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Robustness of networks with dependency topology

YUANSHENG LIN[1,2,3], RUI KANG[1,2], ZHEN WANG[4], ZILONG ZHAO[1,2], DAQING LI[1,2(a)] and SHLOMO HAVLIN[5]

[1] *School of Reliability and Systems Engineering, Beihang University - Beijing 100191, China*
[2] *Science and Technology on Reliability and Environmental Engineering Laboratory - Beijing 100191, China*
[3] *Department of Computer Science, University of California - Davis, CA 95616, USA*
[4] *Center for OPTical IMagery Analysis and Learning (OPTIMAL), Northwestern Poly-technical University Xi'an 710072, China*
[5] *Department of Physics, Bar-Ilan University - Ramat Gan 5290002, Israel*

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Abstract – The robustness of complex networks with dependency links has been studied in recent years. However, previous studies focused mostly on the robustness of networks with dependency relations having local and simple structures, not considering the general cases where global network topology is formed by dependency links. Here, we analyze the percolation properties of network models composed of both connectivity and dependency links, where in addition to the usual connectivity links, dependency links also follow a certain network topology. We perform theoretical analysis and numerical simulations to understand the critical effects of dependency topology on the network robustness. Our results suggest that for a given network topology of connectivity, dependency topology can influence the network robustness, leading to different percolation types. Furthermore, we also give the theoretical analysis and simulation results on different combinations of connectivity topology and dependency topology. Our results may help to design and optimize the network robustness considering the underlying complicated dependency relationships.

Introduction. – Components in critical infrastructures cannot function independently, and usually interact with others through connectivity or dependency links [1–15]. With the aid of connectivity links, nodes can function cooperatively as a network and will stop functioning either through their own failure or when they become disconnected from the giant component of the network. Dependency links show distinct functional relationships: if node A depends on node B, the failure of node B will cause node A to fail directly even if node A is still connected to the giant component [5,6]. For example, nodes in the power grid convey power flow to certain loads, whose faults may induce overloads and failures of other nodes. These successive failures can be represented as certain dependency link between nodes, *i.e.*, failure spreads with a characteristic length [16]. Initiated by random failures or malicious attacks, these dependencies may lead to catastrophic events including blackouts in power grids and jamming in transportation networks [17,18].

Previous studies consider the network robustness with a simple dependency structure. Parshani *et al.* [5] introduced a network model having dependency groups with fixed size 2, see fig. 1(a). Bashan *et al.* [6] generalized Parshani's results and also studied the effects of dependency groups whose sizes follow a normal distribution or a Poisson distribution. Meanwhile, dependency is also considered while studying the robustness of interdependent networks, as significant interactions exist between modern infrastructures [3,4,8,9,11,13,19–22]. Buldyrev *et al.* [3] developed a theoretical framework to understand the robustness of two coupled networks, and dependency is represented as a one-to-one correspondence between two networks, meaning that each node in one network depends on one and only one node in the other network and vice versa. Shao *et al.* [9] proposed a more general network model where interdependent networks may have multiple support-dependencies considering that a node in one

(a) E-mail: daqingl@buaa.edu.cn

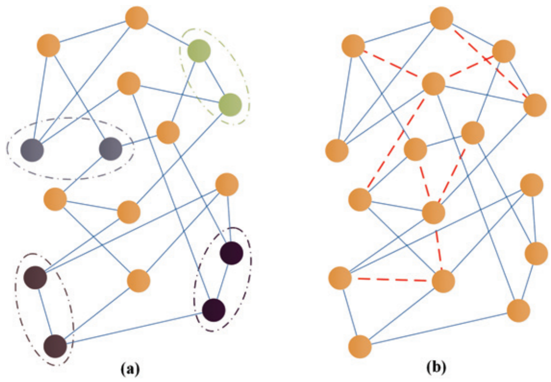

Fig. 1: (Colour online) Network with different dependency structures. (a) Dependency group structure: connectivity network with dependency groups of size 2, studied in [6]. The solid blue lines represent connectivity links, and dependency groups are surrounded by dash-dotted lines; (b) network with topology of dependency links: the solid blue lines represent connectivity links and the red dashed lines represent a network topology of dependency links. Here, the size of dependency groups is not limited as in (a).

network may rely on more than one node in the other network and will function until its last support node fails. Rather than dependencies which are connected randomly, some studies [20,21] analyzed the case where nodes with similar degrees tend to be coupled in an interdependent network and found that this case is more robust to random failures. While the robustness of two coupled networks systems has been studied separately only for dependency coupling [3] and only for connectivity coupling [23] more recently a system where both interdependent and interconnected links exist was also studied [8].

However, in realistic systems, dependencies between nodes can be much more complicated and might form a global network topology, which to the best of our knowledge has not been considered yet. Different from simple and local dependency structures studied earlier (fig. 1(a)), here we analyze the percolation properties of networks with global network topologies of dependency (fig. 1(b)). Our theoretical results are supported by numerical simulations. We present results for different combinations of connectivity and dependency topologies, namely i) ER (Erdös-Rényi) connectivity and ER dependency (ER-ER), ii) ER connectivity and RR (random-regular) dependency (ER-RR), iii) RR connectivity and ER dependency (RR-ER) and iv) RR connectivity and RR dependency (RR-RR). Our results suggest that dependency topology can lead to distinct network robustness, with different percolation thresholds and types of percolation transition.

The paper is organized as follows. In the second section, we will introduce the network model composed of dependency links following a certain topology. In the third section, we show the general framework for the theoretical analysis. In the fourth and fifth sections, we present both the simulation and analytical results of network robustness with different connectivity and dependency topologies. In the last section, we summarize our results.

Model description. – We study the robustness of networks containing different network topologies of connectivity and dependency links. The iterative process between percolation stage and dependency stage is as follows [5,6,24]:

1) Initially, we randomly remove a fraction $1-p$ of nodes from the network as well as their associated links;

2) Percolation stage: in the connectivity network, nodes (and their links) that do not belong to the giant component are also removed;

3) Dependency stage: if a node fails, all the other nodes in the same dependency cluster will fail (they are assumed to depend on each other).

4) The failure of the clusters in step 3 leads to failures of other nodes that become disconnected from the giant component. Then a cascade of failures will continue until no additional failures occur.

Framework for theoretical analysis. – Here we show the theoretical analysis studying the iterations between the percolation stage and the dependency stage in the initial presence of random failures based on the formulism in [5,6].

When we first randomly remove a fraction of $1-p$ of nodes, the fraction of remaining nodes in the connectivity network is defined as $\alpha_{c,1}=p$. Since initial random removal will cause additional nodes to disconnect from the giant component, the remaining fraction of nodes $\beta_{c,1}$ after the first percolation stage is given by $\beta_{c,1}=\alpha_{c,1}E_c(\alpha_{c,1})$, where $E_c(\alpha_{c,1})$ denotes the fraction of nodes belonging to the giant component after a random removal of $1-\alpha_{c,1}$ nodes from the original network.

Before the dependency stage, the total removed fraction of nodes from the original network is $1-\beta_{c,1} = (1-\alpha_{c,1})+(\alpha_{c,1}-\beta_{c,1})$. Accordingly, the remaining fraction of nodes before the dependency stage is $\alpha_{d,1}=\beta_{c,1}$. Due to dependency links, if a node within a dependency cluster fails, the entire dependency cluster will be disabled. The size of the functional part after the first dependency stage is given by $\beta_{d,1}=\alpha_{d,1}E_d(\alpha_{d,1})$, where $E_d(\alpha_{d,1})$ stands for the fraction of nodes that do not depend on failed nodes. The accumulated consequence of the first cascade stage is equivalent to a single random removal of the $1-pE_d(\alpha_{d,1})$ fraction from the original network [5]. In this way, the formulism of the iterative equations can be described as follows:

$$\begin{cases} \alpha_{c,1}=p, & \beta_{c,1}=\alpha_{c,1}E_c\left(\alpha_{c,1}\right),\\ \alpha_{d,1}=pE_c\left(\alpha_{c,1}\right), & \beta_{d,1}=\alpha_{d,1}E_d\left(\alpha_{d,1}\right),\\ \alpha_{c,2}=pE_d\left(\alpha_{d,1}\right), & \beta_{c,2}=\alpha_{c,2}E_c\left(\alpha_{c,2}\right),\\ \alpha_{d,2}=pE_c\left(\alpha_{c,2}\right), & \beta_{d,2}=\alpha_{d,2}E_d\left(\alpha_{d,2}\right). \end{cases} \tag{1}$$

Accordingly, one can get $\alpha_{c,i}$, $\alpha_{d,i}$, $\beta_{c,i}$ and $\beta_{d,i}$ for any stage of the cascading failures:

$$\begin{cases} \alpha_{c,i} = pE_d(\alpha_{d,i-1}), & \beta_{c,i} = \alpha_{c,i}E_c(\alpha_{c,i}), \\ \alpha_{d,i} = pE_c(\alpha_{c,i}), & \beta_{d,i} = \alpha_{d,i}E_d(\alpha_{d,i}). \end{cases} \tag{2}$$

Cascading failures will terminate at the steady state when no further damages occur. In the steady state ($i \to \infty$), $\alpha_{c,i} = \alpha_{c,i+1}$ or $\alpha_{d,i} = \alpha_{d,i+1}$. The final state of the system can be derived by solving the following equation:

$$\begin{cases} \alpha_{c,\infty} = pE_d\left(\alpha_{d,\infty}\right), \\ \alpha_{d,\infty} = pE_c\left(\alpha_{c,\infty}\right). \end{cases} \tag{3}$$

Denoting $\alpha_{d,\infty} = x$ and $\alpha_{c,\infty} = y$, eq. (3) is reduced into

$$x = pE_c\left(pE_d(x)\right). \tag{4}$$

In order to obtain the analytical solutions of eq. (4), we have to derive explicitly both $E_c(T)$ and $E_d(T)$.

For obtaining $E_c(T)$, we can use generating functions [25–28]. For a given network, the generating function of the degree distributions for the connectivity network is defined as follows:

$$G_{c,0}(\xi) = \sum_{k=0}^{\infty} P(k)\xi^k, \tag{5}$$

where $P(k)$ is the degree distribution, *i.e.*, the probability that a node chosen at random has k connectivity links. Similarly, the generating function of the underlining branching process is

$$G_{c,1}(\xi) = G'_{c,0}(\xi)/G'_{c,0}(1) = \sum_{k=1}^{\infty} \frac{kP(k)}{\langle k\rangle}\xi^{k-1}. \tag{6}$$

As described in [26], a random removal of $(1-T)$ fraction of nodes will lead to a new degree distribution of the remaining nodes that has a similar generating function with argument $[1-T(1-\xi)]$. Thus, $E_c(T)$ which is the probability that a randomly chosen functional node belongs to the giant component after the removal of $(1-T)$ nodes is

$$E_c(T) = 1 - G_{c,0}[1-T(1-f)]. \tag{7}$$

Here $f = f(T)$ satisfies the self-consistency and transcendental function

$$f = G_{c,1}[1-T(1-f)]. \tag{8}$$

To calculate $E_d(T)$, we take the following considerations. Dependency topology is made up of many dependency clusters with different sizes. In the network of N nodes, each node has a probability $q(s)$ of belonging to a dependency cluster of size s, thus the number of dependency clusters of size s is $q(s)N/s$. Furthermore, after a random removal of a fraction $(1-T)$ of nodes, a dependency cluster of size s will survive with probability T^s, and the overall number of remaining nodes is given by $\sum_{s=1}^{\infty} q(s)NT^s s/s$. As a consequence, $E_d(T)$ is defined as probability that nodes survive under the removal of a fraction $(1-T)$,

$$E_d(T) = \sum_{s=1}^{\infty} q(s)T^{s-1}. \tag{9}$$

In order to derive $E_d(T)$, we need to derive $q(s)$ for the sizes of dependency clusters. Note that Parshani *et al.* [5] only focused on the special case of fixed size dependency cluster ($q(s)$ is delta function) while Bashan *et al.* [6] generalized the formalism into the case, where $q(s)$ follows a fixed distribution, including Poisson distribution and Gaussian distribution. However, with the presence of dependency topology, $q(s)$ changes with the dependency topology assumed and is usually different from homogeneous distribution assumed in [6].

We next calculate $q(s)$ based on the generating function formalism. We define $H_{d,1}(\eta)$ as the generating function for the sizes of the dependency clusters reached by a random link [25,29,30]:

$$H_{d,1}(\eta) = \eta G_{d,1}(H_{d,1}(\eta)). \tag{10}$$

Here, $G_{d,1}$ and $G_{d,0}$ are defined as the generating functions of the underlining dependency topology. Analogously, the generating function $H_{d,0}(\eta)$ for the size of the dependency cluster to which a randomly chosen node belongs is

$$H_{d,0}(\eta) = \eta G_{d,0}(H_{d,1}(\eta)). \tag{11}$$

According to the definition of $H_{d,0}(\eta)$, $q(s)$ can be given by

$$q(s) = \begin{cases} \dfrac{1}{(s-1)!}\left[\dfrac{\mathrm{d}^{s-1}}{\mathrm{d}\eta^{s-1}}\left(\dfrac{H_{d,0}(\eta)}{\eta}\right)\right]_{\eta=0} = \\ \dfrac{G'_{d,0}\ (1)}{(s-1)!}\left[\dfrac{\mathrm{d}^{s-2}}{\mathrm{d}\eta^{s-2}}[G_{d,1}(\eta)]^s\right]_{\eta=0}, \ s>1, \\ \text{the probability of having dependency zero, } s=1. \end{cases} \tag{12}$$

We evaluate the robustness of a network by calculating the size of the giant component at the steady state:

$$G = \beta_{c,\infty} = \beta_{d,\infty} = yE_c(y) = xE_d(x) = x\sum_{s=1}^{\infty} q(s)x^{s-1} = \sum_{s=1}^{\infty} q(s)x^s. \tag{13}$$

Given $G_{c,0}(\xi)$, $G_{c,1}(\xi)$, $G_{d,0}(\eta)$ and $G_{d,1}(\eta)$, eq. (13) can be solved with the set of equations

$$\begin{cases} x = pE_c(y) = p[1-G_{c,0}[1-y(1-f)]], \\ f = G_{c,1}[1-y(1-f)], \\ y = pE_d(x) = p\sum\limits_{s=1}^{\infty} q(s)x^{s-1}. \end{cases} \tag{14}$$

Results. – The framework for the theoretical analysis presented in the above section can be applied to a network with arbitrary connectivity and dependency topologies. We demonstrate analytically and numerically four

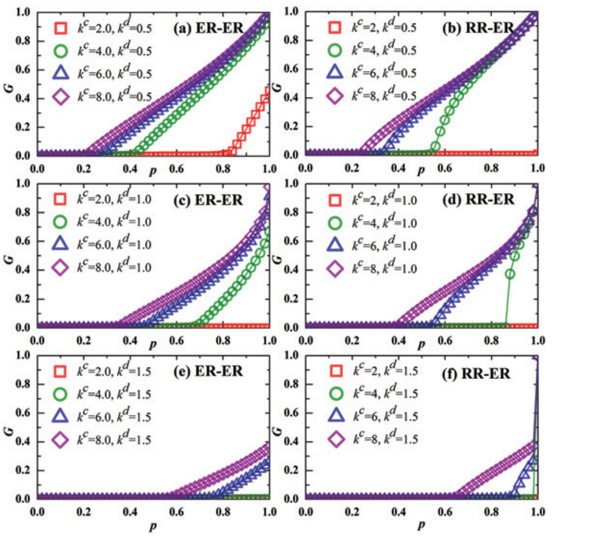

Fig. 2: (Colour online) The size of the largest component, G, as a function of the fraction of non-removed nodes p in networks with different combinations of dependency and connectivity topology: for (a), (c) and (e), ER-ER with $k_d = 0.5$, $k_d = 1.0$ and $k_d = 1.5$, respectively; in (b), (d) and (f), RR-ER with $k_d = 0.5$, $k_d = 1.0$ and $k_d = 1.5$, respectively. Open symbols represent the numerical results of systems of 100000 nodes averaged over 200 realizations. And the solid curves represent the theoretical results attained by solving the set of eqs. (14) and (13).

combinations of connectivity and dependency topologies (ER network and RR network), and discuss their percolation properties. We assume that the average connectivity degree and the average dependency degree are k_c and k_d, respectively.

A) ER networks with dependency network of ER topology (ER-ER). In this case, both connectivity links and dependency links follow the ER topology. In the ER topology, nodes are connected randomly with Poisson degree distribution, then we can get $G_{c,0}(\xi) = G_{c,1}(\xi) = \exp[k_c(\xi - 1)]$. Similarly, for dependency network of ER topology, we get $G_{d,0}(\eta) = G_{d,1}(\eta) = \exp[k_d(\eta - 1)]$. Using eq. (12), we can obtain

$$q(s) = \frac{\exp(-k_d s)(k_d s)^{s-1}}{s!}. \quad (15)$$

In ER dependency topology, eq. (15) is also valid for the special case $s = 1$. Solving the set of eqs. (14) and eq. (13), we obtain an expression for the size of giant component G:

$$G = \sum_{s=1}^{\infty} \frac{\exp(-k_d s)(k_d s)^{s-1} p^s}{s!}(1 - \exp(-k_c G))^s. \quad (16)$$

Note that when an ER network does not have dependencies ($k_d = 0$), eq. (16) coincides with the well-known equation $G = p(1 - \exp(-k_c G))$ [31–33]. Next, we compare our analytical results with simulations. Our simulation results are shown in fig. 2(a), (c), (e) and they agree well with the theoretical predictions. In fig. 2(c), when $k_d = 1$, we

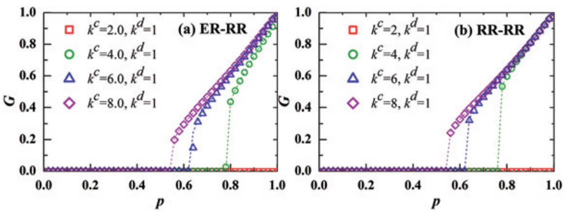

Fig. 3: (Colour online) The size of the largest component, G, as a function of the fraction of remaining nodes p in networks with RR dependency topology: (a) ER-RR; (b) RR-RR.

find that the ER-ER system disintegrates in a continuous percolation process while in the studies of homogeneous dependency groups the transition was usually abrupt [6]. Moreover, for the ER dependency network with other k_d, the phase transitions of ER-ER are also found to be continuous (fig. 2(a), (e)).

B) RR network with dependency network of ER topology (RR-ER). In RR-ER, every node has k_c connectivity links with other nodes, thus $G_{c,0}(\xi) = \xi^{k_c}$ and $G_{c,1}(\xi) = \xi^{k_c - 1}$. And for the ER topology of dependency links, $G_{d,0}(\eta) = G_{d,1}(\eta) = \exp(k_d(\eta - 1))$. We can get

$$G = \sum_{s=1}^{\infty} \frac{\exp(-k_d s)(k_d s)^{s-1}}{s!} x^s, \quad (17)$$

where x could be obtained using eqs. (14). In fig. 2(b), (d), (f), we show the analytical and numerical results for RR-ER. First, it is seen that for the same pair of (k_c, k_d), the RR-ER is more vulnerable compared to ER-ER, *i.e.*, p_c is larger for RR-ER. Furthermore, in contrast to ER-ER here, the percolation process of RR-ER seems to show two kinds of behavior: for a higher density of dependency links ($k_c = 4$, $k_d = 1$) (fig. 2(d)), the system undergoes an abrupt collapse, while for the same k_d, as k_c increases, the system becomes more stable with a continuous phase transition (*e.g.*, $k_c = 6$, $k_d = 1$, in fig. 2(d)).

C) ER network with dependency network of RR topology (ER-RR). Dependency links could form a RR topology in an ER network. In RR topology, each node has k_d dependency links, the corresponding generating functions are $G_{d,0}(\eta) = \eta^{k_d}$ and $G_{d,1}(\eta) = \eta^{k_d - 1}$. Then according to eq. (12), $q(s)$ can also be calculated. Indeed for the special case $k_d = 1$, $q(2) = 1$, G is obtained in the form:

$$G = p^2(1 - \exp(-k_c G))^2, \quad (18)$$

and this is reduced to the case of pairs of dependency nodes discussed in ref. [6]. For the same pair of k_d and k_c, compared with ER-ER (fig. 2(c)), ER-RR is more vulnerable and undergoes an abrupt collapse at critical threshold (fig. 3(a)) which indicates that the change of dependency topology can alter the type of phase transition.

D) RR network with dependency network of RR topology (RR-RR). Next we study the case where nodes in RR connectivity network are also connected by dependency

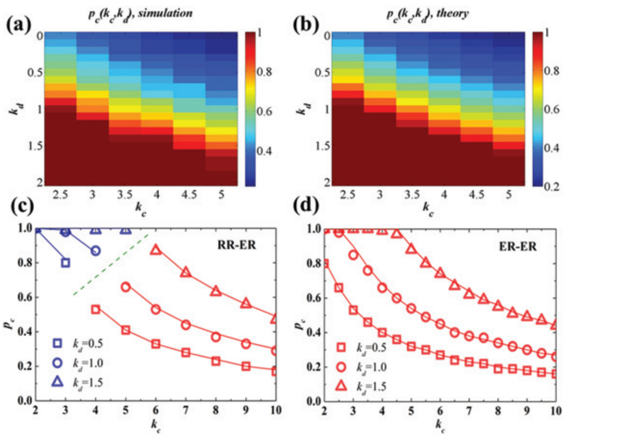

Fig. 4: (Colour online) Dependence of the critical threshold of non-removed nodes p_c on the average connectivity degree (k_c) and the average dependency degree (k_d): (a) ER-ER: simulation results; (b) ER-ER: theoretical results; (c) RR-ER: the blue lines represent the analytical results and the blue symbols represent the simulation results for the first change order cases, while the red lines and the red symbols represent the cases of the continuous transition. The corresponding k_d from left side to the right is 0.5, 1 and 1.5; in (d), we illustrate the values of p_c for ER-ER.

links following the RR topology. So $G_{c,0}(\xi) = \xi^{k_c}$ and $G_{c,1}(\xi) = \xi^{k_c-1}$. For dependency links of RR topology, $G_{d,0}(\eta) = \eta^{k_d}$ and $G_{d,1}(\eta) = \eta^{k_d-1}$. For fig. 3(b) ($k_d = 1$), the results are the same as shown in ref. [24]: the RR network falls apart in an abrupt way. Compared with RR-ER (fig. 2(d)), for $k_d = 1$, we find that RR-RR becomes more vulnerable than RR-ER when $k_c = 6$ or 8 (fig. 3(b)).

Finally, we analyze the dependence of p_c on the average connectivity degree k_c and the average dependency degree k_d in ER-ER and RR-ER. Analytical results are calculated using the graphical method illustrated in the "Methods" section in fig. 5. For different types of phase transitions (abrupt or continuous), the simulation results of p_c are obtained by two distinctive methods: i) for example, in the RR-ER case (fig. 4(c)), for a constant average dependency degree k_d, when k_c is small, the system disintegrates in an abrupt way, and we identify the critical points in simulations when the number of iterative (NOI) failures reaches the maximum [5]; ii) when k_c becomes larger, the system undergoes a continuous transition, the critical point for the transition is identified when the size of the second largest component approaches the maximum. Figure 4(c) shows that for a given dependency topology adding connectivity links will change the network collapse manner (*e.g.*, k_c is 4 and 5) from abrupt to a continuous transition. However, the phase transition of ER-ER is found mainly continuous (fig. 4(d)). In addition, comparing fig. 4(c) with fig. 4(d), for a certain k_d, we find that the p_c for RR-ER is slightly larger than the p_c for ER-ER, which indicates that ER-ER is more robust than RR-ER when they have same number of connectivity and dependency links.

Much attention has been paid to understand the inherent mechanisms that yield continuous or discontinuous phase transition in complex networks. In bootstrap percolation, the type of phase transition can be either continuous or discontinuous, mainly relying on the type of network topology (random or lattice), and on lattice's dimensionality [34]. In 2009, Achlioptas *et al.* [35] proposed the explosive percolation model for abrupt transitions. Then, Cho *et al.* found that the transition type of explosive percolation depends on the bias against certain "bridging" bonds and system's dimensionality [36]. When it comes to the study of cooperative co-infections, Chen *et al.* found that the cooperativity between two diseases could change the epidemic outbreak from continuous into discontinuous. [37]. As most complex systems are coupled together, while studying the cascading failures of interdependent networks, the phase transition is found to be first order [3,4,13,19] while single networks with only connectivity links exhibit a classical continuous phase transition with respect to random failures [38,39]. Also, increasing the fraction of dependency between networks [4] and the strength of interconnectivity links [8] lead to a change of phase transition from second order to first order. In single isolated networks, introducing dependencies between nodes will also change the behaviors of phase transitions. In ref. [5], Parshani *et al.* showed that the coupling strength of dependency between nodes has a great impact on the percolation properties, and high fraction (above certain threshold) of dependency links could lead to cascading failures with abrupt first-order transition. Moreover, Bashan *et al.* found that the phase transition is changing from second order to first order while increasing the size of fixed dependency group or the average size of dependency groups (their size follows normal distribution or Poisson distribution) leading to larger value of percolation threshold [6]. Interestingly, however, as found in the above different combinations of our work, the transition feature of percolation is not only dependent on the dependency strength, but it is also influenced by dependency topology.

Methods. – Next, we show how to distinguish, in our models, the order of percolation transition and identify the corresponding percolation threshold and the size of the giant component. The percolation threshold is the critical fraction, $1 - p_c$, of nodes that will fragment the whole network. Our method is based on graphical solutions of the set of eqs. (14). Analytical results of the size of giant component shown in fig. 2 and fig. 3 are also obtained by this graphical method.

To demonstrate the solution we consider the following cases: ER-ER and ER-RR ($k_c = 4.0, k_d = 1$), RR-ER and RR-RR ($k_c = 6.0, k_d = 1$) as examples to show how to differentiate the types of phase transition. According to eqs. (14), we define

$$D(f) = G_{c,1}[1 - y(1 - f)] - f, \qquad (19)$$

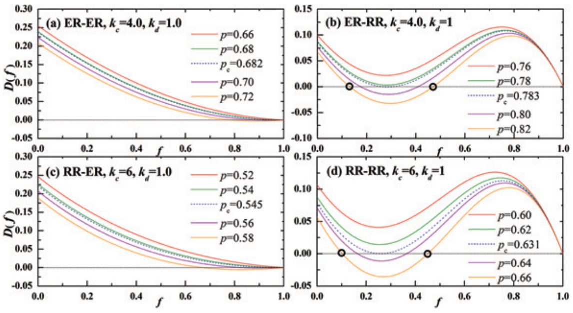

Fig. 5: (Colour online) Graphical solutions of the set of eqs. (14) for the cases of ER-ER, ER-RR, RR-ER and RR-RR. The effective solution for f is given by the intersection point of the curve shown in the graph. For (a), (c), the systems (ER-ER and RR-ER) break down in a continuous transition. In (b), (d), ER-RR and RR-RR disintegrate in an abrupt manner. The dotted lines show the critical value of p_c. The nontrivial intersection points for a p above the critical threshold are marked as black open circles.

and we get the value of f graphically as a function of p that satisfies $D(f) = 0$ as illustrated in fig. 5. In figs. 5(b) and 5(d), for smaller p, $D(f)$ has only one intersection point with $D(f) = 0$ at $f = 1$, which indicates the critical point where the system collapses. When p is increased to a certain value, *e.g.*, $p = 0.783$ in fig. 5(b) or $p = 0.631$ in fig. 5(d), the corresponding curves begin to have three intersection points with $D(f) = 0$, for the first time. This means that, except for the trivial solution ($f = 1$), the system has two other possible states, leading to an abrupt change. Consequently, the value at which the curve is tangential to $D(f) = 0$ is defined as the critical threshold of the phase transition. However, in ER-ER (fig. 5(a)) or RR-ER (fig. 5(c)), the system disintegrates in a continuous percolation transition. Different from the first-order case, $D(f)$ always have only maximal two intersection points with $D(f) = 0$ above the threshold p_c. This graphical method can help to determine the types of phase transitions and identify their corresponding percolation thresholds.

Conclusions. – In this study we examine the percolation properties of networks with dependency links following a certain topology, rather than a simple form of local structure. We perform the theoretical and numerical analysis of percolation properties including the size of the giant component, the order of the phase transition as well as the critical threshold. Particularly, we consider the robustness of the ER network and of the RR network with different network dependency topologies and predict their critical thresholds of phase transitions. We find that for the same connectivity network, different dependency topologies can change the percolation criticality, including the type of phase transition. Especially, our results indicate that the ER network with ER dependency collapses smoothly, while the ER network with RR dependency may break down abruptly. Our results may help to model the complicated failure behaviors in the real systems, and perform the evaluation of system reliability [40].

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