12-1 Hooke's Law

We probably all have some experience with springs. One observation we can make is that it doesn't take much force to stretch or compress a spring a small amount, but the more we try to compress or stretch it, the more force we need. We'll use a model of an ideal spring, in which the magnitude of the force associated with stretching or compressing the spring is proportional to the distance the spring is stretched or compressed.

The equation describing the proportionality of the spring force with the displacement of the end of the spring from its natural length is known as Hooke's law.

$$\vec{F}_{Spring} = -k\,\vec{x}\,. \qquad \text{(Equation 12.1: } \textbf{Hooke's Law}\text{)}$$

The negative sign is associated with the restoring nature of the force. When you displace the end of the spring in one direction from its equilibrium position, the spring applies a force in the opposite direction, essentially in an attempt to return the system toward the equilibrium position (the position where the spring is at its natural length, neither stretched nor compressed). The force applied by the spring is proportional to the distance the spring is stretched or compressed relative to its natural length.

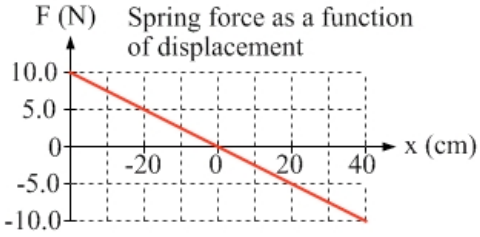

Figure 12.1: A graph of the force applied by a particular spring as a function of the displacement of the end of the spring from its equilibrium position.

The k in the Hooke's law equation is known as the **spring constant**. This is a measure of the stiffness of the spring. If you have two different springs and you stretch them the same amount from equilibrium. The one that requires more force to maintain that stretch has the larger spring constant. Figure 12.1 shows the Hooke's law relationship as a graph of force as a function of the amount of compression or stretch of a particular spring from its natural length.

The Hooke's law relationship is illustrated in Figure 12.2, where $x = 0$ means the spring is neither stretched nor compressed from its natural length. A block attached to spring has been released and is oscillating on a frictionless surface. Free-body diagrams are shown in Figure 12.2, illustrating how the force exerted by the spring on the block depends on the displacement of the end of the spring from its equilibrium position.

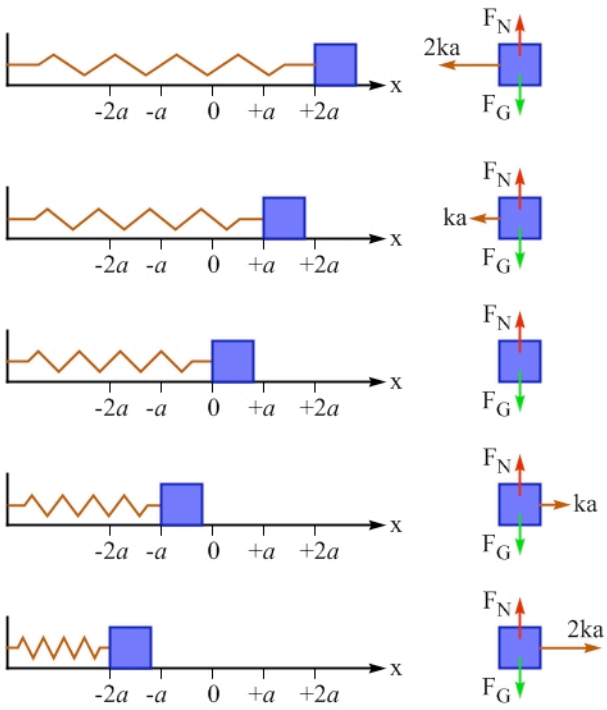

Figure 12.2: A block attached to an ideal spring oscillates on a frictionless surface. By looking at the free-body diagrams of the block when the block is at various positions, we can see that the force applied by the spring on the block is proportional to the displacement of the end of the spring from its equilibrium position, and opposite in direction to that displacement.

EXAMPLE 12.1 – Initial acceleration of a block

A block of mass 300 g is attached to a horizontal spring that has a spring constant of 6.0 N/m. The block is on a horizontal frictionless surface. You release the block from rest when the spring is stretched by 20 cm.

(a) Sketch a diagram of the situation, and a free-body diagram of the block immediately after you release the block.
(b) Determine the block's initial acceleration.
(c) What happens to the block's free-body diagram as the block moves to the left?

SOLUTION

(a) The diagram and free-body diagram are shown in Figure 12.3. After you release the block, only three forces act on the block. The downward force of gravity is balanced by the upward normal force applied by the surface. The third force is the force applied by the spring. The spring force is directed to the left because the end of the spring has been displaced to the right from its equilibrium position.

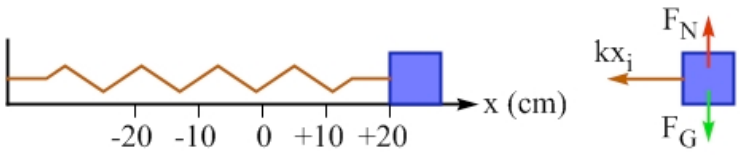

Figure 12.3: A diagram of the block and spring, and the free-body diagram of the block, immediately after the block is released from rest.

(b) Here we can apply Newton's Second Law horizontally, $\sum \vec{F}_x = m\,\vec{a}$, taking right to be the positive *x*-direction. This gives: $-F_{Spring} = m\,\vec{a}$.

Now we can bring in Equation 12.1, $\vec{F}_{Spring} = -k\,\vec{x}$, to get: $-k\,x_i = m\,\vec{a}_i$.

Note that we use only one minus sign in the equation because we're substituting for the magnitude of the spring force only. The one minus sign represents the direction of the spring force, which is to the left. Solving for the block's initial acceleration gives:

$$\vec{a}_i = +\frac{k\,x_i}{m} = -\frac{(6.0\,\text{N/m})(0.20\,\text{m})}{0.30\,\text{kg}} = -4.0\,\text{N/kg}.$$

The initial acceleration is 4.0 N/kg to the left.

(c) As the block moves to the left, nothing changes about the vertical forces, but the spring force steadily decreases in magnitude because the stretch of the spring steadily decreases. Once the block goes past the equilibrium position, the spring force points to the right, and increases in magnitude as the compression increases. The dependence of the spring force on the block's position is shown, for five different positions, in Figure 12.2.

Related End-of-Chapter Exercises: 16, 55.

Essential Question 12.1: Let's say we estimated the time it takes the block in Example 12.1 to reach equilibrium, by assuming the block's acceleration is constant at 4.0 N/kg to the left. Is our estimated time smaller than or larger than the time it actually takes the block to reach the equilibrium point?

Answer to Essential Question 12.1: This estimated time is less than the actual time. The closer the block gets to the equilibrium position, the smaller the force that is exerted on it by the spring, and the smaller the magnitude of the block's acceleration. Because the block generally has a smaller acceleration than the acceleration we used in the constant-acceleration analysis, it will take longer to reach equilibrium than the time we calculated with the constant-acceleration analysis. Thus, remember not to use constant-acceleration equations in harmonic motion situations! We'll learn how to calculate exact times in sections 12-4 to 12-6.

12-2 Springs and Energy Conservation

Now that we have seen how to incorporate springs into a force perspective, let's go on to consider how to fit springs into what we know about energy.

EXPLORATION 12.2 – Another kind of potential energy

Step 1 – ***Attach a block to a spring, and position the block so that the spring is stretched. Let's neglect friction, so when you release the block from rest it oscillates back and forth about the equilibrium position. What is going on with the energy of the system as the block oscillates?*** As the block oscillates, its speed increases from zero to some maximum value, then decreases to zero again, and keeps doing this over and over. The kinetic energy of the system does exactly the same thing, since it is proportional to the square of this speed. Where does the energy go when the kinetic energy decreases, and where does it comes from when the kinetic energy increases?

The energy is stored as potential energy in the spring. This is similar to what happens when we throw a ball up into the air. As the ball rises, the ball's loss of kinetic energy is offset by the gain in the gravitational potential energy of the Earth-ball system, and then that potential energy is transformed back into kinetic energy. Compressed or stretched springs also store potential energy. Such energy is known as **elastic potential energy**.

Step 2 - ***Consider the graph of force, as a function of the displacement of the end of the spring, shown in Figure 12.4. As we did in Chapter 6, defining the change in gravitational potential energy to be the negative of the work done by gravity on an object, find an expression for the change in elastic potential energy as the end of the spring is displaced from its equilibrium position (x = 0) to some arbitrary final position x. Make use of the fact that work is the area under the force-versus-position graph in Figure 12.4.***

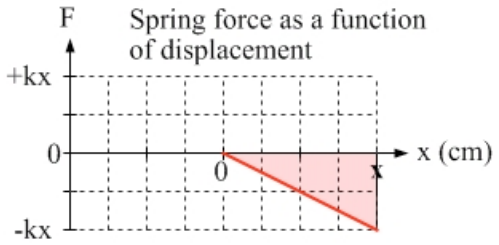

Figure 12.4: The work done by a spring when its end is displaced from the equilibrium position to a point *x* away from equilibrium is represented by the shaded area in the graph.

The area in question is that of the right-angled triangle shown in Figure 12.4. The area is negative because the force is negative the entire time. The area under the curve is given by:

$$\text{area} = -\frac{1}{2}\text{base} \times \text{height} = -\frac{1}{2}x(kx) = -\frac{1}{2}kx^2 .$$

This area represents the work done by the spring. This work is negative because the spring force is opposite in direction to the displacement. Because ΔU_e, the change in the elastic potential energy, is the negative of the work, we have $\Delta U_e = \frac{1}{2}kx^2$ in this case.

Step 3 – ***How much elastic potential energy is stored in the spring when the spring is at its natural length?*** None. If we attach a block to such a spring and release the block from rest, no motion occurs because the system is at equilibrium. There is no transformation of elastic potential energy into kinetic energy because the system has no elastic potential energy when the spring is at its natural length – the equilibrium position is the zero for elastic potential energy.

Step 4 – ***Combine the results from parts 2 and 3 to determine the expression for the elastic potential energy stored in a spring when the end of the spring is displaced a distance x from its equilibrium position.*** In step 3 we found the change in elastic potential energy in displacing the end of the spring from its equilibrium position to a point *x* away from equilibrium to be $\Delta U_e = \frac{1}{2}kx^2$. This change in elastic potential energy is equal to the final elastic potential energy minus the initial elastic potential energy. However, we found the initial elastic potential energy to be zero in step 3, which means the expression for elastic potential energy is simply:

$$U_e = \frac{1}{2}kx^2. \qquad \text{(Equation 12.2: } \textbf{Elastic potential energy}\text{)}$$

Key ideas: Compressed or stretched springs store energy – this is known as elastic potential energy. For an ideal spring, the elastic potential energy is $U_e = \frac{1}{2}kx^2$.

Related End-of-Chapter Exercises: 9, 48.

Now that we know the form of the elastic potential energy equation, we can incorporate springs into the conservation of energy equation we first used in chapter 7:

$$K_i + U_i + W_{nc} = K_f + U_f. \qquad \text{(Equation 7.1)}$$

Graphs of the energies as a function of position are interesting. Consider a block attached to a spring. The block is oscillating back and forth on a frictionless surface, so the total mechanical energy stays constant. An easy way to graph the kinetic energy is to exploit energy conservation, $E = K + U$. Solving for the kinetic energy as a function of position gives:

$$K = E - U = E - \frac{1}{2}kx^2.$$

Graphs of the energies as a function of position are shown in Figure 12.5, for a situation in which the total mechanical energy is 4.0 J. After tracing out the complete energy curves over half an oscillation, the system re-traces these energy-versus-position plots as the block oscillates.

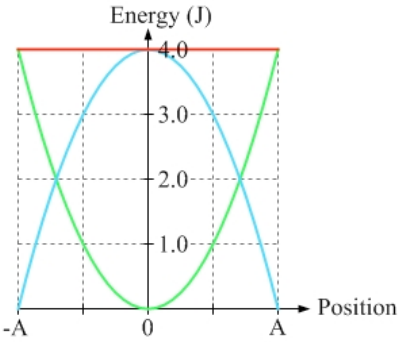

Figure 12.5: Graphs of the system's kinetic energy (zero at -A and A), elastic potential energy (zero at x = 0), and total mechanical energy (constant), as a function of position. The system traces over each of the energy graphs every half oscillation.

Essential Question 12.2: Consider a system consisting of a block attached to an ideal spring. The block is oscillating on a horizontal frictionless surface. When the block is 20 cm away from the equilibrium position, the elastic potential energy stored in the spring is 24 J. What is the elastic potential energy when the block is 10 cm away from equilibrium?

Answer to Essential Question 12.2: To answer this question, we can use the fact that the elastic potential energy is proportional to x^2. Doubling x, the distance from equilibrium, increases the elastic potential energy by a factor of 4. Thus, the elastic potential energy is 6 J when x = 10 cm.

12-3 An Example Involving Springs and Energy

EXAMPLE 12.3 – A fast-moving block

(a) A block of mass m, which rests on a horizontal frictionless surface, is attached to an ideal horizontal spring. The block is released from rest when the spring is stretched by a distance A from its natural length. What is the block's maximum speed during the ensuing oscillations?

(b) If the block is released from rest when the spring is stretched by $2A$ instead, how does the block's maximum speed change?

SOLUTION

(a) Let's begin, as usual, with a diagram of the situation (see Figure 12.6). When will the block achieve its maximum speed? Maximum speed corresponds to maximum kinetic energy, which corresponds to minimum potential energy. The gravitational potential energy is constant, since there is no up or down motion, so we can focus on the elastic potential energy. The elastic potential energy is a minimum (zero, in fact) when the block passes through equilibrium, where the spring is at its natural length. Energy bar graphs for the two points are shown in Figure 12.6.

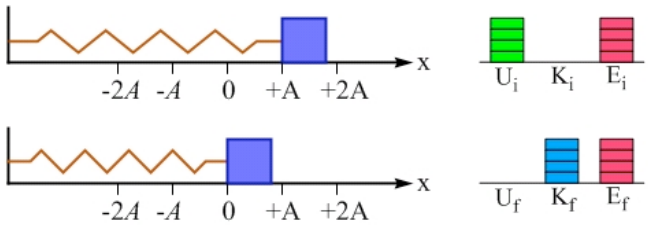

Figure 12.6: Diagrams of the block at the release point and at the equilibrium position, and the corresponding energy bar graphs.

Let's continue with the energy analysis by writing out the conservation of energy equation: $K_i + U_i + W_{nc} = K_f + U_f$. The initial point is the point from which the block is released, while the final point is the equilibrium position.

$K_i = 0$, because the block is released from rest from the initial point.

$W_{nc} = 0$, because there is no work being done by non-conservative forces.

We can neglect gravitational potential energy, because there is no vertical motion. This gives $U_f = 0$, because the elastic potential energy is also zero at the final point.

We have thus reduced the energy equation to: $U_{ei} = K_f$. This gives:

$\frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$. Solving for the maximum speed gives: $v_{\max} = A\sqrt{\frac{k}{m}}$.

Is this answer reasonable? The maximum speed is larger if we start the block farther from equilibrium (where the spring exerts a larger force); if we increase the spring constant (also increasing the force); or if we decrease the mass (increasing acceleration). This all makes sense.

(b) If we start the block from $2A$ away from equilibrium, we simply replace A in our equation above by $2A$, showing us that the maximum speed is twice as large:

$v'_{\max} = 2A\sqrt{\frac{k}{m}}$.

Related End-of-Chapter Exercises: 4, 5.

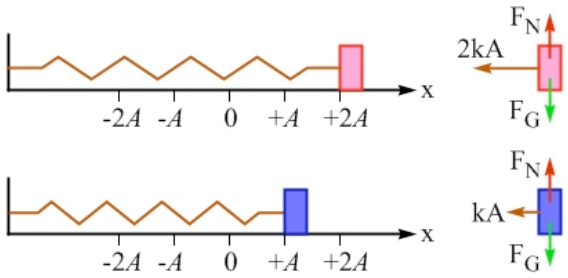

Figure 12.7: Identical blocks attached to identical springs. The blocks are released from rest simultaneously. Block 2, at the top, is released from a distance $2A$ from equilibrium. Block 1 is released from a distance A from equilibrium. The initial free-body diagrams are also shown.

We can make an interesting generalization based on further analysis of the situation in Example 12.3. Take two blocks, one red and one blue but otherwise identical, and two identical springs. Attach each block to one of the springs, and place these two block-spring systems on frictionless horizontal surfaces. As shown in Figure 12.7, we will release one block from rest from a distance A from equilibrium and the other from a distance $2A$ from equilibrium. If the blocks are released simultaneously, which block reaches the equilibrium point first?

Block 2 has an initial acceleration twice as large as that of block 1, because block 2 experiences a net force that is twice as large as that experienced by block 1. The accelerations steadily decrease, because the spring force decreases as the blocks get closer to equilibrium, but we can neglect this change if we choose a time interval that is sufficiently small.

At the end of this time interval, Δt, what is the speed of each block? We're choosing a small time interval so that we can apply a constant-acceleration analysis. Remembering that the blocks are released from rest, so $v_i = 0$, we have:

for block 1, $\vec{v}_1 = \vec{v}_{i1} + \vec{a}_1 \Delta t = \vec{a}_1 \Delta t$;

for block 2, $\vec{v}_2 = \vec{v}_{i2} + \vec{a}_2 \Delta t = \vec{a}_2 \Delta t = 2\vec{a}_1 \Delta t = 2\vec{v}_1$.

What about the distance each block travels? Here we can apply another constant acceleration equation:

for block 1 $\Delta \vec{x}_1 = \vec{v}_{i1} \Delta t + \frac{1}{2} \vec{a}_1 (\Delta t)^2 = \frac{1}{2} \vec{a}_1 (\Delta t)^2$;

for block 2 $\Delta \vec{x}_2 = \vec{v}_{02} \Delta t + \frac{1}{2} \vec{a}_2 (\Delta t)^2 = \frac{1}{2} \vec{a}_2 (\Delta t)^2 = \frac{1}{2} (2\vec{a}_1)(\Delta t)^2 = 2\Delta \vec{x}_1$.

At the end of the time interval, block 1 is $A - \Delta x_1$ from equilibrium and block 2 is exactly twice as far from equilibrium as block 1, at $2A - 2\Delta x_1 = 2(A - \Delta x_1)$ from equilibrium. Thus, after this small time interval has passed, block 2 is still twice as far from equilibrium as block 1, its velocity is twice as large, and its acceleration is twice as large. We could keep the process going, following the two blocks as time goes by, and we would find this always to be true, that block 2's velocity, acceleration, and displacement from equilibrium, is always double that of block 1. This is true at all times, even after the blocks pass through their equilibrium positions to the far side of equilibrium.

This leads to an amazing conclusion – ***that the two blocks take exactly the same time to reach equilibrium*** (and to complete one full cycle of an oscillation). This is because block 2 experiences twice the displacement of block 1, but its average velocity is also twice as large. Because the time is the distance divided by the average velocity, these factors of two cancel out.

Essential Question 12.3: Above we analyzed the situation of two identical (aside from color) blocks, oscillating on identical springs, and found the time to reach equilibrium (or to complete one full oscillation) to be the same. Was that just a coincidence that happened to work out because the starting displacements from equilibrium were in a 2:1 ratio, or can we generalize and say that the time is the same no matter where the block is released?

Answer to Essential Question 12.3: In fact, this result is generally true. As long as the spring is ideal, then ***the time it takes a block to move through one complete oscillation is independent of the amplitude of the oscillation***. The **amplitude** is defined as the maximum distance an object gets from its equilibrium position during its oscillatory motion.

12-4 The Connection with Circular Motion

So far we have looked at how to apply force and energy ideas to springs. Let's now explore an interesting connection between what is called **simple harmonic motion** (oscillatory motion without any loss of mechanical energy), and uniform circular motion.

EXPLORATION 12.4 – Connecting circular motion to simple harmonic motion

Take the two spring-block systems we investigated at the end of the previous section and place them beside a large turntable that is rotating about a vertical axis. Set the constant angular speed of the turntable so that the turntable undergoes one complete revolution in the time it takes the blocks on the springs to move through one complete oscillation. As shown in Figure 12.8, there are two disks on the turntable, one a distance A from the center and the other a distance $2A$ from the center. The blocks are simultaneously released from rest at the instant the disks pass through the position shown in the figure.

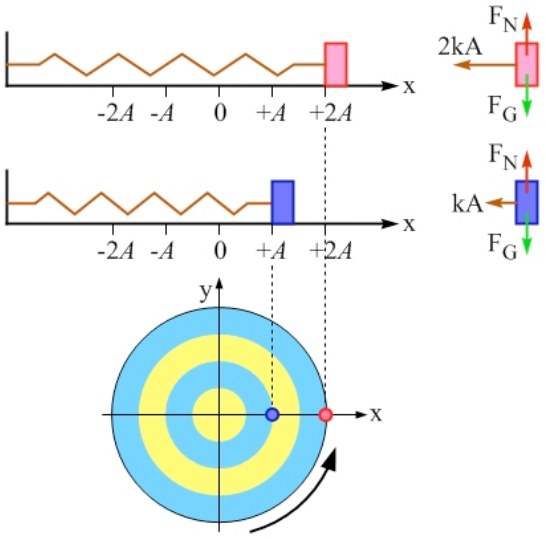

Figure 12.8: Comparing systems – two disks on a rotating turntable and two oscillating block-and-spring systems.

Another amazing thing happens. As the disks spin at constant angular velocity and the blocks oscillate back and forth, the motion of block 1 matches the motion of disk 1, while the motion of block 2 matches the motion of disk 2. The position of the left-hand side of each block ***is at all times*** equal to the x-coordinate of the position of the center of its corresponding disk, taking the origin to be at the center of the turntable and using the x-y coordinate system shown in Figure 12.8.

Step 1 – ***Sketch two separate motion diagrams, one showing the successive positions of disk 1 and the other showing the successive positions of disk 2, as the turntable undergoes one complete revolution. Plot the positions at regular time intervals which, because the disk rotates at a constant rate, correspond to regular angular displacements.*** Motion diagrams for the disks are shown in Figure 12.9, showing positions at 30° intervals.

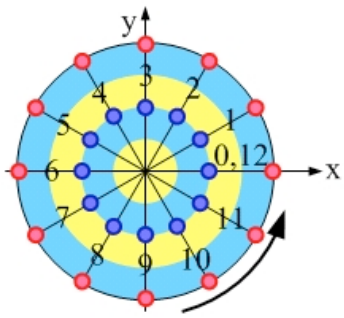

Figure 12.9: Motion diagrams for the two disks, showing their positions at 30° intervals. Because the turntable (and each disk) rotates at a constant rate, these equal angular displacements correspond to the equal time intervals we're used to seeing on motion diagrams.

Step 2 – ***Now add motion diagrams for the two blocks, sketching their positions so they agree with the statement above, that the left-hand side of each block is at all times equal to the x-coordinate of the position of the center of its corresponding disk.*** These motion diagrams are shown in Figure 12.10.

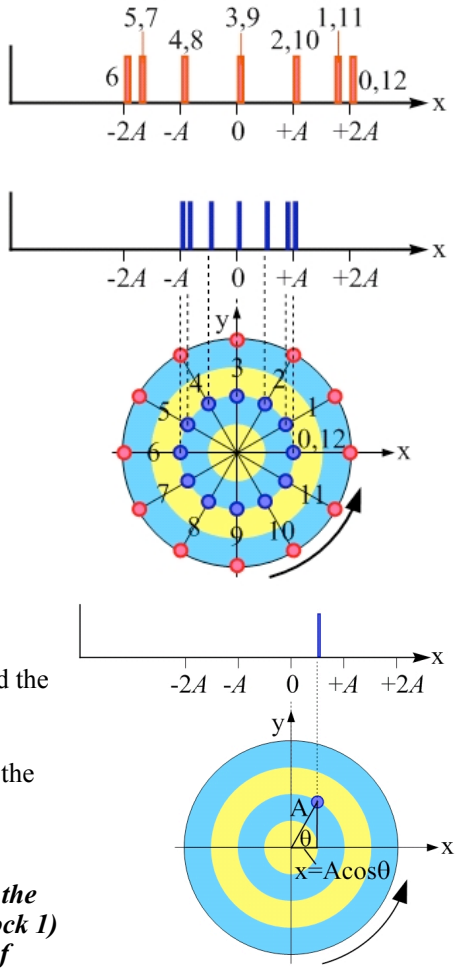

Figure 12.10: Motion diagrams for the two blocks, showing that the motion of a block experiencing simple harmonic motion exactly matches the motion of a well-chosen object experiencing uniform circular motion. The springs have been removed from the picture for clarity, and the motion diagrams for the blocks show the successive positions of the left-hand side of each block during its motion. The motion of a block matches the *x*-component of the motion of the corresponding disk on the turntable.

Step 3 – ***Measuring angles counterclockwise from the positive x-axis, write an equation giving the x-coordinate of disk 1 as a function of time. Hint: first write out the x-coordinate in terms of an arbitrary angle the turntable has rotated through, and then express that angle in terms of time and the turntable's constant angular speed*** ω**.** Figure 12.11 shows the position of the disk 1 when the turntable has rotated through some arbitrary angle θ from its initial position. Its *x*-position at this angle can be found from the adjacent side of the right-angled triangle: $\vec{x} = A\cos(\theta)$. Since the angular velocity is constant, however, and the initial angle $\theta_i = 0$ we can express the angle as: $\theta = \theta_i + \omega t = 0 + \omega t = \omega t$. Substituting this into our expression for the disk's x-position gives: $\vec{x} = A\cos(\omega t)$.

Step 4 – ***Based on the results above, what is the equation giving the x-position of block 1 (actually, the position of the left edge of block 1) as a function of time? What is the equation giving the position of block 2 as a function of time?*** Because the motion of block 1 matches exactly the *x*-component of the motion of disk 1, the equation that gives the disk's *x*-position must also gives the block's *x*-position. Thus, for block 1 we have:

Figure 12.11: The position of block 1 and disk 1 after the turntable has rotated through an arbitrary angle θ.

$$\vec{x} = A\cos(\omega t) \quad . \qquad \text{(Eq. 12.3: } \textbf{Position-versus-time for simple harmonic motion}\text{)}$$

Using the convention introduced earlier in this book, in which a + or – sign is used to represent the direction of a vector in one dimension, the right-hand side of equation 12.3 can be viewed as a vector quantity, with the sign hidden in the cosine. We get a positive sign for some values of time and a negative sign for others. The equation for block 2 is virtually identical to that of block 1, with the only change being the extra factor of 2. For block 2: $\vec{x} = 2A\cos(\omega t)$.

Key ideas: There is an interesting connection between simple harmonic motion and uniform circular motion. One-dimensional simple harmonic motion matches one component of a carefully chosen two-dimensional uniform circular motion. This allows us to write an equation of motion for an object experiencing simple harmonic motion: $\vec{x} = A\cos(\omega t)$. In this context, ω is known as the angular frequency. **Related End-of-Chapter Exercises: 42, 43.**

Essential Question 12.4: We showed above how the position of a block oscillating on a spring matches one component of the position of an object experiencing uniform circular motion. Can we make similar conclusions about the velocity and acceleration of the block on the spring?

Answer to Essential Question 12.4: Absolutely. All aspects of the motion of the oscillating block match one component of the motion of the object experiencing uniform circular motion. If the position of the block is given by $\vec{x} = A\cos(\omega t)$, then its velocity and acceleration are given by:

$$\vec{v}_{x,disk1} = \vec{v}_{block1} = -v\sin(\omega t) = -A\omega\sin(\omega t) \quad \text{and} \quad \vec{a}_{x,disk1} = \vec{a}_{block1} = -\frac{v^2}{A}\cos(\omega t) = -A\omega^2\cos(\omega t).$$

Here, v represents the constant speed of disk 1 as it moves in uniform circular motion.

12-5 Hallmarks of Simple Harmonic Motion

Simple harmonic motion (often referred to as SHM) is a special case of oscillatory motion. An object oscillating in one dimension on an ideal spring is a prime example of SHM. The characteristics of simple harmonic motion include:

- A force (and therefore an acceleration) that is opposite in direction, and proportional to, the displacement of the system from equilibrium. Such a force, that acts to restore the system to equilibrium, is known as a **restoring force**.
- No loss of mechanical energy.
- An angular frequency ω that depends on properties of the system.
- Position, velocity, and acceleration given by Equations 12.3 – 12.5:

$\vec{x} = A\cos(\omega t)$. (Equation 12.3: **Position in simple harmonic motion**)

$\vec{v} = -v_{\max}\sin(\omega t) = -A\omega\sin(\omega t)$. (Equation 12.4: **Velocity in SHM**)

$\vec{a} = -a_{\max}\cos(\omega t) = -\frac{v^2}{A}\cos(\omega t) = -A\omega^2\cos(\omega t)$. (Eq. 12.5: **Acceleration in SHM**)

The above equations apply if the object is released from rest from $\vec{x} = +A$ at $t = 0$. Starting the block with different initial conditions requires a modification of the equations.

Combining Equations 12.3 and 12.5, in any simple harmonic motion system we see that the acceleration is opposite in direction, and proportional to, the displacement:

$\vec{a} = -\omega^2\vec{x}$. (Equation 12.6: **Connecting acceleration and displacement in SHM**)

In general, the angular frequency (ω), frequency (f), and period (T) are connected by:

$\omega = 2\pi f = \frac{2\pi}{T}$. (Eq. 12.7: **Relating angular frequency, frequency, and period**)

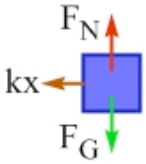

Figure 12.12: The free-body diagram of a block connected to a spring of spring constant k. The block is displaced to the right of the equilibrium point by a distance x.

What determines the angular frequency ω in a particular situation? Let's return to the free-body diagram of a block on a spring, shown in Figure 12.12.

Applying Newton's Second Law horizontally, $\sum\vec{F}_x = m\vec{a}$, we get: $-k\vec{x} = m\vec{a}$.

Re-arranging gives $\vec{a} = -(k/m)\vec{x}$. Comparing this result to the general SHM Equation 12.6 tells us that, for a mass on an ideal spring, $\omega^2 = k/m$, or:

$\omega = \sqrt{\frac{k}{m}}$. (Equation 12.8: **Angular frequency for a mass on a spring**).

This is a typical result, that the angular frequency is given by the square root of a parameter related to the restoring force (or torque, in rotational motion) divided by the inertia.

EXAMPLE 12.5 – Plotting graphs of position, velocity, and acceleration versus time

Once again, let's attach a block to a spring and release the block from rest from a position $\bar{x} = +A$ (relative to $\bar{x} = 0$, which is the equilibrium position). The block oscillates back and forth with a period of $T = 4.00$ s.

(a) Plot graphs of the block's position, velocity, and acceleration as a function of time over two complete oscillations.

(b) Compare the position graph to the velocity graph.

(c) How does the acceleration graph compare to the position graph?

SOLUTION

(a) We can make use of Equations 12.3 – 12.5 to plot the graphs. Before doing so, we can solve for the angular velocity ω, using:

$$\omega = \frac{2\pi}{T} = \frac{2\pi \text{ rad}}{4.00\,\text{s}} = 1.57\,\text{rad/s}.$$

Also, it makes it easier to plot the graphs if we remember that, if the block is released from rest, it returns to its starting point after one period; after half a period it comes instantaneously to rest on the far side of equilibrium; and at times of T/4 and 3T/4 it is passing through equilibrium at its maximum speed. Determining when each graph passes through zero, when it reaches its largest positive and negative values, and then connecting these points with sinusoidally oscillating graphs, gives the results shown in Figure 12.13.

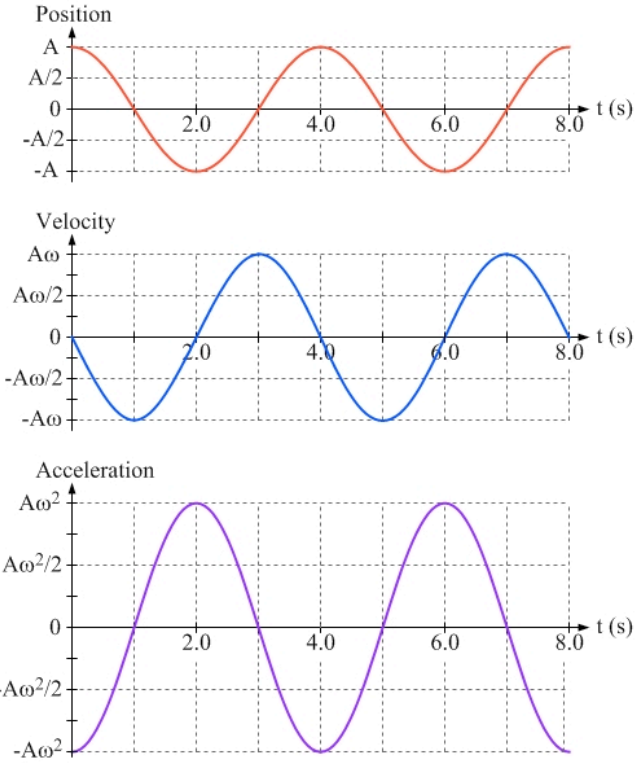

Figure 12.13: Graphs of the position, velocity, and acceleration, as a function of time, of the block in Example 12.5 for two complete oscillations of the block.

(b) Comparing the position and velocity graphs in Figure 12.13, we can see that the block's speed is maximum when the block's displacement from equilibrium is zero. Conversely, the block's speed is zero when the magnitude of the block's displacement from equilibrium is maximized. These observations are consistent with what is taking place with the energy. The kinetic energy is proportional to the speed squared and the elastic potential energy is proportional to the square of the magnitude of the displacement from equilibrium. Kinetic energy is maximum when the elastic potential energy is zero, and vice versa.

(c) Comparing the position and acceleration graphs, we see that one is the opposite of the other, in the sense that when the position is positive the acceleration is negative, and vice versa. This is expected because one of the hallmarks of simple harmonic motion is that $\bar{a} = -\omega^2 \bar{x}$.

Related End-of-Chapter Exercises: 32, 41, 46.

Essential Question 12.5: Return to the situation described in Example 12.5, but now increase the angular frequency by a factor of 2. We can accomplish this by either changing only the spring constant or by changing only the mass. Can we tell which one was changed by looking at the resulting graphs of position, velocity, and/or acceleration as a function of time? Assume that the block is released from rest from the same point it was in Example 12.5, and that the equilibrium position remains the same.

Answer to Essential Question 12.5: We cannot tell. Any one of the three graphs can be used to determine that the angular frequency has changed, because they all involve ω, but none of the graphs can tell us whether we adjusted the spring constant or the mass.

12-6 Examples Involving Simple Harmonic Motion

EXAMPLE 12.6A – Energy graphs

Take a 0.500-kg block and attach it to a spring. We would like the block to undergo oscillations that have a period (the time for one complete oscillation) of 4.00 seconds.

(a) What should the spring constant be?

(b) We'll release the block from rest from a distance *A* from the equilibrium point so that the block has a speed of 4.00 m/s when it passes through equilibrium. Over two complete oscillations, plot the system's elastic potential energy, kinetic energy, and total mechanical energy as a function of time.

SOLUTION

(a) Let's first apply Equation 12.7, $\omega = 2\pi / T$, to find the angular frequency. This gives:

$$\omega = \frac{2\pi \text{ rad}}{4.00\,\text{s}} = \frac{\pi}{2.00}\,\text{rad/s}.$$

Using Equation 12.7, $\omega = \sqrt{\frac{k}{m}}$, we get:

$$k = \omega^2 m = \frac{\pi^2 (0.500)}{4.00}\,\text{rad}^2\,\text{kg/s}^2 = 1.23\,\text{N/m},$$

where we treated the factor of radians as being dimensionless.

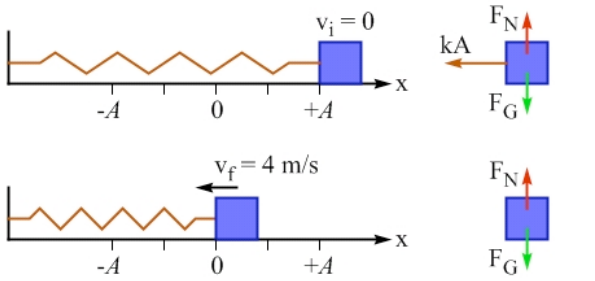

Figure 12.14: A diagram of the block and spring, showing the initial situation and the situation as the block passes through equilibrium.

(b) A diagram of the situation is shown in Figure 12.14. Let's solve for the maximum kinetic energy, which equals the mechanical energy:

$$K_{\max} = \frac{1}{2} m v_{\max}^2 = \frac{1}{2}(0.500\,\text{kg})(4.00\,\text{m/s})^2 = 4.00\,\text{J}.$$

The maximum potential energy is also 4.00 J, because the energy oscillates between potential and kinetic, and the total mechanical energy is conserved.

Using Equation 12.4, $\vec{v} = -v_{\max}\sin(\omega t)$, we can write the kinetic energy as a function of time as $K = \frac{1}{2}mv^2 = \frac{1}{2}mv_{\max}^2 \sin^2(\omega t) = (4.00\,\text{J})\sin^2\left(\frac{\pi}{2.00\,\text{s}}t\right)$.

Because the block takes 4.00 s to complete one oscillation, at $t = 0$ and $t = 4.00$ s it is instantaneously at rest at the starting point. At $t = 2.00$ s (halfway through the cycle) the block is instantaneously at rest on the far side of equilibrium. At each of these times the kinetic energy is 0 and the elastic potential energy is 4.00 J. Conversely, at $t = 1.00$ s and 3.00 s it passes through equilibrium, where the elastic potential energy is zero and the kinetic energy is its maximum value of 4.00 J. Graphs of the various energies as a function of time are shown in Figure 12.15. Note that, at all times, the sum of the kinetic and potential energies is 4.00 J.

Related End-of-Chapter Exercises: 27, 40.

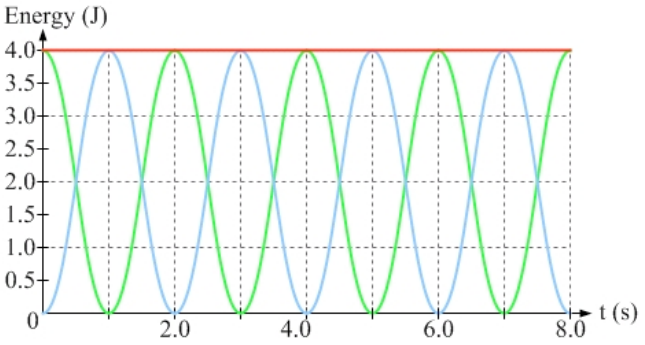

Figure 12.15: Graphs of the block's kinetic energy (zero at t = 0 s), elastic potential energy (zero at t = 1.0 s), and total mechanical energy (constant), as a function of time over 8.00 s, the time for two complete oscillations. The kinetic and potential energies go through two cycles for every one complete oscillation of the block. Compare this graph to Figure 12.5, which shows the energies as a function of position.

EXAMPLE 12.6B – An eighth of the motion

Attach an object to an ideal spring and set it oscillating. The object is released from rest from a distance A from equilibrium. The object travels a distance A to the equilibrium position, ¼ of the entire distance for one complete oscillation, in ¼ of the period. How long does it take the object to travel from A away from equilibrium to $A/2$ from equilibrium, 1/8 of the entire distance covered in one complete oscillation?

SOLUTION

A diagram of the situation is shown in Figure 12.16. Position 1 is where the block is released from rest. Position 2 is halfway between the release point and the equilibrium position, which is position 3.

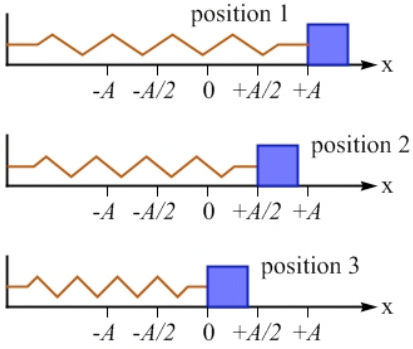

Figure 12.16: A diagram of the position of the block as it moves from its release point (position 1) to the equilibrium position (position 3). Position 2 is exactly halfway between the release point and equilibrium.

Because the block's speed increases as the block approaches equilibrium, the block's average speed as it moves from position 1 to position 2 is less than its average speed as it moves from position 2 to position 3. Thus, because of this low average speed, the time it takes to move from position 1 to position 2 is larger than $T/8$, 1/8 of the total time.

(b) Using Equation 12.3, $\vec{x} = A\cos(\omega t)$, let's find the time it takes. At position 2, the block's position is $+A/2$. Using this in Equation 12.3 gives:

$$+\frac{A}{2} = A\cos(\omega t).$$ Dividing by A gives: $$+\frac{1}{2} = \cos(\omega t).$$

We can use the relation $\omega = 2\pi / T$ to re-write the equation: $$+\frac{1}{2} = \cos\left(\frac{2\pi}{T}t\right).$$

The logical next step is to take the inverse cosine of both sides. Here it is critical to remember that ωt has units of radians. ***Any time we use the three equations involving time (Equations 12.3 – 12.5), we need to work in radians.*** Thus, when we determine $\cos^{-1}(+1/2)$ we will write the result as $\pi/3$ radians instead of 60˚.

Taking the inverse cosine of both sides, then, gives: $$\frac{\pi}{3} = \frac{2\pi}{T}t.$$

This gives us a time of t = $T/6$. As we concluded above, the time is larger than $T/8$.

Related End-of-Chapter Exercises: 34, 50.

Essential Question 12.6: Return to the situation described in Example 12.6A, but now increase the angular frequency by a factor of 2. Let's say we achieve the change in angular frequency by changing either the spring constant or the mass, but not both. Can we tell which one was changed, if the graphs of energy as a function of time still reach a maximum of 4.0 J when the block is released from rest from a distance A from equilibrium?

Answer to Essential Question 12.6: Consider Equation 12.8, $\omega = \sqrt{k/m}$. We can double the angular frequency by increasing the spring constant by a factor of 4, or by decreasing the mass by a factor of 4. Because the object is released from rest, the initial energy is all elastic potential energy, given by $U_i = 0.5kA^2$. We have not changed A, so if the total energy stayed the same we must not have changed the spring constant k. Thus we must have changed the mass.

12-7 The Simple Pendulum

Another classic simple harmonic motion system is the simple pendulum, which is an object with mass that swings back and forth on a string of negligible mass.

EXAMPLE 12.7 – Pendulum speed limit

A ball of mass m is fastened to a string with a length L. Initially the ball hangs vertically down from the string in its equilibrium position. The ball is then displaced so the string makes an angle of θ with the vertical, and then released from rest.

(a) What is the height of the ball above the equilibrium position when it is released?

(b) What is the speed of the ball when it passes through equilibrium?

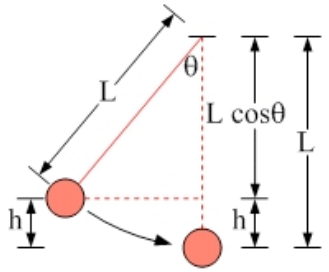

Figure 12.17: A diagram showing the point from which the ball is released and the equilibrium position.

SOLUTION

(a) Consider the geometry of the situation, shown in Figure 12.17. The key to finding the height of the ball is to consider the right-angled triangle in Figure 12.17. The vertical side of the triangle measures $L\cos\theta$. Because the string measures L, the height of the ball above equilibrium is $h = L - L\cos\theta = L(1-\cos\theta)$.

(b) Let's apply energy conservation, starting as usual with: $K_i + U_i + W_{nc} = K_f + U_f$.

The initial point is the release point, while the final point is the equilibrium position.

$K_i = 0$, because the ball is released from rest from the initial point.

$W_{nc} = 0$, because there is no work being done by non-conservative forces.

We can define the ball's gravitational potential energy to be zero at the equilibrium point, giving $U_f = 0$.

The equation thus reduces to $U_i = K_f$, which gives: $mgh = \frac{1}{2}mv_f^2$.

The mass cancels out, so the speed does not depend on the mass. Solving for the speed:

$v_f = \sqrt{2gh} = \sqrt{2gL(1-\cos\theta)}$.

Note that we have seen this $v_f = \sqrt{2gh}$ result before, such as in cases in which an object falls straight down from rest, or when water leaks out a hole in a container.

Related End-of-Chapter Exercises: 11, 29.

EXPLORATION 12.7 – Torques on the pendulum

A simple pendulum consists of a ball of mass m that hangs down vertically from a string. The ball is displaced by an angle θ from equilibrium and released from rest.

Step 1 – ***Draw a free-body diagram for the ball immediately after it is released.*** The free-body diagram is drawn in Figure 12.18). There is a downward force of gravity, and a force of tension directed away from the ball along the string. Using a coordinate system aligned with the string, we can split the force of gravity into components, one component opposite to the tension and the other component giving an acceleration toward the equilibrium position (see Figure 12.18(b)).

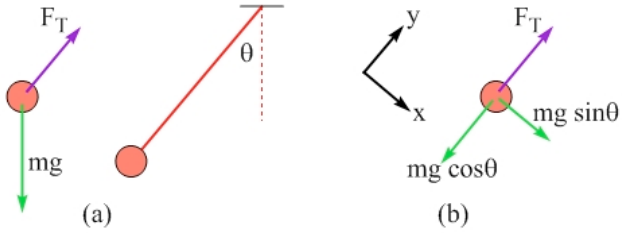

Figure 12.18: Figure (a) shows the free-body diagram, with a force of tension and a force of gravity acting on the ball. Figure (b) shows the force of gravity in components, using a coordinate system in which one axis is parallel to the string.

Step 2 – ***Apply Newton's Second Law for Rotation to find a relationship between the angular acceleration of the ball and the ball's angular displacement (measured from the vertical).*** Take torques about the axis perpendicular to the page passing through the upper end of the string. There is no torque about this axis from the tension or from the component of the force of gravity parallel to the string. The only torque comes from the component of the force of gravity that acts perpendicular to the string. Applying $\tau = rF\sin\phi$, where $r = L$, $F = mg\sin\theta$, and $\phi = 90°$, the torque has a magnitude of $\tau = Lmg\sin\theta$. Taking counterclockwise to be positive, the torque is negative. Applying Newton's Second Law for Rotation, $\sum\vec{\tau} = I\vec{\alpha}$, we get: $-Lmg\sin\theta = I\vec{\alpha}$.

The rotational inertia of the ball is $I = mL^2$, giving: $-Lmg\sin\theta = mL^2\vec{\alpha}$.

Canceling the mass (thus, the mass does not matter) and a factor of L gives:

$$\vec{\alpha} = -\frac{g}{L}\sin\theta \quad .$$ (Equation 12.9: **Angular acceleration of a simple pendulum**)

Step 3 – ***Use the small-angle approximation,*** $\sin\theta \approx \theta$ ***, to find an expression for the angular frequency of the pendulum.*** We can say that $\sin\theta \approx \theta$ if θ is given in radians, and the angle is less than about 10˚(about 1/6 radians). Using the small-angle approximation in Equation 12.9:

$$\vec{\alpha} = -\frac{g}{L}\vec{\theta} \quad .$$ (Equation 12.10: **For a simple pendulum at small angles**)

The general simple harmonic relationship, $\vec{a} = -\omega^2\vec{x}$, can be transformed to an analogous general equation for rotational motion, $\vec{\alpha} = -\omega^2\vec{\theta}$. Equation 12.10 fits this form, so:

$$\omega = \sqrt{\frac{g}{L}} \quad .$$ (Eq. 12.11: **Angular frequency for a simple pendulum at small angles**)

For a pendulum, gravity provides the restoring force, so it makes sense that the angular frequency is larger if g is larger. Conversely, increasing L means the pendulum has farther to travel to reach equilibrium, reducing the angular frequency.

Key ideas: For small-angle oscillations, the motion of a simple pendulum is simple harmonic. Large-angle oscillations are not simple harmonic because the restoring torque is not proportional to the angular displacement. **Related End-of-Chapter Exercises: 57, 58.**

Essential Question 12.7: Compare the free-body diagram of a ball of mass *m,* hanging at rest from a string of length *L*, to that of the same system oscillating as a pendulum, when the ball passes through equilibrium. Make note of any differences between the two free-body diagrams.

Answer to Essential Question 12.7: The two free-body diagrams are shown in Figure 12.19. When the ball is at rest its acceleration is zero. Applying Newton's Second Law tells us that, in this case, the force of tension exactly balances the force of gravity, so $F_T = mg$. When the ball is oscillating it is moving along a circular arc as it passes through the equilibrium position. In this case there is a non-zero acceleration, the centripetal acceleration directed toward the center of the circular arc. To produce the upward acceleration the upward force of tension must be larger than the downward force of gravity. Applying Newton's Second Law shows that the force of tension increases to:

$$F_T = mg + m\frac{v^2}{L}.$$

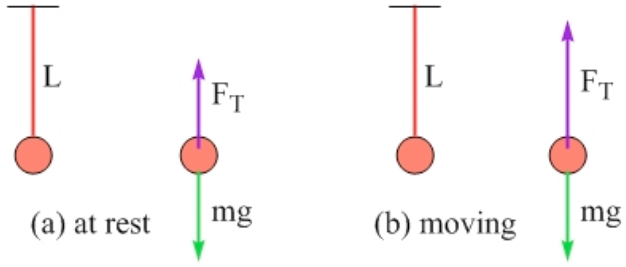

Figure 12.19: Free-body diagrams for the ball on the string when it is (a) at rest, and (b) passing through equilibrium with a speed *v*.

Chapter Summary

Essential Idea

Harmonic oscillations are important in many applications, from musical instruments to clocks and, in the human body, from walking to the creation of sounds with our vocal cords. Even though we have focused on two basic models in this chapter, the block on the spring and the simple pendulum, the same principles apply in many real-life situations.

Springs

An ideal spring obeys Hooke's Law, $\vec{F}_{Spring} = -k\,\vec{x}$. (Equation 12.1: **Hooke's Law**)

k is the spring constant, a measure of the stiffness of the spring.

Springs that are stretched or compressed store energy. This energy is known as elastic potential energy.

$U_e = \frac{1}{2}kx^2$. (Equation 12.2: **Elastic potential energy for an ideal spring**)

When an object oscillates on a spring, the angular frequency of the oscillations depends on the mass of the object and the spring constant of the spring.

$\omega = \sqrt{\frac{k}{m}}$. (Equation 12.8: **The angular frequency of a mass on a spring**)

Simple Harmonic Motion and Energy Conservation

Energy conservation is a useful tool for analyzing oscillating systems. When springs are involved we use elastic potential energy, an idea introduced in this chapter. To analyze a pendulum in terms of energy conservation nothing new whatsoever is needed.

Hallmarks of Simple Harmonic Motion

The main features of a system that undergoes simple harmonic motion include:

- No loss of mechanical energy.
- A restoring force or torque that is proportional to, and opposite in direction to, the displacement of the system from equilibrium.

In this situation the acceleration of the system is related to its position by:

$\vec{a} = -\omega^2 \vec{x}$, (Eq. 12.6: **The connection between acceleration and displacement**)

where the angular frequency ω is generally given by the square root of some elastic property of the system (such as the spring constant) divided by an inertial property (such as the mass).

Time and Simple Harmonic Motion

When we are interested in how a simple harmonic oscillator evolves over time the following equations are extremely useful. These were derived by looking at the connection between simple harmonic motion and one component of the motion of an object experiencing uniform circular motion.

$\vec{x} = A\cos(\omega t)$. (Equation 12.3: **Position in simple harmonic motion**)

$\vec{v} = -A\omega \sin(\omega t)$. (Equation 12.4: **Velocity in simple harmonic motion**)

$\vec{a} = -A\omega^2 \cos(\omega t)$. (Eq. 12.5: **Acceleration in simple harmonic motion**)

Equations 12.3 – 12.5 apply when the object is released from rest at $t = 0$ from a distance A from equilibrium.

In general, the angular frequency (ω), frequency (f), and period (T) are connected by:

$\omega = 2\pi f = \frac{2\pi}{T}$. (Eq. 12.7: **Relating angular frequency, frequency, and period**)

The Simple Pendulum

A simple pendulum, consisting of an object on the end of a string, is another good example of an oscillating system. As long as the amplitude of the oscillations is small (less than about 10˚) and mechanical energy is conserved then the motion is simple harmonic. For larger angles the motion diverges from simple harmonic because the restoring torque is not directly proportional to the angular displacement.

$\omega = \sqrt{\frac{g}{L}}$. (Equation 12.11: **Angular frequency of a simple pendulum**)

End-of-Chapter Exercises

Exercises 1 – 12 are conceptual questions that are designed to see if you have understood the main concepts of the chapter.

1. When a spring is compressed 10 cm, compared to its natural length, the spring exerts a force of 5 N. What is the spring force when the spring is stretched by 10 cm compared to its natural length, instead?

2. A block on a horizontal surface is attached to an ideal horizontal spring, as shown in Figure 12.20. When the block compresses the spring by 10 cm, the spring exerts a force of 10 N on the block. The block is then moved either left or right to a new position, where the force the spring exerts on the block has a magnitude of 20 N. How far has the block been moved? State all possible answers.

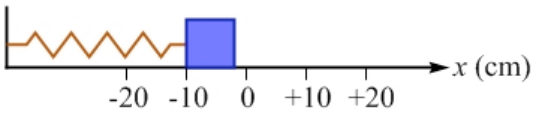

Figure 12.20: A block connected to an ideal horizontal spring. The block initially compresses the spring by 10 cm. For Exercises 2 and 3.

3. A block on a frictionless horizontal surface is attached to an ideal horizontal spring, as shown in Figure 12.20. When the block compresses the spring by 10 cm, the elastic potential energy stored in the spring is 10 J. The block is then moved either left or right to a new position, where the elastic potential energy stored in the spring is 40 J. How far has the block been moved? State all possible answers.

4. A small ball is loaded into a spring gun, compressing the spring by a distance A. When the trigger is pressed, the ball emerges from the gun with a speed v. The ball is loaded into the gun again, this time compressing the spring by a distance $2A$. With what speed will the ball emerge from the gun this time? Justify your answer.

5. A block is attached to a spring and the system is placed on a horizontal frictionless surface with the other end of the spring anchored firmly to a wall. The block is then displaced from equilibrium until 8.0 J of elastic potential energy has been stored in the spring. The block is then released from rest and the block oscillates back and forth about the equilibrium position. Sketch energy bar graphs for this system, showing the elastic potential energy, kinetic energy, and total mechanical energy when the system is (a) at the point where it is released from rest; (b) halfway between that point and the equilibrium position; (c) at the equilibrium position.

6. A block on a spring experiences simple harmonic motion with amplitude A and period T. For one complete oscillation, determine (a) the block's displacement; (b) the total distance traveled by the block; (c) the block's average velocity; (d) the block's average speed; (e) the block's average acceleration.

7. Consider the following four cases. In each case, the block experiences simple harmonic motion of amplitude A.
   Case 1: a block of mass m connected to a spring of spring constant k.
   Case 2: a block of mass m connected to a spring of spring constant $2k$.
   Case 3: a block of mass $2m$ connected to a spring of spring constant k.
   Case 4: a block of mass $2m$ connected to a spring of spring constant $2k$.
   Rank these cases, from largest to smallest, based on (a) their angular frequency; (b) their total mechanical energy; (c) the maximum speed reached by the block during its motion. Your answers should have a form similar to 3>1>2=4.

8. You have three blocks, of mass m, $2m$, and $3m$, and three springs of spring constant k, $2k$, and $3k$. You can attach any one of the blocks to any one of the springs, displace the block from equilibrium by a distance of A, $2A$, or $3A$, and release the block from rest so it experiences simple harmonic motion. Which combination of these three parameters (mass, spring constant, and amplitude) results in oscillations with the largest (a) angular frequency? (b) period? (c) total mechanical energy? (d) speed when the block passes through equilibrium?

9. A block of mass m is connected to a spring with a spring constant k, displaced a distance A from equilibrium. Upon being released from rest, the block experiences simple harmonic motion. Let's say you wanted to double the mechanical energy of the system. (a) Could you accomplish this by changing the mass, but keeping everything else the same? If so, what would the new mass be? (b) Could you accomplish this by changing the spring constant, but keeping everything else the same? If so, what would the new spring constant be? (c) Could you accomplish this by changing the amplitude of the oscillation, but keeping everything else the same? If so, what would the new amplitude be?

10. Repeat Exercise 9, except this time you want to double the angular frequency instead of the energy.

11. Consider the following four simple pendula. In each case, the pendulum experiences simple harmonic motion with a maximum angular displacement θ_{max}, where θ_{max} is small enough that the small-angle approximation can be used.
Case 1: a pendulum consisting of a ball of mass m on a string of length L.
Case 2: a pendulum consisting of a ball of mass m on a string of length $2L$.
Case 3: a pendulum consisting of a ball of mass $2m$ on a string of length L.
Case 4: a pendulum consisting of a ball of mass $2m$ on a string of length $2L$.
Rank these cases, from largest to smallest, based on (a) their angular frequency; (b) their total mechanical energy; (c) the maximum speed reached by the block during its motion. Your answers should have the form 3>1>2=4.

12. Return to Exercise 11. Now rank the cases, from largest to smallest, based on the tension in the string when the ball passes through the equilibrium position.

Exercises 13 – 17 deal with various situations involving ideal springs.

13. A spring hangs vertically down from a support, with a ball with a weight of 6.00 N hanging from the spring's lower end. (a) If the ball remains at rest and the spring is stretched by 20.0 cm with respect to its natural length, what is the spring constant of the spring?

14. Consider again the situation described in Exercise 13. You now take the spring, cut it in half, and hang the same ball from one half of the spring (the other half you don't use at all) so the ball again remains at rest as it hangs vertically from the spring. (a) How much is the spring stretched from its natural length? Briefly justify your answer. (b) How does the spring constant of this new spring compare to the spring constant of the original spring?

15. A small ball with a mass of 50 g is loaded into a spring gun, compressing the spring by 12 cm. When the trigger is pressed, the ball emerges horizontally from the barrel at a height of 1.4 m above the floor. It then strikes the floor after traveling a horizontal distance of 2.5 m. Use $g = 9.8$ m/s^2. Assuming all the energy stored in the spring is transferred to the ball, determine the spring constant of the spring.

16. A block of mass M is connected to a spring of spring constant k. The system is placed on a frictionless horizontal surface and the other end of the spring is firmly fixed to a wall so the block, when displaced from equilibrium a distance A and then released from rest, will experience simple harmonic motion. A second block of mass m is then placed on top of the first block. The coefficient of static friction associated with the interaction between the two blocks is μ_S. What is the maximum value A can be so the blocks oscillate together without the top block slipping on the bottom block?

17. A block of mass $M = 0.800\,\text{kg}$ is connected to a spring of spring constant $k = 2.00\,\text{N/m}$. The system is placed on a frictionless horizontal surface and the other end of the spring is firmly fixed to a wall so the block, when displaced from equilibrium a distance A and then released from rest, will experience simple harmonic motion. A second block of mass $m = 0.600\,\text{kg}$ is then placed on top of the first block. The coefficient of static friction associated with the interaction between the two blocks is $\mu_S = 0.500$. Use $g = 9.80$ m/s^2. (a) What is the maximum value A can be so the blocks oscillate together without the top block slipping on the bottom block? (b) What is the angular frequency in this situation?

Exercises 18 – 23 deal with various aspects of the situation shown in Figure 12.21.

18. A block with a mass of 0.500 kg is released from rest from the top of a ramp that has the form of a 3-4-5 triangle, measuring 3.00 m high and having a base of 4.00 m, as shown in Figure 12.21. The block then slides down the incline, encountering a spring with a spring constant of 5.00 N/m after sliding for 2.50 m. Neglect friction and use g = 9.80 m/s^2. Where does the block reach its maximum speed, at the point it first makes contact with the spring, or at a point higher up the ramp or lower down the ramp than where it first makes contact with the spring? Briefly justify your answer.

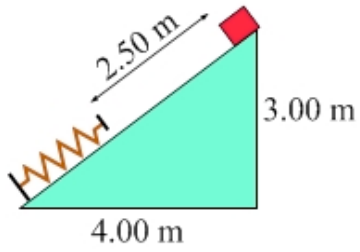

Figure 12.21: A block released from rest from the top of a ramp slides for a distance of 2.50 m before it encounters a spring. For Exercises 18 – 23.

19. Return to the situation described in Exercise 18. Determine (a) how far the block has slid down the ramp when the block reaches its highest speed; (b) the value of this maximum speed.

20. Return to the situation described in Exercise 18. Find the maximum compression of the spring in this situation.

21. Return to the situation described in Exercise 18, but now we'll add friction between the block and the ramp. The coefficient of kinetic friction between the block and the ramp is 0.400. Where does the block reach its maximum speed now, at exactly the same point on the ramp it did in Exercise 18, or at some place higher up or lower down than this point? Briefly justify your answer.

22. Return to the situation described in Exercise 18, but now we'll add friction between the block and the ramp. The coefficient of kinetic friction between the block and the ramp is 0.400. Determine (a) how far the block has slid down the ramp when the block reaches its highest speed; (b) the value of this maximum speed.

23. Return to the situation described in Exercise 18, but now we'll add friction between the block and the ramp. The coefficient of kinetic friction between the block and the ramp is 0.400. Find the maximum compression of the spring in this situation.

Exercises 24 – 30 deal with energy and energy conservation in oscillating systems.

24. A block with a mass of 0.500 kg that is attached to a spring is oscillating back and forth on a frictionless horizontal surface. The period of the oscillations is 2.00 s. When the block is 30.0 cm from its equilibrium position, its speed is 1.20 m/s. What is the amplitude of the oscillations?

25. Consider again the system described in Exercise 24. At a time of $T/4$ after being released from rest, the block is passing through the equilibrium position. At a time of $T/8$ after being released, determine the system's (a) elastic potential energy; (b) kinetic energy.

26. Consider again the system described in Exercise 24, but now we'll make it more realistic. There is a small coefficient of friction associated with the interaction between the block and the surface. This means that, over time, the amplitude of the oscillations decrease until eventually the block comes to rest and remains at rest. Approximately how much work is done by friction on the block during this process?

27. A block is attached to a spring, displaced from equilibrium a distance of 0.800 m, and released from rest. It then oscillates on a frictionless horizontal surface with a period of 4.00 s. At the instant the block is released from rest, the energy in the system is all elastic potential energy, as shown in the set of energy bar graphs in Figure 12.22(a). (a) At how many locations in the subsequent oscillations do we get the set of energy bar graphs shown in Figure 12.22(b)? (b) Determine the distance of each of these locations from the equilibrium position.

28. Return to the situation described in Exercise 27. (a) During one complete oscillation, at how many different times does the energy correspond to the set of energy bar graphs shown in Figure 12.22(b)? (b) Assuming the block is released from rest at $t = 0$, determine all the times during the first complete oscillation when the system's energy corresponds to the energy bar graphs shown in Figure 12.22(b).

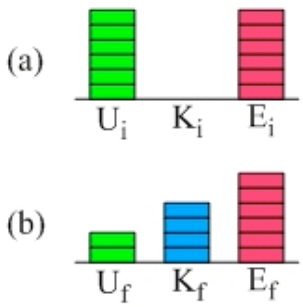

Figure 12.22: Energy bar graphs for a block on a spring at (a) its release point, and (b) some other point. For Exercises 27 and 28.

29. A ball is tied to a string to form a simple pendulum with a length of 1.20 m. The ball is displaced from equilibrium by some angle θ_i and released from rest. In the subsequent oscillations, the ball's maximum speed is 2.50 m/s. (a) From what height above equilibrium was the ball released? (b) What is θ_i ? Use $g = 9.80$ m/s^2.

30. As shown in Figure 12.23 a simple pendulum with a length of 1.00 m is released from rest from an angle of 20° measured from the vertical. When the ball passes through its equilibrium position the string hits a peg, effectively shortening the length of the pendulum to 50.0 cm. (a) How does the maximum height reached by the pendulum on the right compare to the height of the ball at its release point on the left? Justify your answer. (b) What is the maximum angle, measured from the vertical, of the string when the ball is on the right?

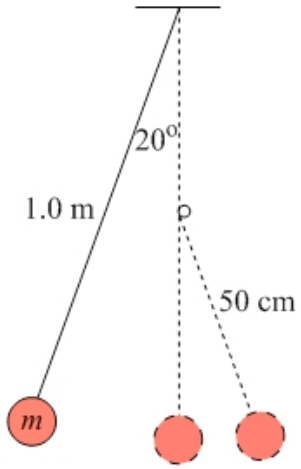

Figure 12.23: A simple pendulum swings down through its equilibrium position, and then hits a peg that effectively shortens its length to 50 cm. For Exercises 30 and 31.

Exercises 31 – 35 deal with time and simple harmonic motion.

31. Return to the situation described in Exercise 30 and shown in Figure 12.23. What is the period of one complete oscillation for this pendulum? Use $g = 9.80$ m/s^2.

32. A block with a mass of 0.600 kg is connected to a spring, displaced in the positive direction a distance of 50.0 cm from equilibrium, and released from rest at $t = 0$. The block then oscillates without friction on a horizontal surface. The first time the block is a distance of 15.0 cm from equilibrium is at $t = 0.200$ s. Determine (a) the period of oscillation; (b) the value of the spring constant; (c) the block's velocity at $t = 0.200$ s; and (d) the block's acceleration at $t = 0.200$ s.

33. Repeat Exercise 32, but now $t = 0.200$ s represents the second time the block is a distance of 15.0 cm from equilibrium.

34. A block on a spring is released from rest from a distance A from equilibrium at $t = 0$. The block then experiences simple harmonic motion with a period T. Determine all the times during the first complete oscillation when the block is a distance $A/4$ from equilibrium.

35. A block with a mass of 0.800 kg is connected to a spring, displaced in the positive direction a distance of 40.0 cm from equilibrium, and released from rest at $t = 0$. The block then oscillates without friction on a horizontal surface. At a time of $t = 0.500$ s, the block is 30.0 cm from equilibrium. If the block's period of oscillation is longer than 0.500 s, determine the spring constant of the spring. Find all possible answers.

Exercises 36 – 39 combine collisions with simple harmonic motion situations.

36. A wheeled cart with a mass of 0.50 kg is rolling along a horizontal track at a constant velocity of 2.0 m/s when it experiences an elastic collision with a second identical cart that is initially at rest, and attached to a spring with a spring constant of 4.0 N/m. This situation is illustrated in Figure 12.24. After the collision, the second cart moves to the right. (a) What is the first cart doing after the collision? Briefly justify your answer. (b) What is the maximum compression of the spring? (c) There is a second collision between the carts. What is each cart doing after the second collision?

Figure 12.24: A cart collides with a second cart that is initially at rest and attached to a spring. For Exercises 36 and 37.

37. A wheeled cart with a mass of 0.50 kg is rolling along a horizontal track at a constant velocity of 2.0 m/s when it experiences a collision with a second identical cart that is initially at rest, and attached to a spring with a spring constant of 4.0 N/m. This situation is illustrated in Figure 12.24. After the collision, the carts stick together and move as one unit. What is the maximum compression of the spring?

38. In a spring version of the ballistic pendulum situation we looked at in Chapter 7, a wooden block with a mass of 0.500 kg is attached to a spring with a spring constant of k = 600 N/m. As shown in Figure 12.25, the system is placed on a frictionless horizontal surface with the block at rest at the equilibrium position. A bullet with a mass of 30.0 g is fired at the block. The bullet gets embedded in the block and, after the collision, the block experiences simple harmonic motion with an amplitude of 15.0 cm. Assuming the bullet's velocity is horizontal at the instant the collision takes place, what is the speed of the bullet just before it hits the block?

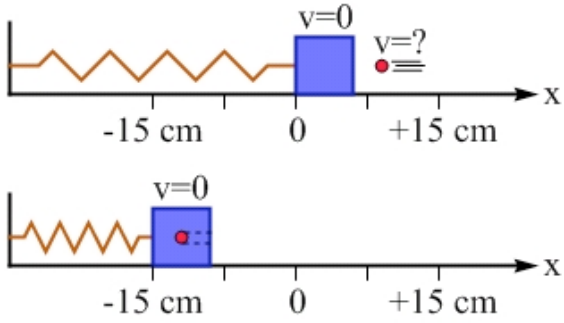

Figure 12.25: A bullet embeds itself in a block. The subsequent oscillations take place with an amplitude of 15.0 cm. For Exercise 38.

39. As shown in Figure 12.26, a ball of mass m is tied to a string to form a simple pendulum. The ball is displaced from equilibrium so the angle between the string and the vertical is 60˚, and is then released from rest. It swings down, and at its lowest point it collides with a second ball of mass 4m that is initially at rest on the edge of a table. If the collision is elastic and the second ball strikes the floor at a point 1.50 m vertically lower and 1.20 m horizontally from where it started, find the length of the string the first ball is attached to.

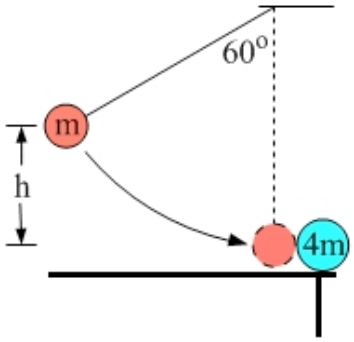

Figure 12.26: A ball of mass m, tied to a string to form a simple pendulum, swings down and collides elastically with a ball of mass 4m. For Exercise 39.

General Problems and Conceptual Questions

40. A block of mass m is connected to a spring of spring constant k, displaced a distance A from equilibrium and released from rest. An identical block is connected to a spring of spring constant 4k and released from rest so its total mechanical energy is equal to that of the first block-spring system. (a) Assuming the blocks are simultaneously released from rest and that they each experience simple harmonic motion horizontally, sketch graphs of the total mechanical energy, elastic potential energy, and kinetic energy as a function of displacement from equilibrium. (b) Repeat part (a), but this time sketch graphs of the three types of energy as a function of time instead.

41. Return to the situation described in Exercise 40. Now plot graphs, as a function of time, of the (a) position; (b) velocity; and (c) acceleration for both of the blocks.

42. You are trying to demonstrate to your friend the connection between uniform circular motion and simple harmonic motion. You have a motorized turntable that spins at a rate of exactly 1 revolution per second, and you glue a ball to the turntable at a distance of 50 cm from the center. You then take a small bucket with a mass of 100 g and connect it to a spring that has a spring constant of 4.00 N/m. The bucket will then oscillate back and forth on a frictionless surface. (a) What mass of sand should you place in the bucket so the period of oscillation of the bucket matches the period of revolution of the ball on the turntable? (b) Assuming you have lined up the equilibrium position of the bucket on the spring with the center of the turntable, what amplitude should you give the bucket so its motion exactly matches one component of the motion of the ball on the turntable?

43. You match the motion of two objects, a ball glued to a turntable that is rotating at a constant angular velocity and a block oscillating on a frictionless surface because it is connected to a spring. (a) If the period of the block's oscillations is 2.5 s, what is the angular frequency of the turntable? (b) If you replace the spring by a spring with double the original spring constant, what should the angular frequency of the turntable be?

44. Consider the two graphs shown in Figure 12.27 for a block that oscillates back and forth on a frictionless surface because it is connected to a spring. The graph on the left shows the block's displacement from equilibrium, as a function of time, for one complete oscillation of the block. The graph on the right shows the elastic potential energy stored in the spring, as a function of time, over the same time period. (a) What is the spring constant of the spring? (b) What is the mass of the block? (c) What is the maximum speed reached by the block as it oscillates?

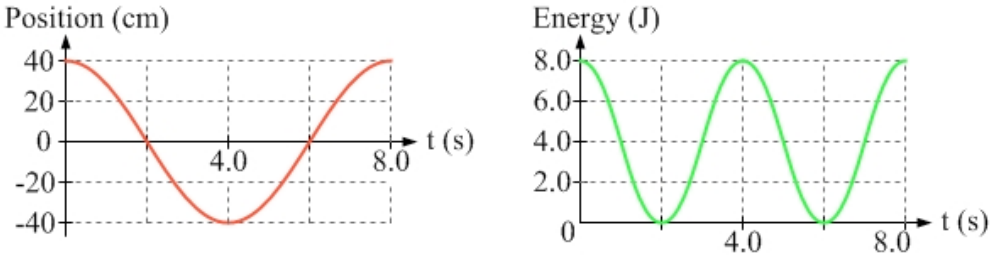

Figure 12.27: Graphs of position as a function of time and elastic potential energy as a function of time for a block oscillating horizontally on a spring. For Exercises 44 and 45.

45. Using only the information available to you in the position vs. time graph shown in Figure 12.27, determine (a) the maximum speed of the oscillating block; (b) the magnitude of the maximum acceleration of the oscillating block.

46. A block with a mass of 500 g is connected to a spring with a spring constant of 2.00 N/m. You start the motion by hitting the block with a stick so that, at $t = 0$, the block is at the equilibrium position but has an initial velocity of 2.00 m/s in the positive direction. The block then oscillates back and forth without friction. Over two complete cycles of the resulting oscillation, plot, as a function of time, the block's (a) position; (b) velocity; and (c) acceleration.

47. Equations 12.3 – 12.5 are ideal for describing the motion of an object experiencing simple harmonic motion after having been displaced from equilibrium and released from rest at $t = 0$. Modify the three equations so they match the motion described in Exercise 46.

48. As shown in Figure 12.28, two springs are separated by a distance of 1.60 m when the springs are at their equilibrium lengths. The spring on the left has a spring constant of 15.0 N/m, while the spring on the right has a spring constant of 7.50 N/m. A block, with a mass of 400 g, is then placed against the spring on the right, compressing it by a distance of 20.0 cm, which also happens to be the width of the block. The block is then released from rest. (a) Assuming all the energy initially stored in the spring is transferred to the block, and that the horizontal surface is frictionless, how long will it take until the block returns to its release point? (b) Repeat the question, but now assume that the block is held against the spring on the left, and released from rest after compressing that spring by 20 cm.

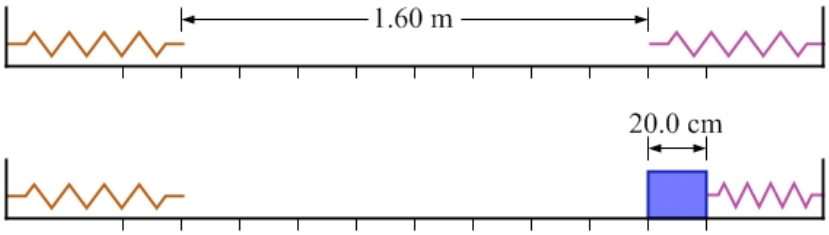

Figure 12.28: Two springs are separated by a distance of 1.60 m. A block is then held against one spring and released from rest. For Exercise 48.

49. A ballistic cart is a cart containing a ball on a compressed spring. In a popular demonstration, the cart is rolled with a constant horizontal velocity past a trigger, which causes the spring to be released, firing the ball vertically (with respect to the cart) into the air. The ball then lands in the cart again 0.60 s later. If the ball has a mass of 22 grams and the spring is initially compressed by 7.5 cm, determine the spring constant of the spring. Assume all the energy stored in the spring is transferred to the ball, and use $g = 9.8\ \text{m/s}^2$.

50. A block with a mass of 0.600 kg is connected to a spring with a spring constant of 4.50 N/m. The block is displaced a distance A from equilibrium and released from rest. How long after being released is the block first (a) at the equilibrium position? (b) at a point $A/4$ from equilibrium?

51. A block on a spring of spring constant $k = 12.0$ N/m experiences simple harmonic motion with a period of 1.50 s. What is the block's mass?

52. Consider again the situation described in Exercise 51. Such a system is used by astronauts in orbit to measure their own masses. Do a web search for "body mass measurement device" (the name of this system) and write a paragraph or two describing how it works.

53. Among the many things that Galileo Galilei is known for are his observations about pendula. Do some research about Galileo and write a paragraph or two describing his contributions to our understanding of the simple pendulum.

54. A particular wooden block floats in water with 30% of its volume submerged. You then push the block farther under the water so that 40% of its volume is submerged. When you let go the block bobs up and down. (a) For simple harmonic motion, there must be a restoring force proportional, and opposite in direction, to the displacement from equilibrium. Considering the net force on the block from combining the buoyant force and the force of gravity, does that net force fit the requirement necessary for simple harmonic motion? (b) Write an expression for the angular frequency of the block's oscillations.

55. Return again to the situation described in Exercise 54. The block has a mass of 0.30 kg. (a) Draw a free-body diagram showing the forces acting on the block immediately after it is released from rest. (b) Using $g = 10$ m/s^2, determine the block's initial acceleration. (c) Describe what happens to the block's free-body diagram as the block moves.

56. You are at the playground with a young boy who has a mass of 20 kg. When the boy is on a swing, you observe that you push him exactly once every 2.0 s. How long are the ropes attaching the swing to its support? Use $g = 9.8$ m/s^2.

57. A simple pendulum consists of a ball with a mass of 0.500 kg attached to a string of length L. The ball is displaced from equilibrium so that, when the ball is released from rest, it is at a level 1.00 m above its equilibrium position, and the string makes a 60˚ angle with the vertical. Use $g = 9.80$ m/s^2. (a) What is the length of the string? (b) Apply energy conservation to find the speed of the ball as it passes through the equilibrium position. (c) Using the small-angle approximation, it can be shown that the maximum speed of the pendulum ball is given by $v_{max} = L\theta_{max}\omega$. Making sure that your units are correct, use this equation to check your answer to part (b). (d) Your results in parts (b) and (c) should be close but should not agree exactly. Comment on which answer is better and why there is any disagreement at all.

58. Return to the situation described in Exercise 57. What is the tension in the string when the ball passes through the equilibrium position?

59. Return to the situation described in Exercise 57. After many oscillations, air resistance and friction eventually bring the pendulum to a stop. What is the total work done by resistive forces in this situation?

60. As shown in Figure 12.29, two simple pendula are identical except that the mass of the ball on one pendulum is 3 times the mass of the ball on the other. Each pendulum has a length of 1.5 m. The pendula are displaced by angles of 20˚, but in opposite directions, and simultaneously released from rest. The balls then experience an elastic collision with one another. (a) What is the velocity of each ball immediately after the first collision? (b) The balls experience a second elastic collision. What is the maximum angular displacement reached by each ball as a direct result of this second collision? (c) Describe, in general, how the motion proceeds after that.

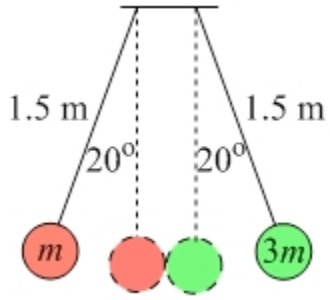

Figure 12.29: Two simple pendula, one with a ball of mass m and the other with a ball of mass $3m$, are displaced by 20˚ and released from rest. They swing down and collide with one another. What happens next? For Exercises 60 and 61.

61. Consider again the situation described in Exercise 60. Determine the time taken by each phase of the motion.

62. A grandfather clock uses a pendulum to keep time. For this exercise, treat the pendulum as a simple pendulum and use $g = 9.80$ m/s^2. (a) How long should the pendulum be if its period needs to be exactly 4 seconds for the clock to keep accurate time? (b) You now have two identical grandfather clocks, both set to keep accurate time on Earth. You take one to the Moon, where the magnitude of the gravitational field is 1/6 what it is on Earth. The clocks are started simultaneously when they both read 12 o'clock. One hour later, the clock on the Earth reads 1 o'clock. What is the time shown on the clock on the Moon at that instant?

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- Simulation: A Simple Pendulum
- Simulation: Two Springs and a Pendulum

Additional Links

- *PhET* simulation: Masses and Springs
- *PhET* simulation: Pendulum Lab