Wave Localization in Complex Networks with High Clustering

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We show that strong clustering of links in complex networks, i.e., a high probability of triadic closure, can induce a localization-delocalization quantum phase transition (Anderson-like transition) of coherent excitations. For example, the propagation of light wave packets between two distant nodes of an optical network (composed of fibers and beam splitters) will be absent if the fraction of closed triangles exceeds a certain threshold. We suggest that such an experiment is feasible with current optics technology. We determine the corresponding phase diagram as a function of clustering coefficient and disorder for scale-free networks of different degree distributions $P(k) \sim k^{-\lambda}$. Without disorder, we observe no phase transition for $\lambda < 4$, a quantum transition for $\lambda > 4$, and an additional distinct classical transition for $\lambda > 4.5$. Disorder reduces the critical clustering coefficient such that phase transitions occur for smaller λ.

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Anderson localization continues to spur excitement although half a century has passed since it was first conceived in the context of electron transport through disordered metals [1]. Since then new systems in which this phenomenon occurs were suggested and verified, such as light in strongly scattering media [2] or photonic crystals [3], acoustical vibrations in glasses [4] or percolation systems [5], and very recently atomic Bose-Einstein condensates in an aperiodic optical lattice [6]. Clearly, new complex topologies can lead to novel physics. Therefore, in this Letter, we investigate the role played by clustering on the localization of waves in an experimentally realizable system of an optical network.

An optical communication network may be considered as a graph with edges representing optical fibers (or waveguides) and nodes representing optical units (essentially beam splitters) that redistribute incoming waves into outgoing fibers. Although constructing such a small network seems experimentally feasible, to the best of our knowledge it has not been performed. Theoretically, the propagation of electromagnetic or electronic waves in two and three dimensional disordered systems was studied with nodes on a lattice and bonds connecting nearest neighbors only [7]. However, if there are almost no losses along the edges, coherent effects are relevant for all edges including those connecting nodes spatially far from each other. Thus transitions in the transport properties of coherent waves on complex networks with long-range links are relevant to typical real-world communication networks [8,9] and can be studied experimentally. Alternatively, one might consider a network of waveguides on the nanoscale similar to photonic lattices [3]. Specifically, we suggest that Anderson localization should be observed upon changing the network topology [see Figs. 1(a) and 1(b)] instead of tweaking the disorder.

Compared with standard lattices, complex scale-free networks have additional degrees of freedom which define the topology of the network [10]. Focusing on the exponent λ of the power-law degree distribution $P(k) \sim k^{-\lambda}$ in scale-free networks and the clustering coefficient C (see below for exact definitions) [11–14], we find that (i) a localization-delocalization transition is induced by in-

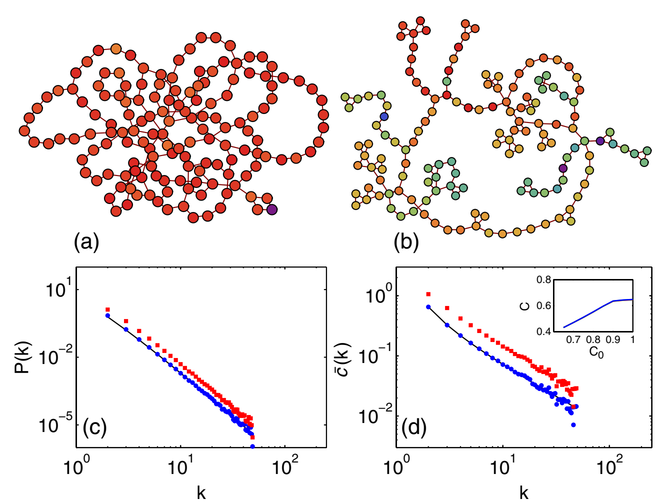

FIG. 1 (color online). Representative pictures of the giant component of scale-free networks ($\lambda = 5$) (a) without and (b) with clustering ($C_0 = 0.6$). Both networks have giant components of similar size ($N \sim 150$); the size of the whole network being $N = 150$ for (a) and $N = 200$ for (b). The logarithmically scaled coloring presents the intensity of a mode with $E \approx 0.2$, dark gray (red) indicating the highest, lighter colors for lower, and black (violet) for the lowest probability. (c) Degree distribution $P(k)$ and (d) clustering coefficients $\bar{C}(k)$ for scale-free networks with $\lambda = 4$ [line in (c)], $C_0 = 0.65$ [line in (d) according to Eq. (1)] and $N = 15\,000$ nodes, averaged over 120 configurations. Blue circles for distributions regarding the whole network and red squares for the giant component (with $\langle N_1 \rangle = 11\,906$ nodes; shifted vertically by a factor of 2). The inset shows that a linear dependence between C_0 and C holds also for the giant component if $C_0 \leq 0.9$.

creasing C even in the absence of on-node (on-site) disorder W for $\lambda > 4$, (ii) the quantum transition point C_q moves to lower values when W is increased (continuous phase diagram), and (iii) the scaling exponent ν is very close to the mean-field value $\nu = 0.5$ for all values of λ and C_q, as may be expected for a system with high spatial dimension [15,16]. We have also verified that similar results hold for networks with homogeneous or Erdös-Renyi-type degree distribution $P(k)$. For $P(k) \sim k^{-\lambda}$ with $\lambda > 4.5$ (approximately) there is an additional distinct classical transition at a clustering coefficient $C_c > C_q$.

Theoretically, one may attempt to study quantum phase transitions using a scattering formulation of the wave propagation [17], or, as we have chosen here, by studying the spectrum of an Anderson model [1] representing the complex network. Usually, diagonal (on-site) or non-diagonal (bond) disorder is introduced to obtain a localization transition [18]. An alternative approach is percolation, i.e., removing some fraction of all sites or bonds. In this case a classical transition [19] in which the infinite cluster breaks into finite pieces is found after the quantum phase-transition [20]. Anderson and quantum percolation transitions, which seems to be in the same universality class, have been studied on different topologies including fractal structures [16], Cayley trees [21], and complex networks [18,22]. In all cases, the transitions were induced either by on-site disorder or by cutting bonds (percolation) and thus changing the degree distribution of the network [22]. Here we show that it is possible to observe a quantum phase transition by changing the clustering of the network without introducing on-site disorder or changing the degree distribution, thus keeping the total number of links constant. We find that clustering drives a localization transition in a way similar to disorder. Both clustering and strong backscattering due to disorder increase the probability of closed loops and thus the probability of interference.

Many random network models have been proposed to reproduce important aspects of real-world networks topologies [10]. The properties of such networks are usually characterized by four quantities: the degree distribution $P(k)$ (distribution of the number of neighbors k per node), the characteristic path length ℓ between two arbitrary nodes (small-world property), the clustering coefficient C (probability of triadic closures), and the assortativity (degree-degree correlations) [23].

Real-world networks exhibit a high clustering coefficient C indicating the presence of many loops on short length scales [10]. This global measure can be achieved by averaging over $C_i = 2T_i/[k_i(k_i - 1)]$ [11], where T_i is the number of triangles passing through vertex i and k_i is its degree. However, since a global C cannot capture specific aspects of the network (e.g., varying degree-degree correlations can lead to networks with different topology but similar C [12]), it was suggested to average C_i within each degree class [9], yielding $\bar{C}(k)$.

To generate scale-free networks with tunable $P(k) \sim k^{-\lambda}$ (see [22] for details) and $\bar{C}(k)$, we have applied the algorithm suggested recently by Serrano and Boguñá [13]. Here we have chosen

$$\bar{C}(k) = C_0(k - 1)^{-1}, \tag{1}$$

with C_0 between 0 (no clustering) and 1 (maximum clustering), which can be obtained without degree-degree correlations [12]. In the following, we will use the parameter C_0 instead of C or $\bar{C}(k)$, since a linear relation holds for $C_0 \leq 0.9$ [see inset in Fig. 1(d); larger C_0 should be treated with care]. We obtained similar, however less reliable, results when generating networks with the algorithm of Volz [14] fixing C instead of $\bar{C}(k)$.

Figures 1(a) and 1(b) show two representative pictures of scale-free networks without and with clustering. Figures 1(c) and 1(d) compare the theoretical $P(k)$ and $\bar{C}(k)$ with the quantities we obtained numerically, considering the whole network or just the giant component. One can see good agreement in both cases. We want to stress that we do not change $P(k)$, the total number of links, and the number of nodes of the whole network, but only its structure by introducing clustering. Basically we rewire the network to achieve a higher clustering. The network can break for high clustering because nodes with low degree aggregate in finite clusters. We checked that the corresponding critical classical coefficient C_c is clearly larger than the critical quantum coefficient C_q if such a classical transition takes place.

Since each triangle represents a very short loop in the network, waves in networks with high clustering will have a high probability to return to the same node and to interfere. Since such interferences are the main reason for quantum localization, one may expect that strong clustering will induce localization. To study wave localization we consider the Anderson Hamiltonian [1],

$$H = \sum_i \epsilon_i a_i^\dagger a_i - \sum_{(i,j)} t_{j,i} a_j^\dagger a_i, \tag{2}$$

where the first part represents the disordered on-site (node) potential (homogeneous distribution $-W/2 < \epsilon_i < W/2$) and the second part describes the transfer between each pair of nodes (i, j). For optical waves, one has $t_{i,j} = \exp(i\varphi_{i,j})$ for connected nodes ($\varphi_{i,j}$ is the optical phase accumulated along the bond) and $t_{i,j} = 0$ for disconnected nodes. For simplicity, we restrict $t_{i,j}$ to random values ± 1; the Hamiltonian thus remains in the orthogonal symmetry class. The extension to unitary symmetry is straightforward. In this scenario, the on-site disorder W results from variations in the optical units (beam splitters) located at the nodes.

By exact diagonalization, we have calculated the eigenvalues of the Hamiltonian (2) on the largest cluster for scale-free networks with various λ and C_0. Figures 1(a) and 1(b) show the intensities corresponding to two eigenmodes. Then we applied level statistics [24] to determine the localization behavior of the modes and to extract the quantum phase-transition points. In disordered systems with extended eigenfunctions the energy spacing distribution $P(s)$ of consecutive eigenvalues (levels) E_i corresponds to the random-matrix theory result, well approximated by the Wigner surmise, $P_{\mathrm{W}}(s) = (\pi/2)s \times \exp(-\pi s^2/4)$. For localized states the level spacings are described by the Poisson distribution, $P_{\mathrm{P}}(s) = \exp(-s)$. For finite systems $P(s)$ is in between $P_{\mathrm{W}}(s)$ and $P_{\mathrm{P}}(s)$. However, it approaches one of them with increasing system size, remaining system-size independent only at the transition point. To determine this point for model parameters λ (exponent of degree distribution), C_0 [Eq. (1)] and W [below Eq. (2)], we study the system-size (N) dependence of

$$\gamma = \frac{\int_2^\infty P(s)ds - \int_2^\infty P_{\mathrm{W}}(s)ds}{\int_2^\infty P_{\mathrm{P}}(s)ds - \int_2^\infty P_{\mathrm{W}}(s)ds}, \qquad (3)$$

where $\gamma \to 0$ with $N \to \infty$ for extended states and $\gamma \to 1$ for localized states [25]. From finite-size scaling arguments [25] we expect that γ around $C_{0,q}$ will not only depend on C_0 but also on the diameter of the network L,

$$\gamma(C_0, W, L) = \gamma(C_{0,q}, W_c, L) + [R_1|C_0 - C_{0,q}| + R_2|W - W_c|]L^{1/\nu}, \qquad (4)$$

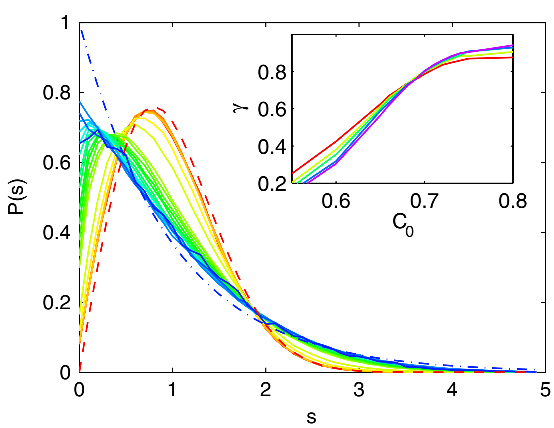

FIG. 2 (color online). Level spacing distribution $P(s)$ for optical modes on scale-free networks with $\lambda = 5$, $N = 12\,500$ and no disorder, $W = 0$. A clear transition from Wigner (dashed red curve) to Poisson (dash-dotted blue curve) behavior is observed as a function of the clustering coefficient prefactor that is increased from $C_0 = 0.0$ (continuous red curve, next to the dashed curve) to $C_0 = 0.90$ (continuous blue curve, next to the dash-dotted curve). Inset: Localization parameter γ [see Eq. (3)] versus C_0 for networks with (from top to bottom on the left) $N = 5000$ (red), $N = 7500$ (light green), $N = 10\,000$ (green), $N = 12\,500$ (blue), and $N = 15\,000$ (purple). A transition from extended modes for small C_0 to localized modes for large C_0 is observed at $C_{0,q} \approx 0.69$. The results are based on eigenvalues around $|E| = 0.2$ and 0.5.

where R_1 and R_2 are constants and $L \propto \ln(a(C_0)N)$ [26]. This relation enables us to obtain the critical clustering coefficient $C_{0,q}$, the critical disorder W_c, and the critical exponent ν. Using Eq. (4) we have determined $C_{0,q}$ and W_c for scale-free networks with various λ. We also checked that equivalent results are obtained if other integral measures of $P(s)$ are studied, e.g., $I_0 = \frac{1}{2}\int_0^\infty s^2 P(s)ds$.

Considering large scale-free networks without disorder ($W = 0$) but varied C_0, Fig. 2 shows $P(s)$ versus s as well as the two limiting cases $P_{\mathrm{W}}(s)$ and $P_{\mathrm{P}}(s)$. One can clearly see that the shape of $P(s)$ changes from Wigner to Poisson with increasing C_0. We thus observe an Anderson-like transition although there are no disorder W and no changes in the degree distribution $P(k)$. The inset of Fig. 2 shows γ for five system sizes versus the clustering strength C_0. One can observe the quantum phase transition at the critical value $C_{0,q} \approx 0.69$ by the crossing of the five curves, indicating a system-size independent critical value of $\gamma_c \approx 0.76$.

Figure 3(a) shows the phase diagram for the transitions from localized (upper right) to extended (lower left) optical modes. The horizontal axis ($C_0 = 0$) corresponds to the case with no clustering studied before by Sade *et al.* [18], where the critical disorder W_c depends on λ. The main new finding of the present study regards the transitions on the vertical axis. Without disorder, the transition to the localized phase occurs at a critical clustering $C_{0,q}$ that depend on λ, i.e., the degree distribution. While even the strongest clustering $C_0 = 1$ cannot achieve such a transition if

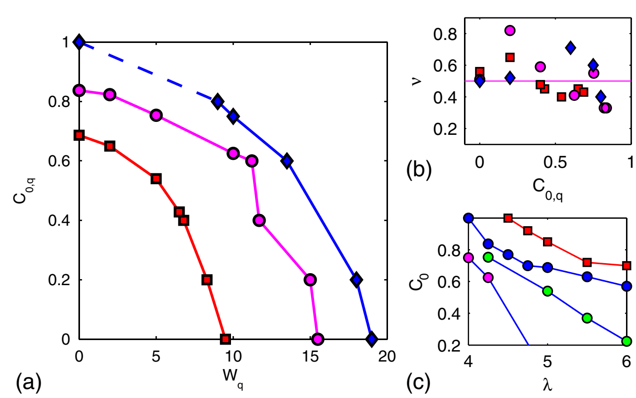

FIG. 3 (color online). (a) Phase diagram for transitions from localized optical modes (upper right) to extended modes in parts of the spectrum (lower left) for different degree distribution exponents λ, $\lambda = 4$ (blue diamonds), 4.25 (magenta circles), and 5 (red squares). Data for $C_0 > 0.9$ are not reliable for network generation reasons, and the error bar for the point at $C_0 = 1$ is about 0.1. (b) Exponent ν for different λ and $C_{0,q}$. The values are, within the error bars (not shown), consistent with the mean-field prediction $\nu = 0.5$. (c) Quantum transitions without disorder (blue circles) and classical transitions (red squares) as a function of the degree exponent λ. In the regime $4 < \lambda < 4.5$ only quantum transitions occur. For $W > 0$ the curves move downwards making quantum transitions possible for $\lambda < 4$ (green circles for $W = 5$ and magenta circles for $W = 10$).

$\lambda < 4$, values of $C_{0,q} < 1$ are observed for $\lambda > 4$. The case $\lambda = 4$ seems to be limiting: this is the broadest degree distribution which allows a quantum phase transition upon increasing clustering.

If variations of C_0 and W are considered, the full phase diagram can be explored. Evidently, smaller values of C_0 are sufficient for quantum phase transitions if $W > 0$. We obtained similar phase diagrams for networks with homogeneous or Erdös-Renyi-type degree distributions (not shown). Within our error bars the critical exponent ν corresponds to the mean-field value $\nu = 0.5$ for infinite dimensions [see Fig. 3(b)] as expected from the Anderson transition [15].

To make sure that the quantum transition is induced by clustering and not by a classical phase transition we determine the corresponding classical critical clustering coefficient $C_{0,c}$. We find no indications of a classical transition for $\lambda < 4.5$; i.e., the giant component is not broken. For $\lambda = 5$ we find $C_{0,c} \approx 0.85$, significantly larger than $C_{0,q} \approx 0.69$ [see insets of Fig. 2 and Fig. 3(c)]. We thus conclude that the quantum transition for $W = 0$ is clearly different from the classical one in two ways: (i) there is no classical transition between $4 < \lambda < 4.5$ although a quantum transition is clearly seen, and (ii) for $\lambda > 4.5$, the quantum transition occurs for lower C_0 values than the classical one. This leaves an intermediate regime ($C_{0,q} < C_0 < C_{0,c} \leq 1$) in which all modes are localized although there is a spanning giant cluster.

In summary, we have shown that quantum phase transitions of wavelike modes (similar to the Anderson transition and to the quantum percolation transition) can be obtained in a complex network without introducing on-site disorder or bond disorder or tampering with the degree distribution (i.e., the number and distribution of links). One only needs to change the clustering coefficient of the network, which corresponds to a rewiring procedure.

We conclude that clustering represents a new degree of freedom that can be used to induce and study phase transitions in complex networks. Comparing systems with different clustering properties might enable one to find the most relevant cause of quantum localization. We propose that the phenomenon should be observable experimentally and relevant in complex coherent optical networks made of fibers and beam splitters. Such experiments will directly probe the influence of complex network topology on the Anderson localization of light [2,3].

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