by the simplified equation (27):

$$\mu = p[1 - 2G(t) + G(t^2)], \qquad (35)$$

where t is the smallest non-negative root of the equation

$$\begin{aligned} t &= (1 - p) + \frac{p}{\langle k\rangle}[(1 + t)G'(t) - tG'(t^2)] \\ &= 1 - p[1 - (1 + t)H(t) + tH(t^2)]. \end{aligned} \qquad (36)$$

The right-hand side of Eq. (36) has zero derivative at $t = 1$, if $G''(1)$ is finite. This condition is equivalent to the existence of the second moment of the degree distribution. Thus one can see [Fig. 1(a,b)] that for finite second moment and small enough p, Eq. (36) has only the trivial solution $t = 1$ corresponding to $\mu = 0$ and, therefore, to the complete disintegration of the networks. As p increases, a nontrivial solution $\mu > 0$ will emerge at $p = p_c$, at which point the right-hand side of Eq. (36) will touch the straight line representing the left-hand side at $t = t_c$; at that point the slope of both lines is equal to 1. Since at $t = 1$ the slope of the right-hand side is zero, t_c must be smaller than 1, and thus the mutual percolation transition is of the first order, where μ changes from zero (for $p < p_c$) to $\mu \geqslant \mu_c > 0$ (for $p \geqslant p_c$). The value of μ_c is given by Eq. (35) computed at $t = t_c$.

An efficient way of finding p_c is to solve Eq. (36) with respect to $1/p$:

$$\frac{1 - (1 + t)H(t) + tH(t^2)}{1 - t} = \frac{1}{p} \qquad (37)$$

and find the maximum of the left-hand side with respect to t (Fig. 2). The left-hand side of Eq. (37) is a curve that changes from $1 - H(0) = 1 - P(1)/\langle k\rangle$ at $t = 0$ to zero at $t = 1$. At $t = 0$ it has a positive slope $1 - [P(1) + 2P(2)]/\langle k\rangle$, so it must have an absolute maximum at $t_c \in (0,1)$. The equation for t_c can be readily obtained by differentiation of Eq. (37):

$$\begin{aligned} 1 - 2H(t_c) + H\left(t_c^2\right) - \left(1 - t_c^2\right)H'(t_c) \\ + 2t_c^2(1 - t_c)H'\left(t_c^2\right) = 0. \end{aligned} \qquad (38)$$

The value of the left-hand side of Eq. (37) at $t = t_c$ gives $1/p_c$. If the value of this maximum is less than 1, then the networks do not have a mutual giant component at any p.

IV. SPECIAL CASES

Figure 1 shows the graphical solutions of Eq. (36) for several special cases of degree distributions of CCNs.

A. Erdős-Rényi networks

For Erdős-Rényi (ER) networks [3,12] $H(t) = \exp[\langle k\rangle(t - 1)]$, and the maximal value of the left-hand side of Eq. (37) monotonically increases with $\langle k\rangle$. This can be readily seen by differentiating Eq. (37) with respect to $\langle k\rangle$. The maximal value reaches 1 at $\langle k\rangle = 1.706\,526$, below which correspondently coupled ER networks disintegrate even without any initial

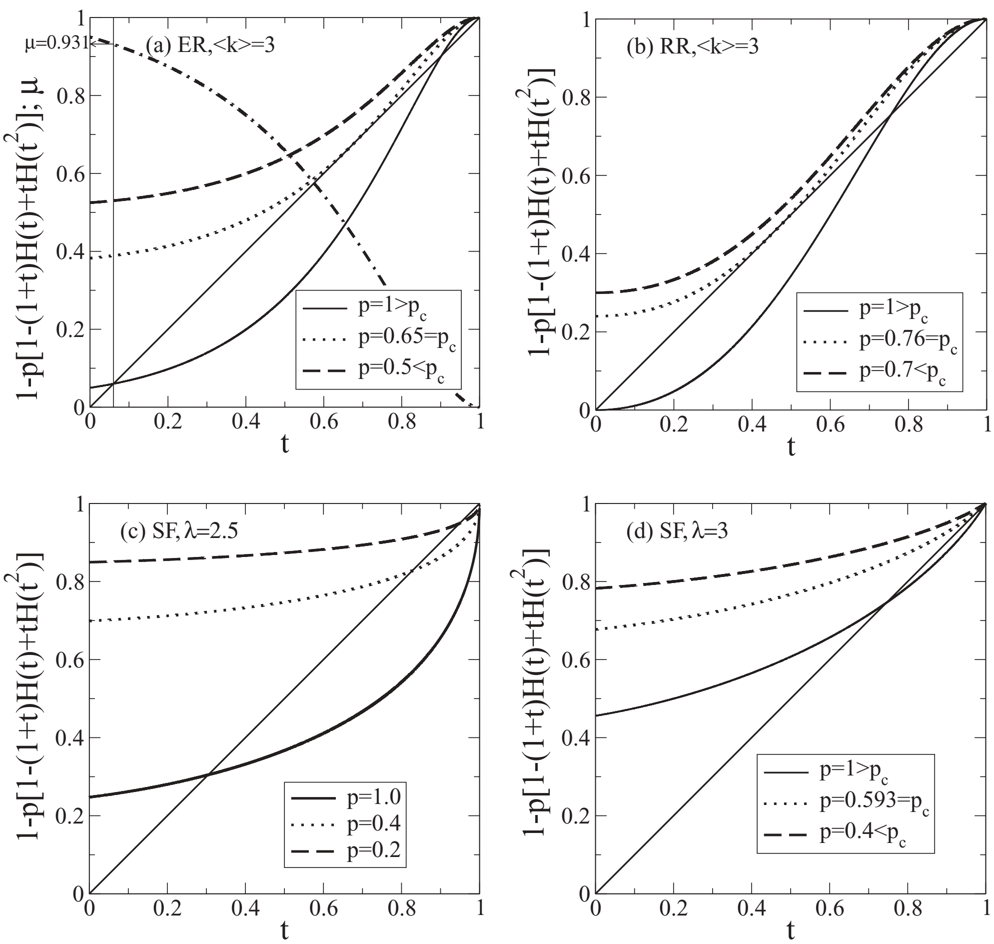

FIG. 1. Graphical solution of Eq. (36) for various special cases of CCNs. (a) ER networks with average degree $\langle k\rangle = 3$. One can see that the monotonically increasing curves representing the right-hand side of Eq. (36) for different p have zero slopes at $t = 1$. The relevant solutions for t are given by the lower intersection points of these curves and a straight line $y = t$ representing the left-hand side of Eq. (36). For $p = 1$, this solution $t = 0.0602$ is indicated by a vertical straight line. The intersection of this vertical line with the plot of Eq. (35) (dot-dash line) gives the mutual giant component $\mu = 0.931$. The critical $p = p_c = 0.649\,9451$ corresponds to a sudden disappearance of the nontrivial solution. (b) RR networks with $\langle k\rangle = 3$. Note that for $p = 1$ the nontrivial solution is $t = 0$, which means that $\mu = 1$. The value of $p_c = 0.758\,751$ is grater than the p_c for ER networks with the same average degree shown in panel (a). (c) Analogous plot for SF networks with $\lambda = 2.5$. It shows that the slope of the curves is infinite for $t \to 1$. One can see that in this case the nontrivial solution exists for any $p > 0$. However, as $p \to 0$, the nontrivial solution $t \to 1$, and, accordingly, $\mu \to 0$ indicating the second-order transition at $p = p_c = 0$. (d) The marginal case of $\lambda = 3$. The slopes of the curves for $t \to 1$ are finite. This means that there is a critical $p = p_c > 0$ at which the slope of the curve becomes equal to 1 at $t \to 1$. For the displayed case of $k_{\min} = 1$, Eq. (44) yields $p_c = 0.593\,284\,56$. The nontrivial solution smoothly approaches 1 as $p \to p_c$. This again implies $\mu \to 0$ (second-order transition).

attack or failure (Fig. 2). Note that the equivalent value of $\langle k\rangle$ for randomly coupled ER networks is 2.4554 [1].

B. Random regular graphs

For a random regular (RR) graph [Fig. 1(b)] in which all the nodes have the same degree $k = \langle k\rangle$, $G(t) = t^{\langle k\rangle}$ and $H(t) = t^{\langle k\rangle-1}$. Then t satisfies

$$t = (1 - p) + p(t^{\langle k\rangle-1} + t^{\langle k\rangle} - t^{2\langle k\rangle-1}) \tag{39}$$

and

$$\mu = p(1 - t^{\langle k\rangle})^2. \tag{40}$$

Equations (39) and (40) can be obtained by simpler methods presented in Ref. [1] for RCNs, since for the case of random regular graphs, the degrees of all the nodes in both networks coincide, and therefore the CCNs and RCNs models are equivalent. Indeed, from Eq. (1) of Ref. [1] it follows in a special case of coinciding degree distributions of the coupled networks that

$$\mu = p[1 - G(t)]^2, \tag{41}$$

where

$$t = 1 - p[1 - G(t)][1 - H(t)]. \tag{42}$$

If $G(t) = t^{\langle k\rangle}$ and $H(t) = t^{\langle k\rangle-1}$, Eqs. (41) and (42) are equivalent to Eqs. (40) and (39), respectively.

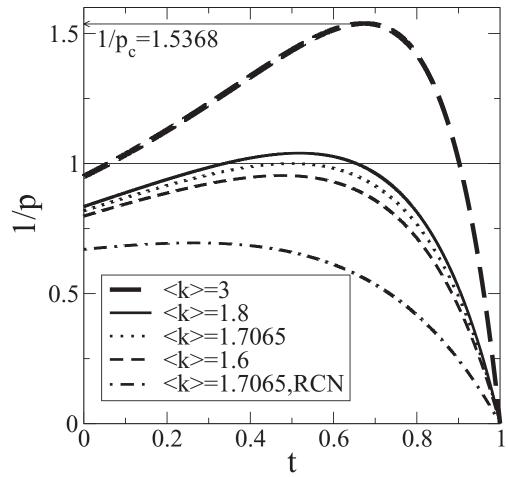

FIG. 2. Graphical solution of Eq. (37) for ER networks with different degree $\langle k\rangle$ illustrating the method of finding p_c. The bold dashed curve corresponds to $\langle k\rangle = 3$ studied in Fig. 1(a). As $\langle k\rangle$ decreases below 1.706, the nontrivial solution corresponding to $p \leqslant 1$ disappears. We also show the behavior of the analogous equation (45) for $\langle k\rangle = 1.706$ for RCNs. In agreement with proposition (3) this curve is always below the curve with the same average degree for CCNs studied here.

C. Scale-free networks

For SF networks with $\lambda < 3$ [Fig. 1(c)], the derivative of the right-hand side of Eq. (36) is infinite at $t = 1$, which means that a nontrivial solution exists at any $p > 0$ since in the vicinity of $t = 1$ the straight line representing the left-hand side of Eq. (36) is always above the curve representing the right-hand side, while for $t = 0$, the curve is always above the line. This means that SF CCNs are as robust as a single SF network for which p_c is always zero.

For the marginal case of $\lambda = 3$ [Fig. 1(d)] $G''(t)$ diverges as $\ln(1-t)$ when $t \to 1$, and thus the left-hand side of Eq. (36) has a finite derivative at $t = 1$. Accordingly $p = p_c > 0$, but the nontrivial solution emerges at $t_c = 1$, so the transition becomes of the second order. For the case of $P(k) = (k_{\min}/k)^2 - [k_{\min}/(k+1)]^2$ for $k \geqslant k_{\min} = 1,2,\ldots$ and $P(k) = 0$ for $k < k_{\min}$ we can find p_c analytically. Indeed, in this case $P(k)$ behaves asymptotically as $2k_{min}^2/k^3$. For $k \to \infty$ the leading term in $G''(t)$ becomes $2k_{\min}^2 t^k/k$, so $G''(t) = -2k_{min}^2 \ln(1-t) + c(t)$, where $c(t)$ is a continuous function. Accordingly, the slope of the right-hand side of Eq. (36) at $t = 1$ becomes $p4k_{\min}^2 \ln(2)/\langle k\rangle$, where

$$\langle k\rangle = k_{\min} + k_{\min}^2 \left(\frac{\pi^2}{6} - \sum_{k=1}^{k_{\min}} \frac{1}{k^2} \right). \quad (43)$$

The critical threshold is thus

$$p_c = \frac{\frac{1}{k_{\min}} + \frac{\pi^2}{6} - \sum_{k=1}^{k_{\min}} \frac{1}{k^2}}{4\ln(2)}. \quad (44)$$

For $k_{\min} = 1$, $p_c = 0.593\,284\,56$; and for $k_{\min} = 2$, $p_c = 0.322\,779\,24$.

D. Effect of the broadness of the degree distribution

It follows from Fig. 1 that for the same $\langle k\rangle = 3$, p_c of the RR networks (0.758 751) is greater than the p_c of the ER networks (0.649 9451). Moreover, for SF networks with $\lambda = 3$ and $k_{\min} = 1$, for which the average degree is $\pi^2/6 < 3$, we have even smaller $p_c = 0.593\,284\,56$. For SF networks with $\lambda = 3$ and $\langle k\rangle = 3$, we can estimate $p_c = 0.35$, which is much smaller than the p_c for the narrower ER and RR degree distribution. For SF networks with $\lambda < 3$, which are even broader, $p_c = 0$ for any $\langle k\rangle$. This is in a complete agreement with the trend observed in percolation of single networks, for which the robustness increases with the broadness of the degree distribution if one keeps $\langle k\rangle$ constant but is opposite to the trend observed in Ref. [1] for RCNs.

In order to investigate the effect thoroughly, we study several classes of degree distributions for a number of values of $\langle k\rangle$. Figure 3 shows p_c as function of $\langle k\rangle$ for RR, ER, uniform, and SF with $\lambda = 3$ degree distributions. For each value of $\langle k\rangle$ the variance of SF degree distribution (∞) is greater than the variance of the uniform degree distribution ($\langle k\rangle(\langle k\rangle+1)/3$), which is greater than the variance of ER degree distribution $\langle k\rangle$, which is greater than the variance of RR degree distribution (0). Indeed, Fig. 3 shows that $p_c(\text{SF}) < p_c(\text{uniform}) < p_c(\text{ER}) < p_c(\text{RR})$. Thus our numerical results suggest that CCNs become more robust if their degree distribution becomes broader (provided the average degree is constant). This behavior is the opposite of the behavior of RCNs.

However, in general, if the measure of broadness is simply the variance of the degree distribution, our statement is incorrect. It is possible to find two distributions with the same variances and average degrees that have different values of p_c. One particular example is the following two distributions: $P_1(0) = 0, P_1(1) = P_1(2) = P_1(3) = P_1(4) = P_1(5) = 1/5$ and $P_2(0) = P_2(3) = 0, P_2(1) = P_2(5) = 1/6,$

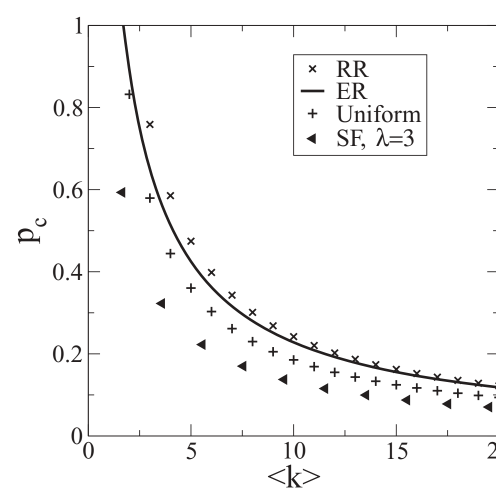

FIG. 3. The values of p_c versus $\langle k\rangle$ for several degree distributions of increasing broadness, namely RR, ER, uniform, and SF with $\lambda = 3$. We define the uniform distribution as follows: $P(k) = 1/(2\langle k\rangle + 1)$ for $k = 0,1,\ldots,2\langle k\rangle$ and $P(k) = 0$ for $k > 2\langle k\rangle$. For SF distribution we use Eqs. (43) and (44), while for other distributions we numerically solve Eq. (38) and use Eq. (37) to find p_c. One can see that p_c decreases(and hence the robustness increases) with the broadness.

$P_2(2) = P_2(4) = 1/3$, which have p_c, respectively, 0.683 099 and 0.683 657.

V. GENERAL IMPLICATIONS ON THE NETWORK ROBUSTNESS

Finally, we will compare the robustness of CCNs and RCNs with the same degree distributions. We will show that in the limit of infinitely large networks (1) the value of p_c for CCNs is always (except for RR networks) smaller than the p_c for RCNs and (2) for the same p, the value of the mutual giant component for CCNs is always (except for RR networks) larger than for RCNs.

In case of the finite networks these propositions are not rigorous since it is possible to find peculiar CCN topologies such that after switching some dependency links the resulting RCN will retain its mutual giant component while the original CCN will completely disintegrate once the same fraction of nodes is deleted in both cases. However, the probability of such exceptions will become negligible for sufficiently large networks.

Equation (42) for the randomly coupled networks can be rewritten as

$$\frac{[1 - H(t)][1 - G(t)]}{1 - t} = \frac{1}{p}. \tag{45}$$

The critical value of p_c for randomly coupled networks can be determined as the inverse maximal value of the left-hand side of Eq. (45). Our proposition (1) is an obvious corollary of the following proposition (3): for any $t \in [0,1]$ the left-hand side of Eq. (37) is greater or equal than the left-hand side of Eq. (45) (Fig. 2). Subtracting Eq. (45) from Eq. (37) and applying relation (2) between $G(t)$ and $H(t)$ we see that the inequality stated in proposition (3) is equivalent to

$$tG'(t^2) - tG'(t) + G(t)G'(1) - G(t)G'(t) \geqslant 0. \tag{46}$$

We will prove Eq. (46) using mathematical induction. We see that for RR graphs for which the degree of every node is equal to m, i.e., $P(m) = 1$, Eq. (46) is satisfied as an equality. Assuming that it is satisfied for any degree distribution such that $P(k) = 0$ for $k < m$ and $k > n \geqslant m$, we will show that it is also satisfied for the degree distribution $\tilde{P}(k) = (1 - b)P(k)$ for any k except for $k = n + 1$, for which $\tilde{P}(n + 1) = b > 0$. The generating function for this new distribution is obviously $\tilde{G} = (1 - b)G + bt^{n+1}$. After elementary algebra we can see that

$$t\tilde{G}'(t^2) - t\tilde{G}'(t) + \tilde{G}(t)\tilde{G}'(1) - \tilde{G}(t)\tilde{G}'(t) \tag{47}$$

$$= (1 - b)[tG'(t^2) - tG'(t) + G(t)G'(1) - G(t)G'(t)] \tag{48}$$

$$+ b(1 - b)[G(t) - t^{n+1}]\{[n + 1 - G'(1)](1 - t^n) + G'(t) - G'(1)t^n\}, \tag{49}$$

which proves Eq. (46) for $\tilde{G}$ provided it is true for G, if we take into account the obvious inequalities $n + 1 > G'(1)$, $1 \geqslant t^n$, $G(t) \geqslant t^{n+1}$, and $G'(t) \geqslant G'(1)t^n$ for any $t \in [0,1]$. This concludes the proof of propositions (3) and (1). Note that the equality sign in these inequalities and hence in inequality (46) is realized only for $t = 1$ and $t = 0$ [if $P(0) = 0$]. Hence proposition (1) always implies strict inequality except for the case of RR graphs.

To prove the proposition (2) we first notice that the smallest positive root of Eq. (37), t_1, is always smaller than the smallest positive root t_2 of Eq. (45). This a is direct consequence of proposition (3). Also we notice that the right-hand side of Eq. (35) is a monotonically decreasing function of t. This can be shown by differentiation and comparing the terms of $G'(t)$ and $tG'(t^2)$ corresponding to the same $P(k)$, namely, $kP(k)t^{k-1} \geqslant kP(k)t^{2k-1}$. Thus $\mu(t_1) > \mu(t_2)$. Finally, we state proposition (4): For the same value of t, the right-hand side of Eq. (35), $\mu(t)$, is greater or equal than the right-hand side of Eq. (41), $\mu_r(t)$. One can prove this proposition using the same induction method we used to prove proposition (3). Combining these two results, $\mu(t_1) > \mu(t_2) \geqslant \mu_r(t_2)$, which concludes the proof of proposition (2).

Thus CCNs are statistically more robust than RCNs with the same $P(k)$ but are still prone to cascade failures and, then, to first-order disintegration (only if $G''(1) < \infty$) as in the case of randomly coupled networks.

VI. SUMMARY

In this work we have studied the problem of failure of CCNs, i.e., coupled networks with coinciding degrees of mutually dependent nodes. We derive new recursive equations [Eqs. (33) and (27)] describing the cascade of failures, which are different from the analogous equations for RCNs studied in Ref. [1]. We also find equations for the size of the mutual giant component [Eqs. (35) and (36)], as well as the efficient way of finding the critical fraction of nodes $p = p_c$ that must survive the initial random failure for the mutual giant component not to vanish, by finding the maximum of Eq. (37).

We show that if the second moment of the degree distribution is finite, CCNs disintegrate in a cascade of failures via a first-order transition at which the mutual giant component suddenly drops from a positive fraction above $p_c > 0$ to zero below p_c. This behavior is analogous to the behavior of RCNs, with the only difference that RCNs disintegrate via a first-order transition even when the second moment of their degree distribution diverges.

Moreover, we show that CCNs are statistically more robust than RCNs with the same degree distribution. In particular, we show that scale-free CCNs with $\lambda < 3$ disintegrate via a second-order phase transition in the same way as noninteracting networks and thus are very resilient against random failure. Namely, the mutual giant component for these networks exists at any $p > 0$ but becomes infinitely small as $p \to 0$. Finally CCNs become more robust if their degree distribution becomes broader (provided the average degree is constant). This behavior is the opposite of the behavior of RCNs.

All our analytical predictions are confirmed by simulations of coupled networks with a large number of nodes ($N \geqslant 10^6$).

Our findings support recent numerical studies of Parshani *et al.* [2], who found that coupled networks with positively correlated degrees of mutually dependent nodes (and not just the present case of fully coincidental degrees) are more robust that their randomly coupled counterparts studied in Ref. [1]. This can be attributed to the fact that the correlation between the degrees of nodes suppresses (or attenuates) the phenomenon of hubs becoming more vulnerable by being dependent on low-degree nodes in a coupled network.

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