Interdependent Networks: Reducing the Coupling Strength Leads to a Change from a First to Second Order Percolation Transition

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We study a system composed from two interdependent networks A and B, where a fraction of the nodes in network A depends on nodes of network B and a fraction of the nodes in network B depends on nodes of network A. Because of the coupling between the networks, when nodes in one network fail they cause dependent nodes in the other network to also fail. This invokes an iterative cascade of failures in both networks. When a critical fraction of nodes fail, the iterative process results in a percolation phase transition that completely fragments both networks. We show both analytically and numerically that reducing the coupling between the networks leads to a change from a first order percolation phase transition to a second order percolation transition at a critical point. The scaling of the percolation order parameter near the critical point is characterized by the critical exponent $\beta = 1$.

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Most of the research on networks has concentrated on the limited case of a single network [1–5] while real world systems are composed from many interdependent networks that interact with one another [6–8]. As a real example, consider a power network and a communication network that are coupled together. The communication nodes depend on the power stations for electricity while the power stations depend on the communication nodes for control [9].

We show that introducing interactions between networks is analogous to introducing interactions among molecules in the ideal gas model. Interactions among molecules lead to the replacement of the ideal gas law by the van der Waals equation that predicts a liquid-gas first order phase transition line ending at a critical point characterized by a second order transition [Fig. 1(a)]. Similarly, interactions between networks give rise to a first order percolation phase transition line that changes at the critical point to a second order transition, as the coupling strength between the networks is reduced [Fig. 1(b)]. Near the critical point the order parameter (the size of the largest component) scales linearly with the distance to the critical point, leading to the critical exponent $\beta = 1$.

In interdependent networks, nodes from one network depend on nodes from another network. Consequently, when nodes from one network fail they cause nodes from another network to also fail. If the connections within each network are different, this may trigger a recursive process of cascading failures that can completely fragment both networks. Recently, Buldyrev *et al.* [10] studied the coupling between two N node networks A and B assuming the following restrictions: (i) Each and every node in network A depends on *one* node from network B and vice versa; (ii) if node A_i depends on node B_i, then node B_i depends on node A_i. They show that for such a model, when a critical fraction of the nodes in one network fail, the system undergoes a first order phase transition due to the recursive process of cascading failures.

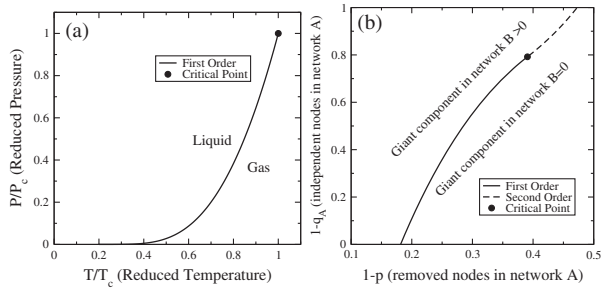

FIG. 1. (a) The van der Waals phase diagram. Along the liquid-gas equilibrium line a first order transition occurs; thus, the order parameter (density) abruptly changes from a low value in the gas phase to a high value in the liquid phase. At the critical point (P_c, T_c) the order parameter changes continuously as a function of temperature (if the pressure is kept constant), but its derivative (compressibility) diverges, characterizing a second order phase transition. (b) The percolation phase transition for two interdependent networks as obtained from the numerical solution of system (8) for $q_B = 1$ and $a = b = 3$. Here $1 - p$, the fraction of removed nodes from network A, plays the role of temperature. (As $1 - p$ increases, the disorder increases.) The fraction $1 - q_A$ of independent nodes in network A plays the role of pressure. (As $1 - q_A$ increases, the stability of network A increases.) Below the critical point, the system undergoes a first order phase transition at which ϕ_∞, the fraction of nodes in the giant (largest) component of network B, abruptly changes from a finite value to zero. As we approach the critical point, $\phi_\infty \to 0$. Above the critical point, the system undergoes a second order transition where the giant component continuously approaches zero.

However, when examining the features of real interdependent networks such as the power network and the communication network presented above, we observe that in practice not all nodes of network A depend on network B and vice versa. We therefore introduce a general model that is applicable to many real networks. This model consists of two networks A and B with the number of nodes N_A and N_B, respectively. Within network A, the nodes are randomly connected by A edges with degree distribution $P_A(k)$, while the nodes in network B are randomly connected by B edges with degree distribution $P_B(k)$. In addition, a fraction q_A of network A nodes depends on the nodes in network B and a fraction q_B of network B nodes depends on the nodes in network A. However, similar to the previous model a node from one network depends on no more than one node from the other network, and if A_i depends on B_j and B_j depends on A_k then $k = i$. We find that for strong coupling (large values of q_A and q_B) the networks undergo a first order transition, while for a weak coupling they undergo a second order phase transition. Even for the case of weak coupling in which a second order percolation transition occurs, the system still disintegrates in an iterative process of cascading failures unlike a regular second order percolation transition in a single network.

The iterative process of cascading failures is initiated by randomly removing a fraction $1 - p$ of network A nodes and all the A edges that are connected to them. Because of the interdependence between the networks, the nodes in network B that depend on removed A nodes are also removed together with the B edges that are connected to them. As nodes and edges are removed, each network breaks up into connected components (clusters). We assume that when the network is fragmented, the nodes belonging to the largest component (giant component) connecting a finite fraction of the network are still functional, while nodes that are parts of the remaining small clusters become nonfunctional. Since each network is connected differently, the nodes that become nonfunctional on each step are different for both networks. This leads to the removal of more dependent nodes from the coupled network and so on.

Next we present the formalism for the cascade process step by step. We define p_A and p_B as the fraction of nodes belonging to the giant components of networks A and B, respectively. The remaining fraction of network A nodes after an initial removal of $1 - p$ is $\psi'_1 \equiv p$. The initial removal of nodes will disconnect additional nodes from the giant cluster. The remaining functional part of network A therefore contains a fraction $\psi_1 = \psi'_1 p_A(\psi'_1)$ of the network nodes. Since a fraction q_B of nodes from network B depends on nodes from network A, the number of nodes in network B that become nonfunctional is $(1 - \psi_1)q_B = q_B[1 - \psi'_1 p_A(\psi'_1)]$. Accordingly, the remaining fraction of network B is $\phi'_1 = 1 - q_B[1 - \psi'_1 p_A(\psi'_1)]$, and the fraction of nodes in the giant component of network B is $\phi_1 = \phi'_1 p_B(\phi'_1)$.

Following this approach we can construct the sequence, ψ_n and ϕ_n, of giant components, and the sequence, ψ'_n and ϕ'_n, of the remaining fraction of nodes at each stage of the cascade of failures. The general form is given by

$$\begin{aligned}
&\psi'_1 \equiv p, \qquad \psi_1 = \psi'_1 p_A(\psi'_1),\\
&\phi'_1 = 1 - q_B[1 - p_A(\psi'_1)p], \qquad \phi_1 = \phi'_1 p_B(\phi'_1),\\
&\psi'_2 = p[1 - q_A(1 - p_B(\phi'_1))], \qquad \psi_2 = \psi'_2 p_A(\psi'_2)\ldots,\\
&\psi'_n = p[1 - q_A(1 - p_B(\phi'_{n-1}))], \qquad \psi_n = \psi'_n p_A(\psi'_n),\\
&\phi'_n = 1 - q_B(1 - p_A(\psi'_n)p), \qquad \phi_n = \phi'_n p_B(\phi'_n). \qquad (1)
\end{aligned}$$

To determine the state of the system at the end of the cascade process we look at ϕ'_m and ψ'_m at the limit of $m \to \infty$. This limit must satisfy the equations $\psi'_m = \psi'_{m+1}$ and $\phi'_m = \phi'_{m+1}$ since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at step m and $m + 1$ are equal. Denoting $\phi'_m = y$ and $\psi'_m = x$, we arrive at a system of two equations with two unknowns:

$$y = 1 - q_B[1 - p_A(x)p], \quad x = p\{1 - q_A[1 - p_B(y)]\}. \quad (2)$$

The model can be solved analytically using the apparatus of generating functions. The generating functions will be defined for network A while similar equations describe network B. As in Refs. [11,12] we will introduce the generating function of the degree distributions $G_{A0}(\xi) = \sum_k P_A(k)\xi^k$. Analogously we will introduce the generating function of the underlying branching processes, $G_{A1}(\xi) = G'_{A0}(\xi)/G'_{A0}(1)$. Random removal of fraction $1 - p$ of nodes will change the degree distribution of the remaining nodes, so the generating function of the new distribution is equal to the generating function of the original distribution with the argument equal to $1 - p(1 - \xi)$ [11]. The fraction of nodes that belongs to the giant component after the removal of $1 - p$ nodes is [12]

$$p_A(p) = 1 - G_{A0}[1 - p(1 - f_A)], \quad (3)$$

where $f_A = f_A(p)$ satisfies a transcendental equation

$$f_A = G_{A1}[1 - p(1 - f_A)]. \quad (4)$$

In the case of two Erdős-Rényi (ER) networks, whose degrees are Poisson distributed [13–15], the problem can be solved explicitly. Suppose that the average degree of the network A is a and the average degree of the network B is b. Then, $G_{A1}(\xi) = G_{A0} = \exp[a(\xi - 1)]$ and $G_{B1}(\xi) = G_{B0} = \exp[b(\xi - 1)]$. Accordingly, $p_B(x) = 1 - f_B$ and $p_A(x) = 1 - f_A$, and therefore system (2) becomes

$$x = p[1 - q_A f_B], \qquad y = 1 - q_B(1 - p[1 - f_A]), \quad (5)$$

where f_A and f_B satisfy the transcendental equations

$$f_A = \exp[ax(f_A - 1)], \qquad f_B = \exp[by(f_B - 1)]. \quad (6)$$

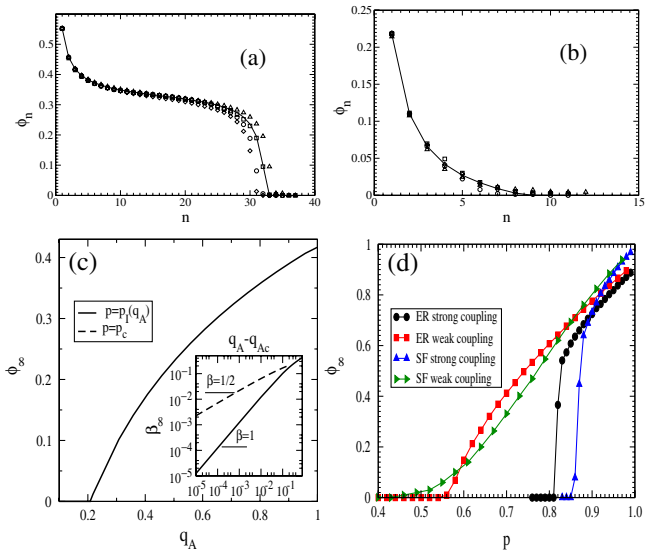

FIG. 2 (color online). ϕ_∞, the fraction of nodes in network's B giant component, after a cascade of failures in ER networks of size $N_A = N_B = 8 \times 10^5$. (a) The value of ϕ_n for every iteration of the first order cascades with the initial parameters $p = 0.7455$, $a = b = 2.5$, $q_A = 0.7$, and $q_B = 0.6$. (b) Similar to (a) but for the second order case with parameters $p = 0.605$, $a = b = 2.5$, $q_A = 0.2$, and $q_B = 0.75$. Symbols represent simulation results for different random realizations of the networks. Solid lines represent the solution of system (5). (c) ϕ_∞, as function of q_A computed at $p = p_I(q_A)$, the line of the first order phase transition, for $a = 3$, $b = 3$, $q_B = 1$. Inset: The same results (solid line) as function of $|q_A - q_{A_c}|$ yield a straight line with slope $\beta = 1$ in double logarithmic scale. If q_A is changed but $p = p_c$ is kept constant we obtain a straight line with slope $\beta = 0.5$ (dashed line). (d) Simulation results for the phase transition of ϕ_∞ as a function of p for ER and SF ($\lambda = 2.7$) networks of size $N = 50$ K. For strong coupling between the networks we observe a jump in ϕ_∞ as expected in the first order phase transition [ER (circle) and SF (up-pointing triangle)]. For weak coupling between the networks the change in ϕ_∞ is gradual as expected for the second and higher order phase transitions [ER (square) and SF (right-pointing triangle)].

The fraction of nodes in the giant components of networks A and B, respectively, at the end of the cascade process is given by $\psi_\infty = p(1 - f_A)(1 - q_A f_B)$ and $\phi_\infty = (1 - f_B)[1 - q_B(1 - p) - pq_B f_A]$. Figure 2 shows excellent agreement between computer simulations of the cascade failures and the numerical results obtained by solving system (1), where $p_A(\psi'_n)$ and $p_B(\phi'_n)$ are computed using Eqs. (3) and (4) and the generating functions of ER networks. Excluding x and y from systems (5) and (6), we obtain a system:

$$f_A = e^{-ap(f_A-1)(q_A f_B-1)},$$
$$f_B = e^{-b\{q_B(1-p[1-f_A])-1\}(f_B-1)}. \quad (7)$$

The first equation can be solved with respect to f_B, and the second equation can be solved with respect to f_A:

$$f_B = \frac{1}{q_A}\left[1 - \frac{\log f_A}{ap(f_A - 1)}\right], \quad f_A \neq 1; \ \forall f_B, f_A = 1,$$
$$f_A = \frac{1}{q_B}\left[\frac{1 + q_B(p-1)}{p} - \frac{\log f_B}{bp(f_B - 1)}\right], \quad f_B \neq 1;$$
$$\forall f_A, f_B = 1. \quad (8)$$

The solutions of system (8) can be graphically presented on a f_A, f_B plane (Fig. 3). The solutions are presented as a crossing of either $f_B(f_A)$ or $f_A = 1$ with $f_A(f_B)$ or $f_B = 1$ and are restricted to the square $0 \leq f_A \leq 1$; $0 \leq f_B \leq 1$. There are three different possible solutions. (i) The solution where the giant components of both networks are zero ($f_A = 1$ and $f_B = 1$) as in Fig. 3(c). (ii) A solution for which only one of the giant components of either network A or B is zero ($f_A = 1$ and $f_B \neq 1$ or $f_A \neq 1$ and $f_B = 1$) as in Fig. 3(d) [or Fig. 3(e)]. (iii) A solution for which both networks have a nonzero giant component ($f_A \neq 1$ and $f_B \neq 1$). This solution is given by the lowest intersection point of the curves in Fig. 3(a). This solution may disappear in two different scenarios.

The first scenario is presented in Fig. 3(b) in which an infinitesimal change $\triangle\vec{z}$ in the vector of the system parameters $\vec{z} = (a, b, q_A, q_B, p)$ may lead to a first order phase transition in which the size of one or both of the giant components changes discontinuously from a finite value to zero: [Fig. 3(a) → Fig. 3(b) → Fig. 3(c) or Fig. 3(d), or Fig. 3(e)]. The condition for the first order phase transition is $\frac{df_B(f_A)}{df_A}\frac{df_A(f_B)}{df_B} = 1$, which corresponds to the touching point of the two curves as in Fig. 3(b). When adding this condition to the two equations in system (8), we can find the three unknowns $f_A = f_{A_I}$, $f_B = f_{B_I}$, and $p = p_I$ for given a, b, q_A, q_B. Fixing a, b, q_B will define a first order phase transition line $p = p_I(q_A)$ as a function of q_A [Fig. 1(b)].

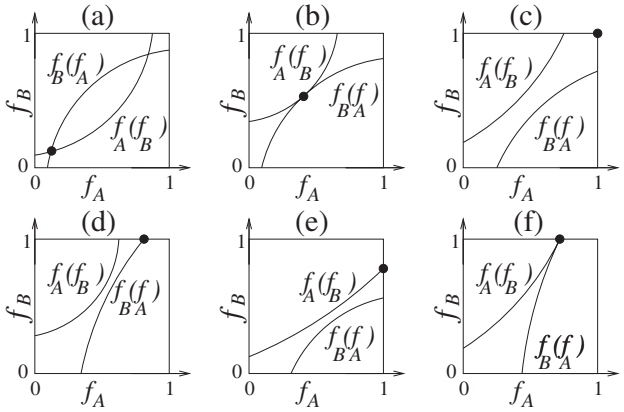

FIG. 3. Illustrations of the different graphical solutions of system (8) (see text for detailed explanation of the different plots).

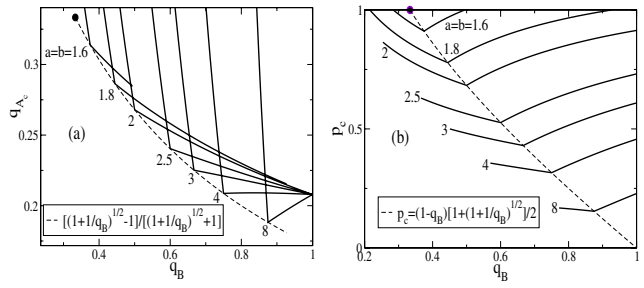

FIG. 4 (color online). The critical point parameters, q_{A_c} (a) and p_c (b), as functions of the coupling strength q_B for ER networks are plotted for different values of $a = b$. (a) For $q_B = 1$ the networks (with large degrees) have the same critical coupling strength $q_{A_c} = 0.207\,94$. The networks with small degrees do not have first order phase transitions (no critical points for q_B), because for large values of q_B, $p_c(q_B) > 1$, which is unphysical. The range of q_B values for which the first order phase transition exists shrinks as $a = b$ decreases and eventually disappears for $a = b = 3/2$, when the critical point exists only for $q_{A_c} = q_B = 1/3$ and $p_c = 1$. This point is marked by a solid circle.

The second scenario is presented in Fig. 3(f). In this case (corresponding to $f_A < 1$, $f_B = 1$, or equivalently to $q_B > 1 - 1/b$) ϕ_∞ continually decreases to zero, while ψ_∞ stays finite. This situation corresponds to the second order phase transition that can be found by substituting $f_B = 1$ into system (8). These two equations allow one to find $f_A = f_{A_{\mathrm{II}}}$, and $p = p_{\mathrm{II}}$, which for fixed a, b, q_B define a line of second order phase transitions $p = p_{\mathrm{II}}(q_A)$ as a function of q_A [Fig. 1(b)].

The line of the first order phase transitions merges with the line of the second order phase transitions in a critical point which can be found by adding to system (8) both the first order condition $\frac{df_B(f_A)}{df_A}\frac{df_A(f_B)}{df_B} = 1$ and the second order condition $f_B = 1$ or $f_A = 1$. These four equations allow us to find the critical parameters $f_B = f_{B_c}$ or $f_A = f_{A_c}$, $p = p_c$, and $q_A = q_{A_c}$ as functions of a, b, q_B. Figure 4 presents the solution for $p_c(q_B)$ and $q_{A_c}(q_B)$ for different values of $a(= b)$. The kink in the solutions occurs when both curves tangentially intersect at $f_A = 1$, $f_B = 1$, which corresponds to $\tilde{q}_B = 1 - 1/b$. The minimal value of p_c occurs exactly at the kink, defining the condition for the first order phase transition as $p_c(\tilde{q}_B) < 1$. Thus the first order transition can exist only in dense networks with sufficiently high average degrees, such that $4(a-1)(b-1) > 1$. Low degree networks must disintegrate in the second order phase transitions.

At the critical point the system can be reduced to a single transcendental Lambert equation. For the simplest case $a/b = q_B = 1$, we find that $f_{A_c} = 1/z$, $q_{A_c} = z - 2$, $p_c = z/[a(z-1)]$, and $\psi_\infty = (3-z)/a$, where $z = \mathcal{W}[\exp(3)] = 2.207\,94$ satisfies the Lambert equation $z\exp(z) = \exp(3)$.

To find the critical exponent β near the critical point, we express the order parameter $\phi_\infty(q_A)$ as function of $q_A > q_{A_c}$ along the transition line $p = p_I(q_A)$ [inset of Fig. 2(c)]. Expanding f_B in series of $x = q_A - q_{A_c}$, we find that $\lim_{x\to 0}(1 - f_B)/x = C > 0$, indicating that $\beta = 1$. Interestingly, if one keeps $p = p_c$ constant and changes only q_A, then $\lim_{x\to 0}(1 - f_B)/\sqrt{x} = C' < 0$ corresponding to $\beta = 1/2$. The inset of Fig. 2(c) confirms our analytical predictions numerically.

Although our analytical theory is developed for ER networks, the same qualitative conclusions hold for randomly connected networks with arbitrary degree distributions, since functions $p_A(x)$ and $p_B(y)$ can be expressed in terms of generating functions of these distributions. Hence an analysis similar to Fig. 3 holds for any degree distributions. Figure 2(d) shows that the first and second order phase transitions exist not only for ER networks but also for scale free (SF) networks characterized by a power law degree distribution.

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