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The effect of spatiality on multiplex networks

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Abstract – Many multiplex networks are embedded in space, with links more likely to exist between nearby nodes than distant nodes. For example, interdependent infrastructure networks can be represented as multiplex networks, where each layer has links among nearby nodes. Here, we model the effect of spatiality on the robustness of a multiplex network embedded in 2-dimensional space, where links in each layer are of variable but constrained length. Based on empirical measurements of real-world networks, we adopt exponentially distributed link lengths with characteristic length ζ. By changing ζ, we modulate the strength of the spatial embedding. When $\zeta \to \infty$, all link lengths are equally likely, and the spatiality does not affect the topology. However, when $\zeta \to 0$ only short links are allowed, and the topology is overwhelmingly determined by the spatial embedding. We find that, though longer links strengthen a single-layer network, they make a multi-layer network more vulnerable. We further find that when ζ is longer than a certain critical value, ζ_c, abrupt, discontinuous transitions take place, while for $\zeta < \zeta_c$ the transition is continuous, indicating that the risk of abrupt collapse can be eliminated if the typical link length is shorter than ζ_c.

Introduction. – Interdependent and multiplex networks have been studied mainly on random topologies where analytic calculations are possible [1–13]. However, since many real-world complex systems —such as power grids and transportation systems— are embedded in space, it is important to understand how the underlying space and the strength of the embedding impact the interdependent networks [14–18]. This is particularly important when dealing with critical infrastructure which is heavily influenced by spatial constraints [19–21].

For single networks, several models have been proposed to describe spatial effects [16,22–30]. In lattice-based models, links are only formed to nearest or next nearest neighbors, which are regularly spaced. In random geometric models, links are formed to all neighbors within some distance [31,32]. In models of power grid topology, links are formed with the m nearest neighbors, statically [33] or as a generative model [34]. Some models utilize a cost function [14,35–38] or a characteristic distance distribution [39–42] to determine link lengths.

Dependency and spatial embedding are basic physical properties which appear together in a wide range of systems such as social networks [43–46], financial networks [47–49], brain networks [50] and many other systems [16]. Previous research on the robustness of spatially embedded interdependent networks considered coupled lattices with dependency links of geographic length up to r, a system parameter [51]. The model was also studied under partial dependency [52–54], for general networks formed of interdependent lattices [55] for interdependent resistor networks with process-based dependency [56] and in the presence of healing [57]. However, with this model it was not possible to modulate spatial effects and study the influence of the strength of the embedding directly. In addition to providing a way to measure the effects of spatiality on multiplex networks, our model presents an alternative and more realistic way to study interdependent networks.

Our chief focus is on the the effect of the strength of the embedding (as reflected in ζ) on the robustness of the multiplex. We find that, though increasing ζ *decreases* p_c (the percolation threshold) in single networks —making them more robust— it has the opposite effect on multiplex networks. Increasing ζ *increases* p_c for multiplex networks until a critical length ζ_c where p_c is maximal. At ζ_c, the percolation transition changes from a continuous transition to an abrupt transition and the multiplex network becomes susceptible to cascading failures which

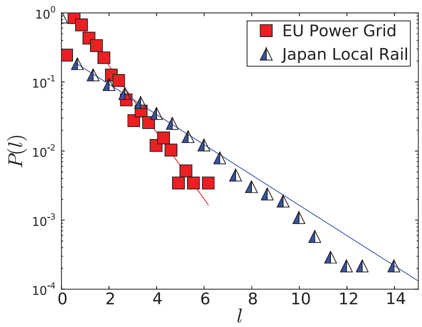


Fig. 1: (Color online) Examples of real-world networks with links of characteristic length. We examine the distribution of the geographic lengths of the edges in both the European power grid [58] (1851 edges) and the inter-station local railway lines in Japan [59] (20745 edges). These networks have links of characteristic length. Longer links are exponentially unlikely, as indicated by the linear drop on the semi-logarithmic plot. For visibility, we measure the lengths in units of effective minimum length, which we take as the peak of the distribution (mode length). The normalization value ($l = 1$) corresponds to 3.7 km (power) and 1.0 km (rail). The characteristic length is 4.8 km (power) and 1.2 km (rail) if measured as the mean or 3.3 km (power) and 2.0 km (rail) if measured as the inverse slope of the fit. The Japan local railway data is formed from the complete railway network from [59] with bullet train lines and internal station tracks removed.

spread via a nucleation process. Applied to interdependent infrastructure, this surprisingly implies that longer links make the system more vulnerable even though for single-layer networks they improve robustness.

Model. – In our model, the link lengths are exponentially distributed (see figs. 1 and 2(a)), similar to the Waxman model for a single network [39,60] or the spatial multiplex of Halu *et al.* [18]. With link lengths l distributed with probability $P(l) \sim \exp(-l/\zeta)$, we modulate the strength of the spatial effects in the multiplex by changing the characteristic link length (ζ). We determine the "strength" of the spatial embedding in terms of the deviation in network topology from a random network [61]. When $\zeta \to \infty$, all link lengths are equally likely and the spatiality does not affect the topology. However, when ζ is smaller, the overall topology exhibits strong deviations from randomness [61]. Thus high values of ζ reflect weak spatiality and low values reflect strong spatiality. For intermediate values, we have intermediate spatial embedding, as observed in real-world networks [15,16,39], see fig. 1.

In this manuscript, we focus on the case in which the multiplex consists of two layers, each with the same number of nodes, characteristic link length ζ and average degree $\langle k \rangle$ (see fig. 2(a)) and construct the network as follows. We begin by assigning $N = L^2$ nodes integer (x, y) coordinates with $x, y \in [0, 1, \ldots, L)$. To construct the links in each layer, we select a source node at random with coordinates (x_0, y_0) and draw a length l with probability $P(l) \sim e^{-l/\zeta}$. We choose the permitted link length $(\Delta x, \Delta y)$ which is closest to fulfilling $l = \sqrt{\Delta x^2 + \Delta y^2}$, select one of the eight length-preserving permutations $(\Delta x \leftrightarrow -\Delta x, \Delta y \leftrightarrow -\Delta y, \Delta x \leftrightarrow \Delta y)$ uniformly at random and make a link to node (x_1, y_1) with $x_1 = x_0 + \Delta x$, $y_1 = y_0 + \Delta y$ (using periodic boundary conditions). Due to the periodic boundary conditions, link lengths longer than $L/2$ are not physical and are re-drawn if produced by the distribution. This process is executed independently in each layer and is continued until the desired number of links $(N\langle k \rangle/2)$ is obtained. However, because they are constructed independently, the links in each layer are different (as demonstrated in fig. 2(a)) and this disorder enables the critical behavior which we describe

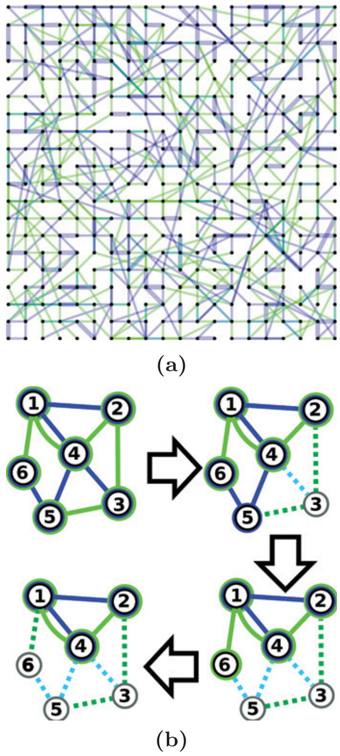


Fig. 2: (Color online) Spatially embedded multiplex networks. (a) The nodes occupy regular locations in two-dimensional space while the links in each layer (blue and green) have lengths that are exponentially distributed with characteristic length $\zeta = 3$ and are connected at random. (b) Cascading failures in multiplex networks. In the first stage, all of the nodes are in the giant component of both layers (blue links and green links) and the mutual giant connected component (MGCC) consists of the entire network. An initial attack on Node 3 causes it and its links to fail. This detaches Node 5 from the giant component of the green links, and in the next step it and its links fail. After the failure of Node 5, Node 6 is no longer in the giant component of the blue links. After Node 6's failure we find that the remaining nodes are in the giant component of both layers. We note that the MGCC is not simply the intersection of the giant components in the separate layers. If that were so, Node 6 would remain in the MGCC after the failure of Node 3, which is not the case.

below. We focus on values of $1 < \zeta < L/2$ because when ζ is very small or large the integer coordinates and finite system size make the exponential length distribution unsatisfiable.

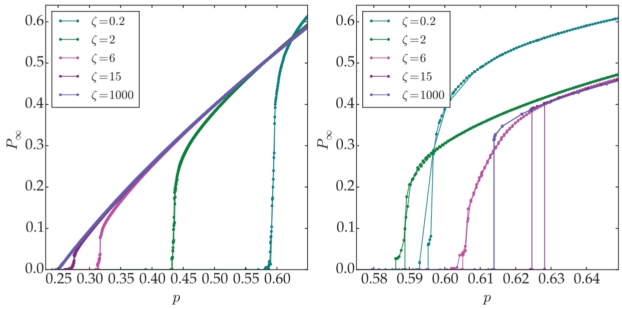
We perform site percolation by randomly removing a fraction $1 - p$ of the nodes from the system and finding the mutual giant connected component (MGCC) [1]. The MGCC is determined as the largest set of nodes for which every node has a path to every other node in each layer and each layer's path can use links from its layer only. For example, we can interpret each node as a geographic entity which is linked via two types of links to other nodes. Each node requires both of its constituents to function and each constituent requires connectivity within its layer. The nodes could represent neighborhoods or cities which require electricity and communications links, or local businesses which require links to customers and suppliers.

When a node is removed, its links in both layers are removed with it, causing further damage in every layer. However, since the links are not the same in each layer, after a node is removed, there will be some nodes that are connected to the giant component in one layer but are disconnected in the other layer. Since the node functionality (membership in the MGCC) requires connectivity in *both* layers, such nodes will fail, causing further damage in the system. This leads to the cascading failures as demonstrated in figs. 2(b) and fig. 6.

Results. – Since we are not aware of a discussion of the percolation properties of this topology for single networks, we briefly describe those properties here. In the limit of $\zeta \to 0$, the only permitted links will be to nearest neighbors (because links of length < 1 are not accessible) and a square lattice is recovered. As such, in the case of $\langle k \rangle = 4$, we recover the standard 2-dimensional percolation behavior with $p_c \approx 0.5927$ [62,63] (fig. 3(a)). As ζ increases, the robustness increases and in the limit $\zeta \to \infty$, all lengths are equally likely to be drawn and we approximately recover Erdős-Rényi topology with $p_c = 1/\langle k \rangle = 0.25$ (see figs. 3(a) and 4(b)). Thus, we have a single parameter ζ which allows us to smoothly transition from lattice to random topology.

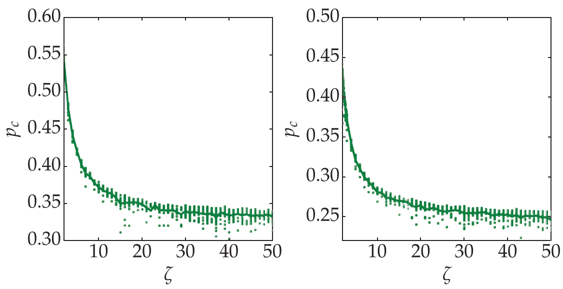
In our model the average link length remains finite for all finite values of ζ. In contrast to small-world networks [25], where even low rewiring probability brings the system close to the infinite-dimensional limit [64–67], in our model the effective dimensionality of the system remains (for finite ζ and in the limit of $L \to \infty$) equal to 2, as expected from universality principles. We also note that for all values of ζ, a single network undergoes a second-order transition (fig. 3(a) and figs. 5(c) and (d)).

In multiplex networks, where connectivity to the giant component in both layers is required, cascading failures emerge [1,3,4,11,68]. For $\zeta \ll 1$, large cascading failures do not emerge. This is because the multi-layer structure is the same in both networks: any node that is connected in one layer is likely to be connected in the other because the links



(a) single-layer (b) multiplex

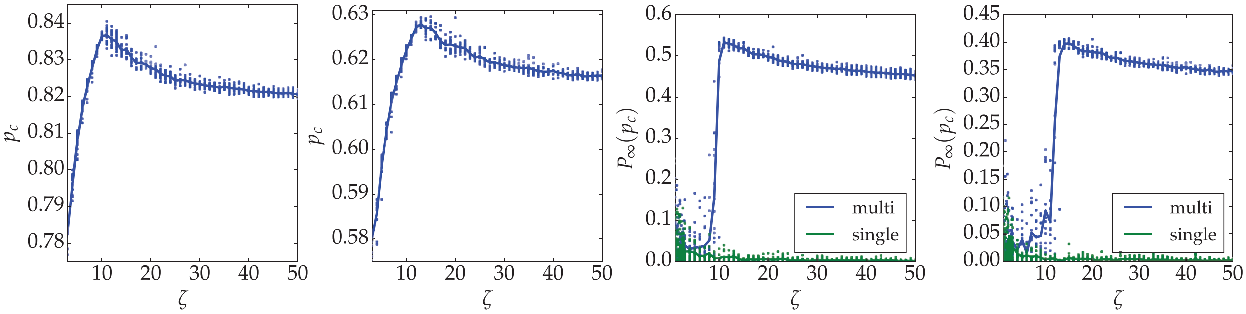
Fig. 3: (Color online) Percolation of (a) single and (b) multiplex networks with links of characteristic geographic length ζ. The fraction of nodes in the largest connected component (P_∞) as a function of p, the fraction of nodes remaining after a random removal. (a) In single networks, the transition is always continuous. The value of p_c decreases quickly and monotonically with increasing ζ (cf. fig. 4). (b) In multiplex networks, the transition is comparable to single networks for $\zeta = 0.2$ ($p_c \approx 0.5927$) but p_c *increases* as ζ increases. The maximal value is reached at ζ_c and the transition becomes discontinuous. The case of $\zeta = 1000$ is essentially random, with $p_c \approx 2.4554/\langle k \rangle = 0.61385$ [1]. Lines represent individual realizations with $\langle k \rangle = 4$ and $L = 4000$.



(a) $\langle k \rangle = 3$ (b) $\langle k \rangle = 4$

Fig. 4: (Color online) Dependence of p_c on ζ for single networks. The percolation threshold p_c drops quickly as a function of ζ and by $\zeta \approx 10$ it is already very close to $p_c = 1/\langle k \rangle$, the value from Erdős-Rényi networks.

are mostly the same. Conversely, any node that would be disconnected in one network is highly likely to be disconnected in the other network anyway, regardless of the requirement of connectivity in both layers. The absence of cascading failures in cases of high intersimilarity/overlap has been documented extensively elsewhere [5,18,69–72]. As ζ increases, correlation between the links in each layer decreases and cascading failures become possible. However, as long as ζ is short, the cascades remain confined to the vicinity of the random node removals and do not trigger global collective failures. Because the cascade damage is limited to the vicinity of the removal, the size of the MGCC decreases steadily as the removal fraction increases, and the system undergoes a second-order percolation transition. Once ζ reaches a critical length ζ_c, a damage front emerges which propagates through the



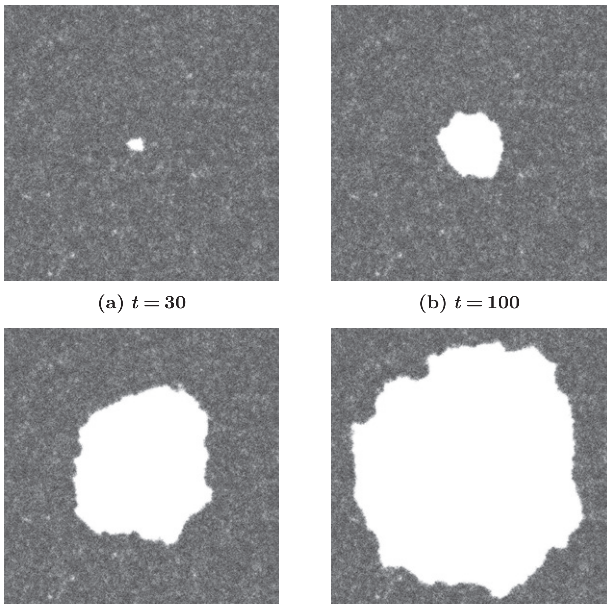
(a) $\langle k \rangle = 3$ (b) $\langle k \rangle = 4$ (c) $\langle k \rangle = 3$ (d) $\langle k \rangle = 4$

Fig. 5: (Color online) The effect of the characteristic link length ζ on the percolation threshold and size of the giant component at criticality. (a), (b): the percolation threshold in multiplex networks increases until it reaches a peak at ζ_c and then decreases slightly to its asymptotic value of $2.4554/\langle k \rangle$. (c), (d): the size of the order parameter P_∞ at p_c. The low values of P_∞ at p_c for single-layer networks, indicate that the transition is continuous. For continuous transitions, percolation is determined when a cluster spans from top to bottom of the underlying lattice. Multiplex networks have a second-order transition for low values of ζ but at ζ_c this jumps to a large fraction of the system size, indicating a discontinuity in P_∞ and an abrupt transition. $L = 4000$ with at least 10 realizations of each system. The averages and raw data are plotted.

whole system, leading to abrupt (first-order) transitions as shown in figs. 5 and 6. This cascading failure is different from the cascading failures observed in random networks because the global transition is driven by a propagating damage front (fig. 6) beginning in a single location, and not the hybrid transition caused by a global branching process of failures in interdependent random networks [4,68]. Hence we see no scaling in P_∞ near criticality and we would characterize the transition as first-order, similar to well-known nucleation transitions like the freezing of water. As ζ becomes even longer, p_c decreases and slowly approaches its asymptotic value of $2.4554/\langle k \rangle$ as known from interdependent Erdős-Rényi networks [1], (Figs. 5(a) and (b)). The abrupt transition in this limit is the hybrid transition of random interdependent networks [68]. Surprisingly, we find that even though the dimensionality of each network layer does does not change as ζ is varied, the strength of the spatial embedding as captured in the characteristic link lengths has a profound impact on the robustness of the multiplex.

Cascading failures which spread through the system have been observed in interdependent lattices with dependency links of large finite length [51,53,54,73]. In interdependent lattices, a dependency link from one network to another induces damage in the other network, by definition. As the length of dependency links in the system increases, that damage will be carried farther and a critical mass of damage can emerge via a nucleation process, and then spread, causing abrupt system collapse. Connectivity links, on the other hand, do not induce damage, but rather provide functionality to the nodes.

We find that, despite the fact that the connectivity links supply support, they can also induce failure propagation. Connectivity link lengths which are above a critical length, ζ_c, but much smaller than the total system size are sufficient to cause cascading which spread through the system. The first order transition that is observed in this case lacks scaling behavior just above p_c (fig. 3(b)) and is characterized by a slow spreading process (fig. 6).



(a) $t = 30$ (b) $t = 100$ (c) $t = 200$ (d) $t = 300$

Fig. 6: Dynamic evolution of cascading failures. Here we show the nodes of the multiplex network, colored black if functional and white if not. At criticality, a large whole emerges due to fluctuations. The nodes near the edge of the hole lose many links, increasing their likelihood of being disconnected in the next step. This causes the propagation of this spinodal interface until the system disintegrates. Nucleation-driven phase transitions, a hallmark of first-order transitions like freezing water, have been observed in interdependent lattices with dependency lengths of finite length [51,54]. ($L = 4000$, $\zeta = 15$, $\langle k \rangle = 4$.)

Surprisingly, intermediate spatial embedding ($\zeta \approx \zeta_c$) makes the system more susceptible to cascading failures.

This is due to the fact that, because the damage from an emergent hole is relayed by the cascade dynamics to the neighborhood of its interface, the nodes near the edge of the hole are far more likely to become disconnected. Similar results have been found in social networks, where it was shown that high modularity makes viral cascades more likely to occur due to the increased likelihood of multiple exposure to the information [74–76].

In single-layer networks, p_c *decreases* monotonically as ζ increases from $\zeta \approx 0.5$ to the limit of $\zeta = \infty$ (fig. 4). In contrast, in multiplex networks, p_c increases until ζ_c and then decreases monotonically thereafter (fig. 5). The peak in $p_c(\zeta)$ at ζ_c is due to the fact that the transition requires an initial hole which emerges from random fluctuations. The size of that hole is described by the percolation correlation length above criticality, $\xi(p)$. The hole size required to trigger the transition increases with ζ [51,73] while the size of emergent holes above the percolation threshold, $\xi(p)$, decreases with p [62]. This would indicate that the smaller ζ is, the smaller the critical hole needs to be and that p_c would increase monotonically as ζ decreases. However, when $\zeta < \zeta_c$ there is not enough space between the emergent hole and the extent of the damage propagation (ζ) for the network to disintegrate and the small emergent holes remain in place [51,53,54,73].

Measuring overlap. – When $\zeta \to 0$, the fixed spacing of the nodes forces all of the links to be to nearest neighbors only. In this case, both networks are identical and the MGCC is the same as the single network giant component. The fraction of common connectivity links between two interdependent networks or between two layers in a multiplex network is called intersimilarity [69,70] or overlap [5,18,71,72]. The cascading failures and abrupt transitions which characterize interdependent networks increase as overlap decreases.

In multiplex networks with links of characteristic length, the extent of the overlap can be estimating by considering the probability that, given the same source node, two links lead to the same target node. In the continuum limit this is proportionate to the probability that the links have the same length and the same direction. The system is isotropic by construction so the directional condition is simply $1/2\pi r$, the size of a ring of radius r. We obtain

$$P(\text{overlap}) \sim \int_1^\infty \frac{P^2(r)}{2\pi r}\mathrm{d}r \sim \frac{1}{\zeta^2}\int_1^\infty \frac{e^{-2r/\zeta}}{2\pi r}\mathrm{d}r \sim \frac{1}{\zeta^2}. \tag{1}$$

The exponent of -2 agrees with the analysis by Halu *et al.* [18]. Empirically, we find that the scaling in our system is scale-free, with an exponent of -2 with a logarithmic correction (fig. 7). The logarithmic correction from the continuum calculation (eq. (1)) is due to the fact that with the discretization of space that we introduce, links that would otherwise have been distinct are unified and the critical exponent is reduced from pure scaling of -2. This can be verified by calculating eq. (1) as a Riemann sum with r values spaced according to the lattice, which gives the same logarithmic correction.

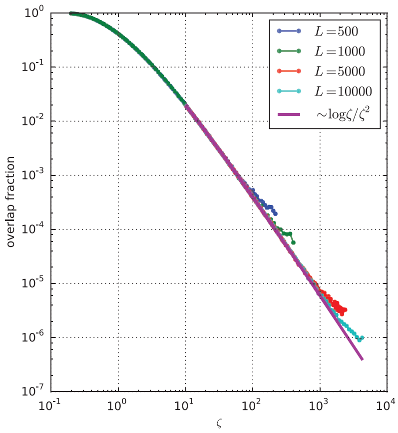


Fig. 7: (Color online) The fraction of overlapping nodes. The fraction of overlapping nodes is determined as the number of common links across both layers divided by the total number of links in each layer. When $\zeta \ll 1$, the overlap is maximal and the networks are identical (for $\langle k \rangle = 4$, as in this figure). As ζ increases, the fraction decreases with an exponent of -2, with a logarithmic correction.

Unlike studies of random multilayer networks with overlap [69–72,77] or other inter-network correlations [78–80], decreased overlap alone is not enough to enable the critical behavior observed here. It is only the combination of the disorder (as indicated by decreased overlap) with the spatially embedded links that enables the distinctive abrupt spreading transition which we observe here.

Conclusion. – The model that we present here allows for continuous variation of the "strength" of the spatial embedding (by changing the characteristic length) in single and multiplex networks, while not affecting the dimension of the topology in each layer, which remains equal to the embedded space. We find that multiplex networks with intermediate spatiality ($\zeta_c < \zeta \ll \infty$) are more vulnerable than both extreme spatial embedding ($\zeta \ll 1$) and no spatial embedding ($\zeta \to \infty$). Regarding interdependent infrastructure and other real-world systems, we conclude that shorter links (with $\zeta < \zeta_c$) can make the system more robust and eliminate the risk of abrupt transitions. This is in marked contrast to the conclusions based on considering a single layer only, where longer links *always* improve robustness.

Outlook. – This research also provides an important new direction for the study of interdependent spatial networks. Previous research on spatially embedded interdependent networks [51–55,73] have used two-dimensional lattices with dependency links connecting nodes from one network to the other. The dependency links were also affected by the spatial embedding via the restriction that they have length of up to r, a system parameter. This model led to many important results, but left several important issues unaddressed. First, the topology of real-world spatially embedded networks is not lattice-like or even strictly planar and it was not clear that results derived on lattices would accurately describe real-world topologies. Second, the assumption that dependency links are longer than connectivity links does not correspond with what we would expect from many real systems, including critical infrastructure: It is not reasonable to expect a communications station to get power from a distant power plant and not the one nearest to it. Here we address these problems by modeling spatially embedded interdependent networks as multiplex networks where the dependency relationship is to the nearest node in the other layer and the connectivity links are of finite characteristic length but not uniform or regular.

In this paper, we have studied our model in the special case of two networks embedded in two-dimensional space. We expect more general realizations of the model to be studied in the future. For example, multiplex bond percolation [81] and networks of spatial networks would be important future directions [55]. In the study of n interdependent lattices, it was shown that the discontinuous transition appears for lower values of r when there are more networks [55] and we would expect a similar decrease of ζ_c for n-layered systems. Similarly, because the distinctive nucleation at criticality (fig. 6) is the cause of the abrupt transition, we expect that similar features will appear when networks are embedded in higher-dimensional space (up to the critical dimension $d = 6$) [62].

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REFERENCES

[1] Buldyrev S. V., Parshani R., Paul G., Stanley H. E. and Havlin S., *Nature*, **464** (2010) 1025.
[2] Gao J., Buldyrev S. V., Stanley H. E. and Havlin S., *Nat. Phys.*, **8** (2012) 40.
[3] Peixoto T. P. and Bornholdt S., *Phys. Rev. Lett.*, **109** (2012) 118703.
[4] Baxter G. J., Dorogovtsev S. N., Goltsev A. V. and Mendes J. F. F., *Phys. Rev. Lett.*, **109** (2012) 248701.
[5] Bianconi G., *Phys. Rev. E*, **87** (2013) 062806.
[6] Nicosia V., Bianconi G., Latora V. and Barthélémy M., *Phys. Rev. Lett.*, **111** (2013) 058701.
[7] De Domenico M., Solé-Ribalta A., Cozzo E., Kivelä M., Moreno Y., Porter M. A., Gómez S. and Arenas A., *Phys. Rev. X*, **3** (2013) 041022.
[8] Radicchi F. and Arenas A., *Nat. Phys.*, **9** (2013) 717.
[9] Kivelä M., Arenas A., Barthélémy M., Gleeson J. P., Moreno Y. and Porter M. A., *J. Complex Netw.*, **2** (2014) 203.
[10] Boccaletti S., Bianconi G., Criado R., Del Genio C., Gómez-Gardeñes J., Romance M., Sendina-Nadal I., Wang Z. and Zanin M., *Phys. Rep.*, **544** (2014) 1.
[11] Danziger M. M., Bashan A., Berezin Y., Shekhtman L. M. and Havlin S., *An introduction to interdependent networks*, in *NDES*, edited by Mladenov V. and Ivanov P., *Communications in Computer and Information Science Series*, Vol. **438** (Springer) 2014, pp. 189–202.
[12] Lee K.-M., Min B. and Goh K.-I., *Eur. Phys. J. B*, **88** (2015) 48.
[13] Danziger M. M., Shekhtman L. M., Bashan A., Berezin Y. and Havlin S., *Vulnerability of Interdependent Networks and Networks of Networks*, in *Interconnected Networks*, edited by Garas A. (Springer) 2016, pp. 79–99.
[14] Gastner M. T. and Newman M. E., *Eur. Phys. J. B*, **49** (2006) 247.
[15] Li D., Kosmidis K., Bunde A. and Havlin S., *Nat. Phys.*, **7** (2011) 481.
[16] Barthélémy M., *Phys. Rep.*, **499** (2011) 1.
[17] Morris R. G. and Barthélémy M., *Phys. Rev. Lett.*, **109** (2012) 128703.
[18] Halu A., Mukherjee S. and Bianconi G., *Phys. Rev. E*, **89** (2014) 012806.
[19] Rinaldi S., Peerenboom J. and Kelly T., *IEEE Control Syst.*, **21** (2001) 11.
[20] Helbing D., *Nature*, **497** (2013) 51.
[21] Li D., *Complex Systems and Networks* (Springer) 2016, Chapt. *Resilience of Spatial Networks*, pp. 79–106.
[22] Doar M. and Leslie I., *How bad is naive multicast routing?*, in *Proceedings of INFOCOM'93* (IEEE) 1993, pp. 82–89.
[23] Wei L. and Estrin D., *The trade-offs of multicast trees and algorithms*, in *Proceedings of the International Conference on Computer Communications and Networks (ICCCN), September 1994*, pp. 17–24.
[24] Zegura E. W., Calvert K. L. and Donahoo M. J., *IEEE/ACM Trans. Netw.*, **5** (1997) 770.
[25] Watts D. J. and Strogatz S. H., *Nature*, **393** (1998) 440.
[26] Penrose M., *Random Geometric Graphs*, Vol. **5** (Oxford University Press, Oxford) 2003.
[27] Kleinberg J., *The Small-World Phenomenon: An Algorithmic Perspective*, in *Proceedings of Theory of Computing* (ACM) 2000, pp. 163–170.
[28] Kosmidis K., Havlin S. and Bunde A., *EPL*, **82** (2008) 48005.
[29] Li D., Li G., Kosmidis K., Stanley H. E., Bunde A. and Havlin S., *EPL*, **93** (2011) 68004.
[30] McAndrew T. C., Danforth C. M. and Bagrow J. P., *Phys. Rev. E*, **91** (2015) 042813.
[31] Rozenfeld A. F., Cohen R., ben Avraham D. and Havlin S., *Phys. Rev. Lett.*, **89** (2002) 218701.

[32] Bradonjić M., Hagberg A. and Percus A. G., *Giant Component and Connectivity in Geographical Threshold Graphs*, in *Algorithms and Models for the Web-Graph* (Springer) 2007, pp. 209–216.

[33] Hines P., Blumsack S., Cotilla Sanchez E. and Barrows C., *The Topological and Electrical Structure of Power Grids*, in *Proceedings of HICSS* (IEEE) 2010, pp. 1–10.

[34] Deka D. and Vishwanath S., *Generative growth model for power grids*, in *Proceedings of SITIS* (IEEE) 2013, pp. 591–598.

[35] Manna S. S. and Kabakioglu A., *J. Phys. A: Math. Gen.*, **36** (2003) L279.

[36] Popović M., Štefančić H. and Zlatić V., *Phys. Rev. Lett.*, **109** (2012) 208701.

[37] Emmerich T., Bunde A. and Havlin S., *Phys. Rev. E*, **89** (2014) 062806.

[38] Ren Y., Ercsey-Ravasz M., Wang P., González M. C. and Toroczkai Z., *Nat. Commun.*, **5** (2014) 5347.

[39] Waxman B. M., *IEEE J. Sel. Areas Commun.*, **6** (1988) 1617.

[40] Wang Z., Thomas R. J. and Scaglione A., *Generating Random Topology Power Grids* (Institute of Electrical and Electronics Engineers) 2008, pp. 183–183.

[41] Grassberger P., *J. Stat. Mech.: Theory Exp.* (2013) P04004.

[42] Gao J., Zhou T. and Hu Y., *Sci. Rep.*, **5** (2015) 14662.

[43] Backstrom L., Sun E. and Marlow C., *Find Me if You can*, in *Proceedings of WWW'10* (ACM, New York, NY, USA) 2010, pp. 61–70.

[44] Onnela J.-P., Arbesman S., Gonzlez M. C., Barabsi A.-L. and Christakis N. A., *PLoS ONE*, **6** (2011) e16939.

[45] Hu Y., Havlin S. and Makse H. A., *Phys. Rev. X*, **4** (2014) 021031.

[46] Levy M. and Goldenberg J., *Physica A*, **393** (2014) 418.

[47] Degryse H. and Ongena S., *J. Finance*, **60** (2005) 231.

[48] Bargigli L., di Iasio G., Infante L., Lillo F. and Pierobon F., *Quant. Finance*, **15** (2015) 673.

[49] Kenett D. Y. and Havlin S., *Mind Soc.*, **14** (2015) 155.

[50] Bullmore E. and Sporns O., *Nat. Rev. Neurosci.*, **13** (2012) 336.

[51] Li W., Bashan A., Buldyrev S. V., Stanley H. E. and Havlin S., *Phys. Rev. Lett.*, **108** (2012) 228702.

[52] Bashan A., Berezin Y., Buldyrev S. V. and Havlin S., *Nat. Phys.*, **9** (2013) 667.

[53] Danziger M. M., Bashan A., Berezin Y. and Havlin S., *Interdependent Spatially Embedded Networks: Dynamics at Percolation Threshold*, in *Proceedings of SITIS* (IEEE) 2013, pp. 619–625.

[54] Danziger M. M., Bashan A., Berezin Y. and Havlin S., *J. Complex Netw.*, **2** (2014) 460.

[55] Shekhtman L. M., Berezin Y., Danziger M. M. and Havlin S., *Phys. Rev. E*, **90** (2014) 012809.

[56] Danziger M. M., Bashan A. and Havlin S., *New J. Phys.*, **17** (2015) 043046.

[57] Stippinger M. and Kertész J., *Physica A*, **416** (2014) 481.

[58] Zhou Q. and Bialek J., *IEEE Trans. Power Syst.*, **20** (2005) 782.

[59] National Land Information Division, National Spatial Planning and Regional Policy Bureau, MILT of Japan, *National railway data*, `http://nlftp.mlit.go.jp/ksj/gml/datalist/KsjTmplt-N02.html` (2012).

[60] Bradde S., Caccioli F., Dall'Asta L. and Bianconi G., *Phys. Rev. Lett.*, **104** (2010) 218701.

[61] Jalan S. and Bandyopadhyay J. N., *EPL*, **87** (2009) 48010.

[62] Bunde A. and Havlin S., *Fractals and Disordered Systems* (Springer-Verlag, Inc. New York) 1991.

[63] Ziff R., *Phys. Rev. Lett.*, **69** (1992) 2670.

[64] Newman M. E. J. and Watts D. J., *Phys. Rev. E*, **60** (1999) 7332.

[65] Barthélémy M. and Amaral L. A. N., *Phys. Rev. Lett.*, **82** (1999) 3180.

[66] Amaral L. A. N., Scala A., Barthélémy M. and Stanley H. E., *Proc. Natl. Acad. Sci. U.S.A.*, **97** (2000) 11149.

[67] Newman M. E. J., Jensen I. and Ziff R. M., *Phys. Rev. E*, **65** (2002) 021904.

[68] Zhou D., Bashan A., Cohen R., Berezin Y., Shnerb N. and Havlin S., *Phys. Rev. E*, **90** (2014) 012803.

[69] Parshani R., Rozenblat C., Ietri D., Ducruet C. and Havlin S., *EPL*, **92** (2010) 68002.

[70] Hu Y., Zhou D., Zhang R., Han Z., Rozenblat C. and Havlin S., *Phys. Rev. E*, **88** (2013) 052805.

[71] Cellai D., López E., Zhou J., Gleeson J. P. and Bianconi G., *Phys. Rev. E*, **88** (2013) 052811.

[72] Li M., Liu R.-R., Jia C.-X. and Wang B.-H., *New J. Phys.*, **15** (2013) 093013.

[73] Berezin Y., Bashan A., Danziger M. M., Li D. and Havlin S., *Sci. Rep.*, **5** (2015).

[74] Centola D., *Science*, **329** (2010) 1194.

[75] Weng L., Menczer F. and Ahn Y.-Y., *Sci. Rep.*, **3** (2013) 2522.

[76] Nematzadeh A., Ferrara E., Flammini A. and Ahn Y.-Y., *Phys. Rev. Lett.*, **113** (2014) 088701.

[77] Valdez L. D., Macri P. A., Stanley H. E. and Braunstein L. A., *Phys. Rev. E*, **88** (2013) 050803.

[78] Watanabe S. and Kabashima Y., *Phys. Rev. E*, **89** (2014) 012808.

[79] Reis S. D. S., Hu Y., Babino A., Andrade J. S. jr., Canals S., Sigman M. and Makse H. A., *Nat. Phys.*, **10** (2014) 762.

[80] Radicchi F., *Phys. Rev. X*, **4** (2014) 021014.

[81] Hackett A., Cellai D., Gómez S., Arenas A. and Gleeson J. P., *Phys. Rev. X*, **6** (2016) 021002.