

## Scale-Free Networks Are Ultrasmall

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We study the diameter, or the mean distance between sites, in a scale-free network, having  $N$  sites and degree distribution  $p(k) \propto k^{-\lambda}$ , i.e., the probability of having  $k$  links outgoing from a site. In contrast to the diameter of regular random networks or small-world networks, which is known to be  $d \sim \ln N$ , we show, using analytical arguments, that scale-free networks with  $2 < \lambda < 3$  have a much smaller diameter, behaving as  $d \sim \ln \ln N$ . For  $\lambda = 3$ , our analysis yields  $d \sim \ln N / \ln \ln N$ , as obtained by Bollobas and Riordan, while for  $\lambda > 3$ ,  $d \sim \ln N$ . We also show that, for any  $\lambda > 2$ , one can construct a deterministic scale-free network with  $d \sim \ln \ln N$ , which is the lowest possible diameter.

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It is well known [1–4] that random networks, such as Erdős-Rényi networks [5,6] as well as partially random networks such as small-world networks [7], have a very small average distance (or diameter) between sites, which scales as  $d \sim \ln N$ , where  $N$  is the number of sites. Since the diameter is small even for large  $N$ , it is common to refer to such networks as “small-world” networks. Many natural and manmade networks have been shown to possess a scale-free degree distribution, including the Internet [8], World Wide Web [3,9], metabolic [10] and cellular networks [11], and trust cooperation networks [12].

The question of the diameter of such networks is fundamental in the study of networks. It is relevant in many fields regarding communication and computer networks, such as routing [13], searching [14], and transport of information [13]. All those processes become more efficient when the diameter is smaller. It also might be relevant to subjects such as the efficiency of chemical and biochemical processes and spreading of viruses, rumors, etc., in cellular, social, and computer networks. In physics, the scaling of the diameter with the network size is related to the physical concept of the dimensionality of the system and is highly relevant to phenomena such as diffusion, conduction, and transport. The anomalous scaling of the diameter in those networks is expected to lead to anomalies in diffusion and transport phenomena on those networks. In this Letter we study the diameter of scale-free random networks and show that it is significantly smaller than the diameter of regular random networks. We find that scale-free networks with  $2 < \lambda < 3$  have diameter  $d \sim \ln \ln N$  and thus can be considered as “ultra small-world” networks.

We define the diameter of a graph as the average distance between any two sites on the graph (unlike the usual mathematical definition of the largest distance between two sites). Since no embedding space is defined for those networks, the distance denotes the shortest path between two sites (i.e., the smallest number of followed links needed to reach one from the other). If the network is

fragmented we are interested only in the diameter of the largest cluster (assuming there is one).

To estimate the diameter we study the radius of such graphs. We define the radius of a graph as the average distance of all sites on the graph from the site with the highest degree in the network (if there is more than one, we arbitrarily choose one of them). The diameter of the graph,  $d$ , is restricted to

$$r \leq d \leq 2r, \quad (1)$$

where  $r$  is the radius of the graph, defined as the average distance  $\langle l \rangle$  between the highest degree site (the origin) and all other sites.

A scale-free graph is a graph having degree distribution, i.e., the probability that a site has  $k$  connections:

$$p(k) = ck^{-\lambda}, \quad k = m, m+1, \dots, K, \quad (2)$$

where  $c \approx (\lambda - 1)m^{\lambda-1}$  is a normalization factor, and  $m$  and  $K$  are the lower and upper cutoffs of the distribution, respectively. The ensemble of such graphs has been defined in [15]. However, we refer here to the ensemble of scale-free graphs with the “natural” cutoff  $K = mN^{1/(\lambda-1)}$  [16–18].

We begin by showing that the lower bound on the diameter of any scale-free graph with  $\lambda > 2$  is of the order of  $\ln \ln N$ ; then we show that for random scale-free graphs with  $2 < \lambda < 3$  the diameter actually scales as  $\ln \ln N$ . It is easy to see that the lowest diameter for a graph with a given degree distribution is achieved by the following construction: Start with the highest degree site, and then in each layer attach the next highest degree sites until the layer is full. By construction, loops occur only in the last layer. This structure is somewhat similar to a graph with assortative mixing [19]—since high degree sites tend to connect to other high degree sites.

In this kind of graph, the number of links outgoing from the  $l$ th layer (sites at distance  $l$  from the origin),  $\chi_l$ , equals the total number of sites with degree between  $K_l$ ,

which is the highest degree of a site not reached in the  $l$ th layer, and  $K_{l+1}$ , which is the same for the  $l + 1$  layer. See Fig. 1. This can be described by the following equation:

$$\chi_l = N \int_{K_{l+1}}^{K_l} p(k)dk \approx m^{\lambda-1} N K_{l+1}^{1-\lambda}. \quad (3)$$

The number of links outgoing from the  $l + 1$  layer equals the total number of links in all the sites between  $K_l$  and  $K_{l+1}$  minus one link at every site, which is used to connect to the previous layer:

$$\chi_{l+1} = N \int_{K_{l+1}}^{K_l} (k - 1)p(k)dk \approx \frac{\lambda - 1}{\lambda - 2} m^{\lambda-1} N K_{l+1}^{2-\lambda}. \quad (4)$$

Solving those recursion relations with the initial conditions  $K_0 = N^{1/(\lambda-1)}$  and  $\chi_0 = K_0$  leads to

$$\chi_l = a^{(\lambda-1)(1-u^l)} N^{1-u^{l+1}}, \quad (5)$$

where  $a = (\lambda - 1)/(\lambda - 2)m$ ,  $u = (\lambda - 2)/(\lambda - 1)$ , and

$$K_l = m(\chi_l/N)^{1/1-\lambda}. \quad (6)$$

To bound the radius,  $r$ , of the graph, we assume that the low degree sites are connected randomly to the center. We choose some degree  $1 \ll k^* \approx (\ln \ln N)^{1/(\lambda-1)}$ . We can use Eq. (6) to show that if  $l_1 \approx \ln \ln N / \ln(\lambda - 2)$ , then  $K_{l_1} < k^*$ , so all sites with degrees  $k \geq k^*$  would have been reached with probability 1 in the first  $l$  layers. On the other hand, if we start uncovering the graph from any site—

provided it belongs to the giant component—then at a distance  $l_2$  from this site there are at least  $l_2$  bonds. The probability that none of those bonds lead to a site of degree  $k^*$  decays as  $[1 - k^* p(k^*)/\langle k \rangle]^{l_2}$ . So, taking  $k^{*\lambda-1} \ll l_2 \ll \ln \ln N$ , we definitely reach a site of degree at least  $k^*$  at a distance  $l_2$  from almost every site. Since  $l = l_1 + l_2$ , all those sites are at a distance of order  $\ln \ln N$  from the highest degree site; this is the behavior of the radius of the graph. Thus,  $\ln \ln N$  is a lower bound for the diameter of scale-free networks, and by applying this approach one can generate scale-free networks with this diameter, for  $\lambda > 2$ . For  $\lambda = 2$  the construction is somewhat similar to the condensate obtained in [20].

In the following, we present analytical arguments showing that the behavior of  $d \sim \ln \ln N$  is actually achieved in random noncorrelated scale-free graphs with  $2 < \lambda < 3$ . Noncorrelated networks are networks in which the degree of a site reached by following a link is independent of the degree of the site at the other side of that link. One can view the process of uncovering the network (which is the same as building it) by following the links one at a time. For simplicity, let us start with the site with the highest degree (which is also guaranteed to belong to the giant component), whose degree is proportional to  $N^{1/(\lambda-1)}$  [16,17]. Next we expose the layers,  $l = 1, 2, 3, \dots$ , one at a time. To this end, we consider the graph as built from one large developing cluster and sites which have not yet been reached (they can also belong to the giant component or not). A similar consideration has been used by Molloy and Reed [21].

After layer  $l$  is explored the distribution of the unreached sites changes (since most high degree sites are reached in the first layers). To take account for this, we assume that the  $l$ th layer has  $\chi_l$  outgoing links. The distribution of degrees after uncovering some of the edges changes to  $P'(k) \approx P(k) \exp(-k/K_l)$  [21]. In the limit of large  $N$  and large  $K_l$  we assume that after exploring this layer the highest degree of the unvisited sites is of order  $K_l$ , where  $\chi_l$  and  $K_l$  are functions of  $l$  that will be determined later.

Let us now consider the  $l + 1$  layer. There is a new threshold function, that is, the new distribution of unvisited sites is like a step function—almost  $P(k)$  for  $k < K_{l+1}$  and almost 0 for  $k > K_{l+1}$ . The reason is as follows: A site with degree  $k$  has a probability of  $p = k/(N\langle k \rangle)$  to be reached by following a link [22]. If there are  $\chi_l$  outgoing links then if  $p\chi_l > 1$  we can assume that (in the limit  $N \rightarrow \infty$ ) the site will be reached in the next level with probability 1. Therefore, all unvisited sites with degree  $k > N\langle k \rangle/\chi_l$  will be surely reached in the next layer. On the other hand, almost all the unvisited sites with degree  $k < N\langle k \rangle/\chi_l$  will remain unvisited in the next layer—therefore, their distribution will remain almost unchanged. From those considerations, the highest degree of the unexplored sites in the  $l + 1$  layer is determined by

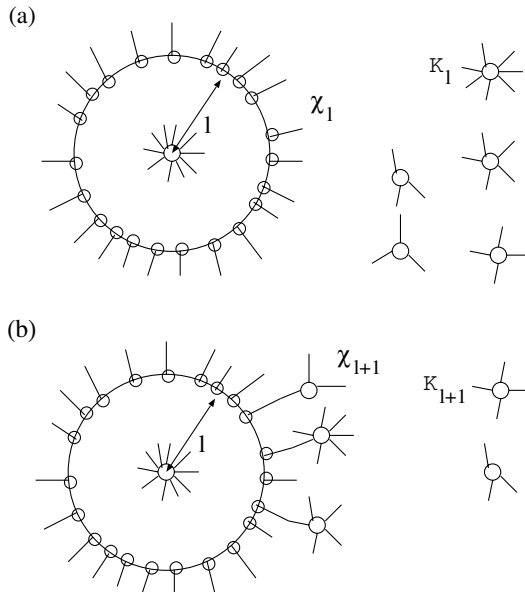


FIG. 1. Illustration of the exposure process. The large circle denotes the exposed fraction of the giant component, while the small circles denote individual sites. The sites on the right have not been reached yet. (a) The structure after the exposure of the  $l$ th layer. (b) The structure after the exposure of the  $l + 1$  layer.

$$K_{l+1} \approx N\langle k \rangle / \chi_l. \quad (7)$$

In the  $l$ th layer the number of loops, i.e., the number of links connecting two sites of the  $l$ th layer and the number of sites in the  $l + 1$  layer connected to more than one site in the  $l$ th layer, is proportional to  $\chi_l^2 / (\langle k \rangle N)$ . Since as long as  $\chi_l$  is not of order  $N$ , this fraction is smaller in order than  $\chi_l$ , we can safely assume that loops can be neglected until the last shells have been reached. Similar arguments have been used in [16].

In the  $l + 1$  layer, all sites with degree  $k > N\langle k \rangle / \chi_l$  will be exposed. Since the probability of reaching a site via a link is proportional to  $kP(k)$ , the average degree of sites reached by following a link is  $\kappa \equiv \langle k^2 \rangle / \langle k \rangle$  [16]. For scale-free graphs,  $\kappa$  can be approximated by [16]

$$\kappa = \left( \frac{\lambda - 2}{\lambda - 3} \right) \left( \frac{K^{3-\lambda} - m^{3-\lambda}}{K^{2-\lambda} - m^{2-\lambda}} \right). \quad (8)$$

This is the average degree for sites reached in this layer, whose degree is  $k < N\langle k \rangle / \chi_l$ . Therefore,  $\kappa$  should be calculated using the new cutoff (7), from (8) follows  $\kappa \sim K_{l+1}^{3-\lambda}$ .

Using the above considerations, the number of outgoing links from the  $l + 1$  layer can be calculated. To this end we consider the total degree of all sites reached in the  $l + 1$  level. This includes all sites with degree  $k$ ,  $K_{l+1} < k < K_l$ , as well as other sites with average degree proportional to  $\kappa - 1$  links (the  $-1$  is due to one link going inwards). Thus, the value of  $\kappa$  is calculated using the cutoff  $K_{l+1}$ . Loops within a layer and multiple links connecting the same site in the  $l + 1$  layer can be neglected since as long as the number of sites in the layer are of order less than  $N$ , they are negligible in the limit  $N \rightarrow \infty$ . The two contributions can be written as the sum of two terms:

$$\chi_{l+1} = N \int_{K_{l+1}}^{K_l} (k - 1) p(k) dk + \chi_l [\kappa(K_{l+1}) - 1]. \quad (9)$$

Noting that  $p(k) \propto k^{-\lambda}$  and that  $\kappa \propto K^{3-\lambda}$  [16], it follows that  $\chi_{l+1} \propto N K_{l+1}^{2-\lambda}$  [where both terms in Eq. (9) contribute the same order]. This can be written as a second recurrence equation:

$$\chi_{l+1} = ANK_l^{2-\lambda}, \quad (10)$$

where  $A = \langle k \rangle m^{\lambda-2} / (3 - \lambda) = (\lambda - 1)m / [(\lambda - 2) \times (3 - \lambda)]$ .

Solving the Eqs. (7) and (10) yields the result,

$$\chi_l \sim A^{[(\lambda-2)^l - 1] / (\lambda - 3)} N^{1 - [(\lambda-2)^{l+1}] / (\lambda - 1)}, \quad (11)$$

where  $\chi_l$  is the number of outgoing links from the  $l$ th layer. Equation (7) then leads to

$$K_l \sim A^{[(\lambda-2)^{l-1} - 1] / (3 - \lambda)} N^{[(\lambda-2)^l] / (\lambda - 1)}. \quad (12)$$

Using the same considerations that follow Eq. (6), one can deduce that here also

$$d \approx \ln \ln N / \ln(\lambda - 2). \quad (13)$$

Our result, Eq. (13), is consistent with the observations that the distance in the Internet network is extremely small and that the distance in metabolic scale-free networks is almost independent of  $N$  [10]. These results can be explained by the fact that  $\ln \ln N$  is almost a constant over many orders of magnitude. Our arguments show, however, that for a fixed distribution and very large values of  $N$  no scale-free graph with  $\lambda > 2$  can have a constant diameter. However, for  $\lambda = 2$ , since the highest degree site has order  $N$  links, we expect that for this case  $d \approx \text{const}$ .

For  $\lambda > 3$  and  $N \gg 1$ ,  $\kappa$  is independent of  $N$ , and since the second term of Eq. (9) is dominant, Eq. (9) reduces to  $\chi_{l+1} = (\kappa - 1)\chi_l$ , where  $\kappa$  is a constant depending only on  $\lambda$ . This leads to the known result  $\chi_l \approx C(N, \lambda)(\kappa - 1)^l$  and the radius of the network  $l \propto \ln N$  [23].

For  $\lambda = 3$ , Eq. (9) reduces to  $\chi_{l+1} = \chi_l \ln \chi_l$ . Taking the logarithm of this equation one obtains  $\ln \chi_{l+1} - \ln \chi_l = \ln \ln \chi_l$ . Defining  $g(l) = \ln \chi_l$  and approximating the difference equation by the differential equation  $g' = \ln g$ . This equation cannot be solved exactly. However, taking  $u = \ln g$  the equation reduces to

$$l = \int_{\ln \ln \sqrt{N}}^{\ln \ln N} e^{u - \ln u} du. \quad (14)$$

The lower bound is obtained from the highest degree site for  $\lambda = 3$ , having degree  $K = m\sqrt{N}$ . Thus,  $\chi_0 = m\sqrt{N}$ . The upper bound is the result of searching  $l$  for which  $\chi_l \sim N$  with lower order corrections. The integral in Eq. (14) can be approximated by the steepest descent method, leading to

$$l \approx \ln N / (\ln \ln N), \quad (15)$$

assuming  $\ln \ln N \gg 1$ .

The above result, Eq. (15), has been obtained rigorously for the maximum distance in the Barabasi-Albert (BA) model [24], having  $\lambda = 3$  (for  $m \geq 2$ ) [25]. Although the result in [25] is for the largest distance between two sites, their derivation makes it clear that the average distance will also behave similarly. For  $m = 1$  in the Barabasi-Albert model the graph turns into trees and the behavior of  $d \sim \ln N$  is obtained [25,26]. It should be noted that for  $m = 1$  the giant component in the random model contains only a fraction of the sites (while for  $m \geq 2$  it contains all the sites—at least in the leading order). The BA model, on the other hand, is fully connected for every  $m$ . This might explain why exact trees and BA trees are different from generalized random graphs.

Our derivation is valid for uncorrelated networks. For assortative networks [19] the diameter is expected to be even smaller, as mentioned earlier. For disassortative networks we would expect the odd layers to hold high degree nodes and the even layers to hold low degree nodes, so it is

plausible that the scaling of the diameter is the same, with some possible constant factor  $\leq 2$ . Note that this argument may not be valid for disassortative networks with  $m = 1$ , where many dead ends exist.

In summary, we have shown that scale-free graphs have diameter  $d \sim \ln \ln N$ , which is smaller than the  $d \sim \ln N$  behavior, expected for regular random graphs. For every  $\lambda > 2$  scale-free graphs can be built to have a diameter of order  $d \sim \ln \ln N$ . If random scale-free graphs are considered, only for  $2 < \lambda < 3$ , the behavior  $d \sim \ln \ln N$  is obtained, while for  $\lambda > 3$  the usual result  $d \sim \ln N$  is recovered.

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*Note added.*—After this manuscript [27] was submitted, two other manuscripts were submitted that confirm our results [28,29].

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