10-1 Rotational Kinematics

Kinematics is the study of how things move. Rotational kinematics is the study of how rotating objects move. Let's start by looking at various points on a rotating disk, such as a compact disc in a CD player.

EXPLORATION 10.1 - A rotating disk

Step 1 – ***Mark a few points on a rotating disk and look at their instantaneous velocities as the disk rotates.*** Let's assume the disk rotates counterclockwise at a constant rate. Even though the rotation rate is constant, we observe that each point on the disk has a different velocity. The instantaneous velocities of five different points are shown in Figure 10.1. Points at the same radius have equal speeds, but their velocities are different because the directions of the velocities are different. We also observe that the speed of a point is proportional to its distance from the center of the disk.

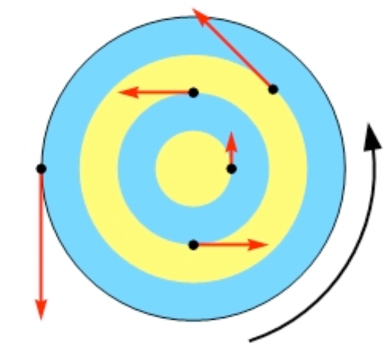


Figure 10.1: Instantaneous velocities, shown as red arrows, of various points on a disk rotating counterclockwise.

Step 2 – ***Plot the paths followed by various points on the disk as the disk spins through 1/8th of a full rotation.*** Figure 10.2 shows that each point travels on a circular arc, and the distance traveled by a particular point increases as the distance of that point from the center increases.

Step 3 – ***What is the same for all the arcs shown in Figure 10.2?*** One thing that is the same is the angle (45°, in this case) the points move through, measured from the center of the disk. This leads to an interesting conclusion. Maybe the parameters we used (position, velocity, and acceleration) to study projectile and one-dimensional motion are not the most natural parameters to use when describing rotational motion. For instance, in a given time interval every point on the rotating disk has a unique displacement, yet each point has the same angular displacement.

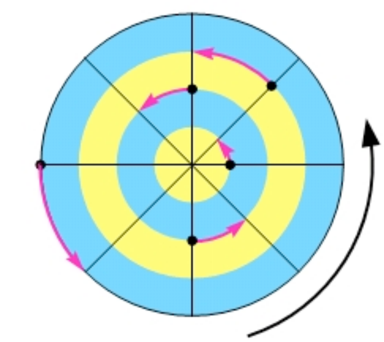


Figure 10.2: The circular arcs followed by five different points on the disk, as the disk moves through 1/8th of a full rotation.

Step 4 – ***Is there a connection between the distance traveled by a particular point in a given time interval (let's call this the arc length, s) and the corresponding angle, θ, of the arc the point moves along?*** Absolutely. The connection between arc length and angle is:

$$s = r\theta \qquad \text{(Equation 10.1: \textbf{Arc length})}$$

where r is the radius of the arc (the distance from the point to the center). Note that the angle must be in units of radians in this equation, and that π radians $= 180^\circ$.

There is an equivalent relationship for speed. Each point on the disk has a unique velocity, but each point moves through the same angle in a given time interval. Thus, every point has the same angular velocity, a quantity we symbolize using the Greek letter omega, $\vec{\omega}$. Angular velocity is related to angular displacement $\Delta\vec{\theta}$ in the same way that velocity is related to displacement. $\vec{v} = \Delta\vec{r} / \Delta t$, so for the angular variables we have:

$$\vec{\omega} = \frac{\Delta\vec{\theta}}{\Delta t}. \qquad \text{(Equation 10.2: \textbf{Angular velocity})}$$

Step 5 – ***What is the connection between the instantaneous velocity of a point on the rotating disk and the disk's angular velocity?*** The instantaneous velocity of a point is called the tangential velocity, $\vec{v}_T$, because the direction of the velocity is always tangential to the circular path followed by the point. At a given instant in time, every point on the disk has the same angular speed ω (this is the magnitude of the angular velocity, $\vec{\omega}$). As we noted in Step 1, however, the speed of a particular point is proportional to *r,* its distance from the center. The connection between the tangential speed v_T , and the angular speed ω is:

$$v_T = r\omega .$$ (Eq. 10.3: **Connecting tangential speed to angular speed**)

Step 6 – ***How do we describe motion when the rotating object is not rotating at a constant rate?*** If the rotating disk spins with constant angular velocity we can fully describe the motion of any point on the disk using the parameters described above (and time, *t*). If the angular velocity is changing, however, such as if the disk is speeding up or slowing down, we need an additional parameter to describe motion. This is the angular acceleration $\vec{\alpha}$, the rotational equivalent of the acceleration, defined as:

$$\vec{\alpha} = \frac{\Delta\vec{\omega}}{\Delta t} .$$ (Equation 10.4: **Angular acceleration**)

Key ideas for rotational motion: To describe rotational motion, we use the rotational variables $\theta, \omega,$ and α . These are more natural variables to use, instead of the more familiar *r*, *v*, and *a*, because every point on a rotating object has the same angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$, while each point has unique values of position, velocity, and acceleration.
Related End-of-Chapter Exercises: 1, 43.

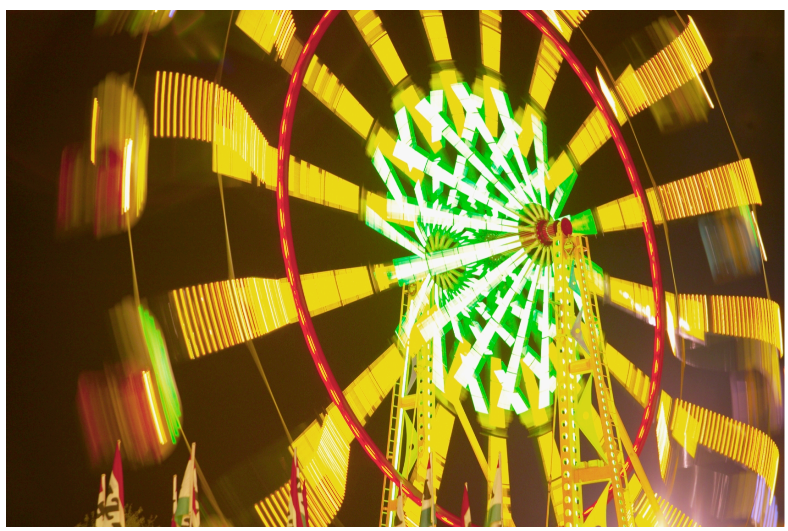


Figure 10.3: In this time exposure image of a rotating Ferris wheel, note how the tracks left by the parts farther away from the center are longer than those made by parts closer to the center. Comparing this picture to the diagram of the rotating disk in Figure 10.2, where we see the length of the tracks increasing as the distance from the center increases, we can understand why parts farther from the center of the Ferris wheel leave longer tracks on the photograph.
Photo credit: Gisele Wright / iStockPhoto.

Essential Question 10.1: If you travel a distance of 1.0 m as you walk around a circle that has a radius of 1.0 m, through what angle have you walked? Comment on the units here.

Answer to Essential Question 10.1: Let's re-arrange equation 10.1 to $\theta = s/r$. Thus, an arc length that is equal to the radius corresponds to an angle of 1.0 radian, which is about 57°. If the arc length and the radius have units of meters, the units cancel on the right side of the equation and we have units of radians on the left side. This violates the general rule that units have to match on two sides of an equation. We have two ways around this. One way is to treat the radian as dimensionless. Another way is to define the radius as having units of meters/radian.

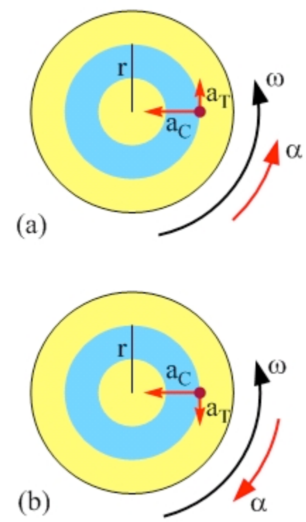
10-2 Connecting Rotational Motion to Linear Motion

The angular variables we defined in Section 10-1 are vectors, so they have a direction. In which direction is the angular velocity of the disk shown in Figure 10.2? If we all observe the disk from the same perspective we can say that the direction is counterclockwise. In practice, we will generally use clockwise or counterclockwise to specify direction. In actuality, however, the direction is given by the **right-hand rule**. When you curl the fingers on your right hand in the direction of motion and stick out your thumb, your thumb points in the direction of the angular velocity. This is straight up out of the page for the disk in Figure 10.2.

EXPLORATION 10.2 – Connecting angular acceleration to acceleration

We can connect the magnitudes of the acceleration and angular acceleration in the same way that the distance traveled along an arc is connected to the angle ($s = r\theta$) and the speed is connected to the angular speed ($v = r\omega$). How?

Imagine yourself a distance *r* from the center of a rotating turntable, moving with the turntable. If the turntable has a constant angular velocity, you have no angular acceleration, but you have a centripetal acceleration, $\vec{a}_C = v^2/r$, directed toward the center of the turntable. The angular acceleration, α, cannot be connected to the centripetal acceleration by a factor of *r*, because $\alpha = 0$ in this case.



You have a non-zero angular acceleration if the turntable (and you) speeds up or slows down. If the turntable speeds up, the acceleration has two components (see Figure 10.4(a)), a centripetal acceleration $\vec{a}_C$ toward the center, and a component tangent to the circular path, which is called the tangential acceleration $\vec{a}_T$. If the turntable slows down, then the tangential acceleration reverses direction (see Figure 10.4(b)), as does the angular acceleration (because the angular velocity is decreasing instead of increasing). Thus, the magnitude of the tangential acceleration is directly related to the magnitude of the angular acceleration:

$a_T = r\alpha$ (Eq. 10.5: **Connecting tangential and angular accelerations**)

Figure 10.4: If you are rotating with a turntable as it speeds up (a) or slows down (b), your acceleration has two components, a centripetal component directed toward the center and a tangential component $\vec{a}_T$.

Key idea for angular acceleration: The angular acceleration $\vec{\alpha}$ is directly related to the tangential acceleration $\vec{a}_T$ (the component of acceleration tangent to the circular path), and is not related to the centripetal acceleration $\vec{a}_C$. **Related End-of-Chapter Exercises: 44, 45.**

Equations for motion with constant angular acceleration

In Chapter 2, we considered one-dimensional motion with constant acceleration, and used three main equations to analyze motion. The analogous equations for rotational motion are summarized in Table 10.1. Note the parallels between the two sets of equations.

Straight-line motion equation		Analogous rotational motion equation	
$v = v_i + a\,t$	(Equation 2.9)	$\omega = \omega_i + \alpha\,t$	(Equation 10.6)
$x = x_i + v_i\,t + \frac{1}{2}a\,t^2$	(Equation 2.11)	$\theta = \theta_i + \omega_i\,t + \frac{1}{2}\alpha\,t^2$	(Equation 10.7)
$v^2 = v_i^2 + 2a\,\Delta x$	(Equation 2.12)	$\omega^2 = \omega_i^2 + 2\alpha\,\Delta\theta$	(Equation 10.8)

Table 10.1: Each kinematics equation has an analogous rotational-motion equation.

EXAMPLE 10.2 – Drawing a motion diagram for rotational motion

A turntable starts from rest, and has a counterclockwise angular acceleration of $(\pi/3)\text{ rad/s}^2$. Sketch a motion diagram for an object 1.0 m from the center that rotates with the turntable, plotting its position at 0.50 s intervals for the first 3.0 s.

SOLUTION

Let's use equation 10.7 to find the object's angular position at 0.50-second intervals. The object starts at the position shown by the red circle in Figure 10.5 – the horizontal line will be the origin. Take counterclockwise to be positive, and then set up a table (see Table 10.2) summarizing what we know. This is similar to what we did for one-dimensional motion.

Parameter	Value
Positive direction	Counterclockwise
Initial position	$\theta_i = 0$
Initial angular velocity	$\omega_i = 0$
Angular acceleration	$\alpha = +(\pi/3)\text{ rad/s}^2$

Table 10.2: Summarizing the initial information about the object.

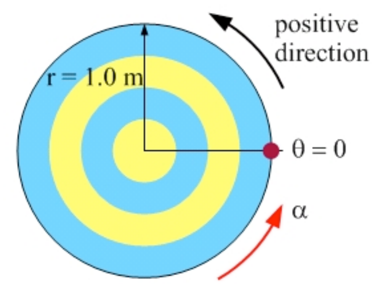


Figure 10.5: The initial situation for the rotating object.

Using the values from Table 10.2, Equation 10.7 simplifies to: $\theta = \theta_i + \omega_i\,t + \frac{1}{2}\alpha\,t^2 = 0 + 0 + \frac{1}{2}\left(\frac{\pi}{3}\text{rad/s}^2\right)t^2 = +\left(\frac{\pi}{6}\text{rad/s}^2\right)t^2$.

Substituting different times into this equation gives the angular position of the object at the times of interest, as summarized in Table 10.3.

Time (s)	0	0.50	1.00	1.50	2.00	2.50	3.00
Angular position (radians)	0	$+\pi/24$	$+\pi/6$	$+3\pi/8$	$+2\pi/3$	$+25\pi/24$	$+3\pi/2$
Angular position (°)	0	+7.5	+30	+67.5	+120	+187.5	+270

Table 10.3: The angular position of the object at 0.50-second intervals.

Using the information in Table 10.3, we can sketch a motion diagram for the object. The motion diagram is shown in Figure 10.6.

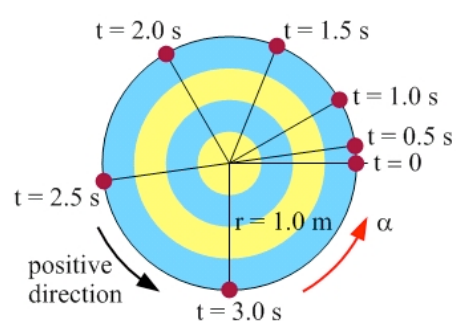


Figure 10.6: A motion diagram for an object moving with an accelerating turntable, showing the position at 0.5-second intervals.

Essential Question 10.2: If we repeated Example 10.2, for an object at a radius of 0.5 m from the center of the turntable, what would change in Table 10.3? Assume the object has an angular position of zero at $t = 0$.

Answer to Essential Question 10.2: Because Table 10.3 deals with angular variables, which are independent of the radius, nothing would change in the table.

10-3 Solving Rotational Kinematics Problems

EXPLORATION 10.3A – Unrolling the motion

Return to the situation from Example 10.2. Let's take the object's path, which is a circular arc (see Figure 10.5), and unroll it so it is a straight line. How would we analyze this straight-line motion? Unrolling the circular arc from Figure 10.6 gives the straight-line motion situation shown in Figure 10.7. Let's define the line the object moves along to be the *x*-axis. The origin is the object's initial position, and the positive direction is to the right. This situation should look familiar, because it is an excellent example of one-dimensional motion with constant acceleration, as we studied in Chapter 2.

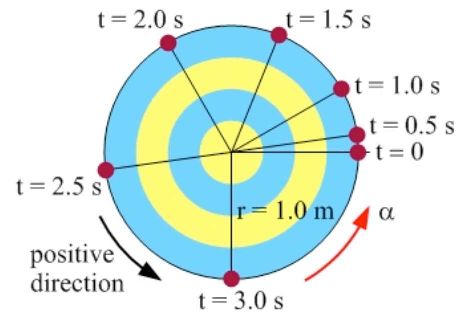


Figure 10.6: A motion diagram for an object moving with an accelerating turntable, showing the position at 0.5-second intervals.

For the rotational situation, the distance traveled is the length of the circular arc the object moves along. After unrolling the arc to get a straight line, we can use Equation 10.1, $s = r\theta$, to find the arc length corresponding to the distance traveled from the origin. When using this equation, use angles in radians. Table 10.4 builds on Table 10.3, bringing in a row for the arc length *s*, which is the same as the position, *x*, for the equivalent one-dimensional motion. Because we have the special case $r = 1.0$ m, s and θ are numerically equal, and differ only in their units.

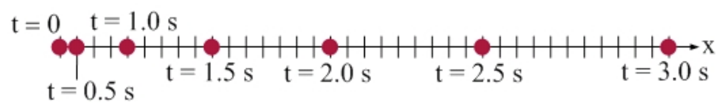


Figure 10.7: The straight-line motion resulting from straightening the circular arc traveled by the object in Figure 10.6.

Time (s)	0	0.50	1.00	1.50	2.00	2.50	3.00
Angular position, θ (rad)	0	$+\pi/24$	$+\pi/6$	$+3\pi/8$	$+2\pi/3$	$+25\pi/24$	$+3\pi/2$
***s* or *x* (m)**	0	$+\pi/24$	$+\pi/6$	$+3\pi/8$	$+2\pi/3$	$+25\pi/24$	$+3\pi/2$

Table 10.4: Determining the arc length, and the displacement in the corresponding 1-dimensional motion situation, for the object on the turntable.

Key idea: Rotational motion with constant angular acceleration is analogous to one-dimensional motion with constant acceleration. **Related End-of-Chapter Exercises: 2, 39.**

Based on Example 10.2 and Exploration 10.3A, let's write down a general method for solving rotational kinematics problems. The method parallels the method used in Chapter 2 for solving one-dimensional kinematics problems.

A General Method for Solving a Rotational Kinematics Problem

1. Draw a diagram of the situation.
2. Choose an origin to measure positions from, and mark it on the diagram.
3. Choose a positive direction, and mark this on the diagram with an arrow.
4. Organize what you know, and what you're looking for. Making a date table is a useful way to organize the information.
5. Think about which of the constant-acceleration equations to apply, and then set up and solve the problem. The three main equations are:

$$\omega = \omega_i + \alpha\, t\,. \qquad \text{(Equation 10.6)}$$

$$\theta = \theta_i + \omega_i\, t + \frac{1}{2}\alpha\, t^2\,. \qquad \text{(Equation 10.7)}$$

$$\omega^2 = \omega_i^2 + 2\alpha\, \Delta\theta\,. \qquad \text{(Equation 10.8)}$$

EXPLORATION 10.3B – Graphs for rotational motion

Plot a set of graphs showing, as a function of time, the angular acceleration, the angular velocity, and the angular position of the object on the turntable we considered in Exploration 10.3A. How does this set of graphs compare to graphs showing, as a function of time, the acceleration, velocity, and position of the equivalent straight-line motion situation that we considered in Exploration 10.3A?

The angular acceleration is constant, with a value of $+(\pi/3)\ \text{rad/s}^2$. The graph of the angular acceleration is the horizontal line shown at the top of Figure 10.8.

To graph angular velocity as a function of time, we can use Equation 10.6, $\omega = \omega_i + \alpha\, t$. Substituting values for the initial angular velocity and the angular acceleration gives: $\omega = 0 + (\pi/3)\ \text{rad/s}^2 \times t = (\pi/3)\ \text{rad/s}^2 \times t$.

This function is a straight line, starting from the origin, with a constant slope, as shown in the middle graph of Figure 10.8.

To graph the angular position as a function of time we can use Equation 10.7, as in Exploration 10.2, to get

$$\theta = \theta_i + \omega_i\, t + \frac{1}{2}\alpha\, t^2 = 0 + 0 + \left(\frac{\pi}{6}\text{rad/s}^2\right)t^2 = +\left(\frac{\pi}{6}\text{rad/s}^2\right)t^2\,.$$

Recall that values of the angular position as a function of time are given in Table 10.3, and repeated in 10.4, so those points can be plotted on a graph and a smooth curve drawn through them. The result is the quadratic graph shown at the bottom of Figure 10.8.

Note that, because r = 1.0 m, we can actually use these same graphs to represent the acceleration, velocity, and position of the equivalent straight-line motion that we considered in the previous Exploration. We would need to change the units and the labels on the three y-axes, but the graphs would otherwise look identical.

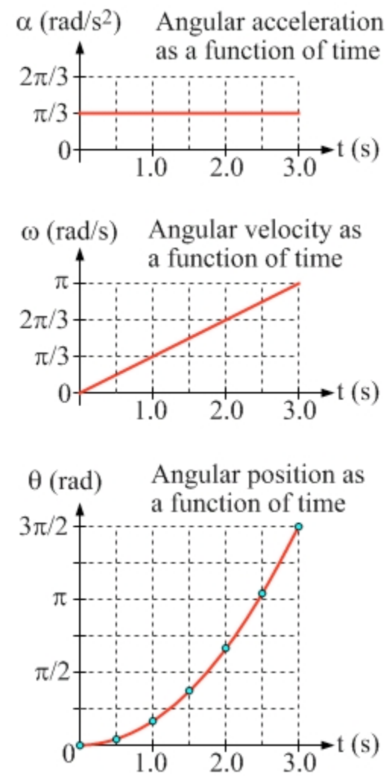


Figure 10.8: Graphs of the angular acceleration, angular velocity, and angular position for the object rotating with the turntable, all as a function of time.

Key idea: Plotting graphs of the angular acceleration, angular velocity, and angular position confirm the idea that rotational motion with constant angular acceleration is analogous to straight-line motion with constant acceleration, because the graphs in these two different situations have the same form. **Related End-of-Chapter Exercises: 40, 41.**

Essential Question 10.3: In Exploration 10.3B, we considered how to transform graphs for rotational motion into graphs for straight-line motion, but we did this with the special case of r = 1.0 m. What additional changes would be necessary if the radius r had a different value? Say, for example, that r = 3.0 m.

Answer to Essential Question 10.3: Because the straight-line motion variables are related to the equivalent rotational variables by a factor of r (e.g., $v = r\omega$), changing the value of r requires changing the graphs for the straight-line motion by a factor equal to the numerical value of r. One way to do this is to draw the lines on each graph exactly as before, but re-scale each y-axis. In the case of r = 3.0 m, for instance, each number on each y-axis would be multiplied by a factor of 3.0. A second approach is to keep the scales on the axes the same as before but move the graphs. For instance, the graph of velocity vs. time, which is given by the equation $v = +(\pi / 3) \text{ m/s}^2 \times t$ when r = 1.0 m, would be given by $v' = +(\pi \text{ m/s}^2)\, t$ when r = 3.0 m.

10-4 Torque

If an object is at rest, how can we get it to rotate? If an object is already rotating, how can we change its rotational motion? We answered equivalent questions about straight-line motion by saying "Apply a net force!" Let's now consider the rotational equivalent of force.

EXPLORATION 10.4 – Turning a revolving door

From an overhead view, a revolving door looks like a + sign mounted on a vertical axle. The door can spin freely, clockwise or counterclockwise, about its center.

Step 1 – ***Consider the three cases illustrated in Figure 10.9, in which a force (the red arrow) is applied to a revolving door. In each case, determine the direction the door will start to rotate, assuming it starts from rest.***

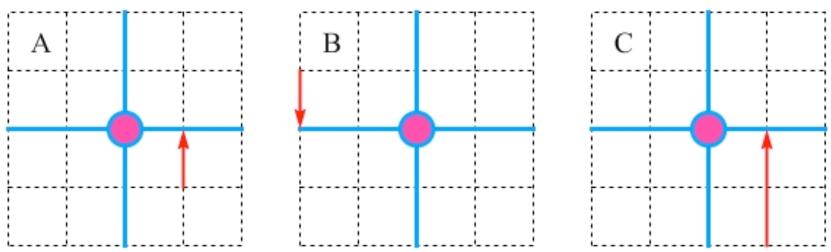


Figure 10.9: Three cases of forces applied to a revolving door, shown from an overhead perspective.

Although the direction of the force in case B is opposite to that in cases A and C, in each case the door will rotate counterclockwise. If you are ever confused about the direction an object will tend to rotate, place your pen or pencil on the diagram and hold it at the axis of the object, in this case at the center. Then push on the object in the direction, and at the location, of the applied force and see which way the object spins. Knowing the direction of a force applied to an object is not enough to determine the direction of rotation; we also need to know where the force is applied in relation to the axis of rotation.

Step 2 – ***Rank the three cases based on how quickly the revolving door spins, from largest to smallest, assuming the door is initially at rest.*** In case C, the door will rotate more quickly than in case A, because the applied force in C is twice as large as that in A while everything else (the point at which the force is applied, and the direction of the force) is equal. The door in case B also rotates faster than that in A because, even though the force has the same magnitude, in case B the force is applied further from the axis of rotation. Applying a force farther from the axis of rotation generally has a larger effect on the rotation of an object, which you have probably experienced. If you have ever come to a door where it was not obvious which side was connected to the hinges, and given the door a push on the edge where the hinges were, you most likely came close to running straight into the door as it opened very slowly in response to your push. Applying the same force at the edge of the door furthest from the hinges, however, is far more effective at opening the door.

The comparison that is hardest to rank is that between B and C. In case C the applied force is twice as large as that in B, but the force in B is applied twice as far from the axis of rotation as that in C. Which effect is more important? It turns out that these effects are equally important, so cases B and C are equivalent. The overall ranking is B=C>A.

The point of this discussion is that the angular acceleration of the door is proportional to both the applied force and the distance of the applied force from the axis of rotation. Let's now consider whether the direction at which the force is applied makes any difference.

Step 3 – ***Consider the three cases shown in Figure 10.10. Rank these three cases based on the revolving door's angular acceleration, from largest to smallest.***

Let's split the forces in cases D and E into components, as shown in Figure 10.11. How do the components of the force influence the door in each case? If you've ever tried to open a door by exerting a force parallel to the door itself, you'll know that this is completely ineffective. Similarly, the parallel components in cases D and E do absolutely nothing to affect the door's rotation. Only the perpendicular components, which have a magnitude of $F \sin\theta$, affect the rotation. Because these components are smaller than *F*, the magnitude of the perpendicular force in case A, ranking the three cases gives A>D=E.

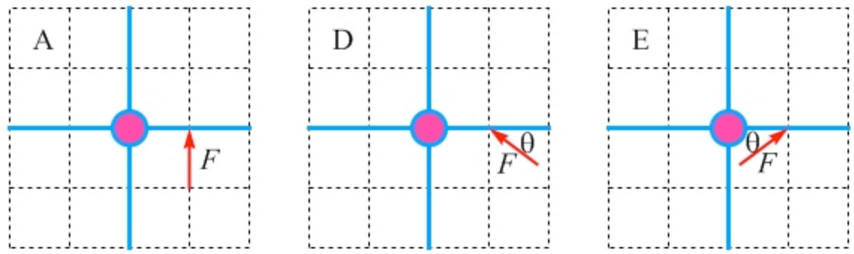


Figure 10.10: Three cases involving the same magnitude force applied at the same point on a revolving door, but applied in different directions.

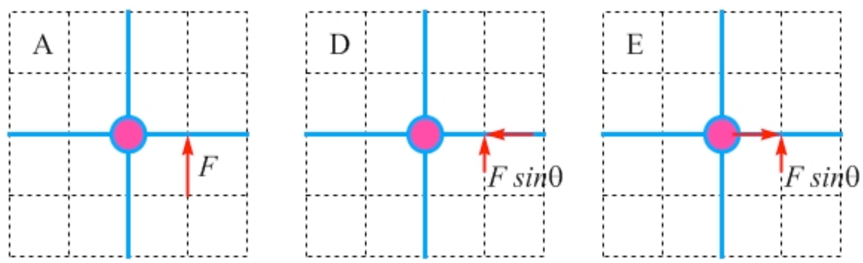


Figure 10.11: Splitting the force in case D, and case E, into components parallel to the door and perpendicular to the door.

Key ideas: The angular acceleration of a door depends on three factors: the magnitude of the applied force; the distance from the axis of rotation to where the force is applied; and the direction of the applied force. **Related End-of-Chapter Exercises: 48, 49.**

In Exploration 10.4, we learned about the rotational equivalent of force, which is torque.

The name for the rotational equivalent of force is **torque**, which we symbolize with the Greek letter tau (τ). Whereas a force is a push or a pull, a torque is a twist. A torque can result from applying a force. The torque resulting from applying a force *F* at a distance *r* from an axis of rotation is:

$$\tau = r F \sin\theta \,. \qquad \text{(Equation 10.9: \textbf{Magnitude of the torque})}$$

The angle *θ* represents the angle between the line of the force and the line the distance *r* is measured along.

Essential Question 10.4: Make a list of common household items or tools that exploit principles of torque.

Answer to Essential Question 10.4: Quite a number of tools and gadgets exploit torque, in the sense that they enable you to apply a small force at a relatively large distance from an axis, and the tool converts that into a large force acting at a relatively small distance from an axis. Examples include scissors, bottle openers, can openers, nutcrackers, screwdrivers, crowbars, wrenches, wheelbarrows, and bicycles.

10-5 Three Equivalent Methods of Finding Torque

EXPLORATION 10.5 – Three ways to find torque

A rod of length L is attached to a wall by a hinge. The rod is held in a horizontal position by a string that is tied to the wall and attached to the end of the rod, as shown in Figure 10.12.

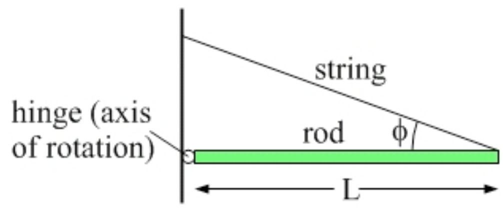


Figure 10.12: A rod attached to a wall at one end by a hinge, and held horizontal by a string.

Step 1 – ***In what direction is the torque applied by the string to the rod, about an axis that passes through the hinge and is perpendicular to the page?*** As we did in previous chapters, it's a good idea to draw a free-body diagram of the rod (or at least part of a free-body diagram, as in Figure 10.13) to help visualize what is happening. For now the only force we'll include on the free-body diagram is the force of tension applied by the string (we'll go on to look at all the forces applied to the rod in Exploration 10.8). Try placing your pen over the picture of the rod. Hold the pen where the hinge is and push on the pen, at the point where the string is tied to the rod, in the direction of the force of tension. You should see the pen rotate counterclockwise. Thus, we can say that the torque applied by the string, about the axis through the hinge, is in a counterclockwise direction.

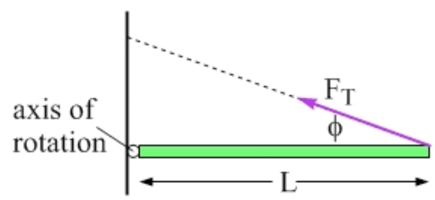


Figure 10.13: A partial free-body diagram for the rod, showing the force of tension applied to the rod by the string.

Note that we are dealing with direction for torque much as we did for angular velocity. The true direction of the torque can be found by curling your fingers on your right hand counterclockwise and placing your hand, little finger down, on the page. When you stick out your thumb it points up, out of the page. This is the true direction of the torque, but for simplicity we can state directions as either clockwise or, as in this case, counterclockwise.

Now we know the direction of the torque, relative to an axis through the hinge, applied by the string, let's focus on determining its magnitude.

Step 2 – ***Measuring the distance r in Equation 10.9 along the bar, apply Equation 10.9 to find the magnitude of the torque applied by the string on the rod, with respect to the axis passing through the hinge perpendicular to the page.***

Finding the magnitude of the torque means identifying the three variables, r, F, and θ, in Equation 10.9. In this case we can see from Figure 10.13 that the distance r is the length of the rod, L; the force $\vec{F}$ is the force of tension, $\vec{F}_T$; and the angle θ is the angle between the line of the force (i.e., the string) and the line the distance r is measured along (the rod), so θ is the angle ϕ in Figure 10.13. In this case, then, applying Equation 10.9 tells us that the magnitude of the torque is $\tau = L F_T \sin\phi$.

Step 3 – ***Now, determine the torque, about the axis through the hinge that is perpendicular to the page, by first splitting the force of tension into components, and then applying Equation 10.9.*** Which set of axes should we use when splitting the force into components? The most sensible coordinate system is one aligned parallel to the rod and perpendicular to the rod, giving the two components shown in Figure 10.14. Because the force component that is parallel to the rod is directed at the hinge, where the axis goes through, that component gives a torque of zero (it's like trying to open a door by pushing on the door with a force directed at the line passing through the hinges). Another way to prove that the force is zero is to apply Equation 10.9 with an angle of 180˚, which means multiplying by a factor of sin(180˚), which is zero.

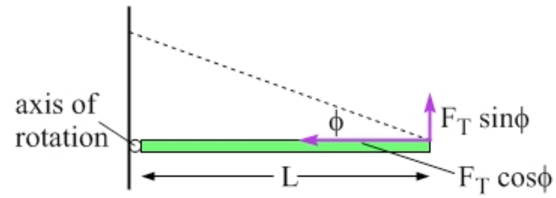


Figure 10.14: Splitting the force of tension into a component parallel to the rod, and a component perpendicular to the rod.

The torque from the force of tension is associated entirely with the perpendicular component of the force of tension. Now, identifying the three pieces of Equation 10.9 gives a force magnitude of $F = F_T \sin\phi$; a distance measured along the rod of $r = L$, and an angle of $\theta = 90^\circ$ between the line of the perpendicular force component and the line we measured r along. Because $\sin(90^\circ) = 1$, applying Equation 10.9 tells us that the magnitude of the torque from the tension, with respect to our axis through the hinge, is $\tau = L(F_T \sin\phi)\sin(90^\circ) = L F_T \sin\phi$. This agrees with our calculation in Step 2.

Step 4 – ***Instead of measuring r along the rod, draw a line from the hinge that meets the string (the line of the force of tension) at a 90˚ angle. Apply Equation 10.9 to find the magnitude of the torque applied by the string on the rod, with respect to the axis passing through the hinge, by measuring r along this line.***

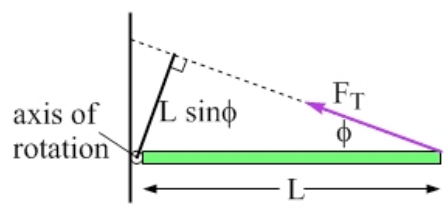


Figure 10.15: A diagram showing the lever arm, in which the distance used to find torque is measured from the axis along a line perpendicular to the line of the force.

As we can see from Figure 10.15, the r in this case is not L, the length of the rod, but is instead $L\sin\phi$. This result comes from applying the geometry of right-angled triangles. The magnitude of the force, F , is F_T , the magnitude of the full force of tension, and the angle between the line we measure r along and the line of the force is 90˚. This is known as the **lever-arm method** of calculating torque, where the lever-arm is the perpendicular distance from the axis of rotation to the force. Applying Equation 10.9 gives the magnitude of the torque as $\tau = (L\sin\phi) F_T \sin(90^\circ) = L F_T \sin\phi$, agreeing with the other two methods discussed above.

Key idea for torque: We can find torque in three equivalent ways. It can be found using the whole force and the most obvious distance; after splitting the force into components; or by using the lever-arm method in which the distance from the axis is measured along the line perpendicular to the force. Use whichever method is most convenient in a particular situation. **Related End-of-Chapter Exercises: 7, 23, 50.**

Essential Question 10.5: Torque can be calculated with respect to any axis. In Exploration 10.5, what is the torque, due to the force of tension, with respect to an axis passing through the point where the string is tied to the wall? In each case, assume the axis is perpendicular to the page.

Answer to Essential Question 10.5: The torque, from the tension, is zero with respect to any axis that passes through the string, because the line of the force (the string, in this case) passes through an axis that lies on the string. It is important to remember that the torque (both its direction and magnitude) associated with a force depends on the particular axis of rotation the torque is being measured with respect to.

10-6 Rotational Inertia

In Chapter 3, we found that an object's acceleration is proportional to the net force acting on the object:

$$\vec{a} = \frac{\sum \vec{F}}{m}. \qquad \text{(Equation 3.1: \textbf{Connecting acceleration to net force})}$$

A similar relationship connects the angular acceleration of an object to the net torque acting on it:

$$\vec{\alpha} = \frac{\sum \vec{\tau}}{I}. \qquad \text{(Eq. 10.10: \textbf{Connecting angular acceleration to net torque})}$$

Thus, the angular acceleration of an object is proportional to the net torque acting on the object. The I in the denominator of Equation 10.10 is known as the rotational inertia, which is the rotational equivalent of mass.

We have already looked at how the angular acceleration $\vec{\alpha}$ is the rotational equivalent of the acceleration $\vec{a}$, and how torque, $\vec{\tau}$, is the rotational equivalent of force, $\vec{F}$. The I in the denominator of Equation 10.10 must therefore be the rotational equivalent of the mass, m. I is known as the **rotational inertia**, or the **moment of inertia**. In the same way that mass is a measure of an object's tendency to maintain its state of straight-line motion, an object's rotational inertia is a measure of the object's tendency to maintain its rotational motion. Something with a large mass is hard to get moving, and it is also hard to stop if it is already moving. Similarly, if an object has a large rotational inertia it is difficult to start it rotating, and difficult to stop if it is already rotating.

One question to consider is, are rotational inertia and mass the same thing? In other words, does an object's mass, by itself, determine the rotational inertia? Let's check the units of rotational inertia. Re-arranging Equation 10.10, we find that rotational inertia has units of torque units (N m) divided by angular acceleration units (rad/s^2). Remembering that the newton is equivalent to kg m/s^2, and that we can treat the radian as being dimensionless, we find that rotational inertia has units of kg m^2. Rotational inertia depends on more than just mass, it depends on both mass and, somehow, length squared. Let's investigate this further.

EXPLORATION 10.6 – Rotational inertia

Consider a ball of mass M mounted at the end of a stick that has a negligible mass, and a length L (which is large compared to the ball's radius). The other end of the stick is pinned so the stick can rotate freely about the pin.

Step 1 – ***If the ball and stick are held horizontal and then released from rest, what is the ball's initial acceleration?*** The ball's initial acceleration is $\vec{g}$, the acceleration due to gravity. The force of the stick acting on the ball only becomes non-zero after the ball starts moving. We should also draw a diagram to help analyze the situation. The diagram is shown in Figure 10.16.

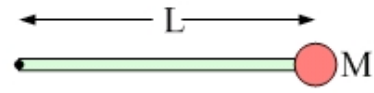


Figure 10.16: The initial position of the ball and stick. The system can rotate about an axis passing through the left end of the stick.

Step 2 – ***What is the ball's initial angular acceleration?***

The angular acceleration can be found from the equation $a = r\alpha$. Here $r = L$, the length of the stick, so we have $\vec{\alpha} = g / L$, directed clockwise.

Step 3 – ***What is the torque acting on the ball at the instant it is released?***

Here we can draw a free-body diagram of the ball, shown in Figure 10.17. Initially the only force acting on the ball is the force of gravity, Mg directed down. Considering an axis perpendicular to the page and passing through the pin, the torque is $\vec{\tau} = L\,Mg$, directed clockwise.

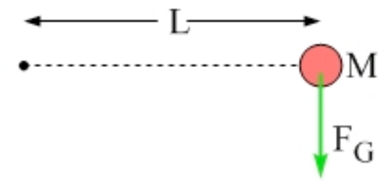


Figure 10.17: The free-body diagram of the ball immediately after the system is released from rest.

Step 4 – ***Using Equation 10.10, and the results from steps 2 and 3, determine the rotational inertia of the ball relative to the axis passing through the pin.***

Re-arranging Equation 10.10 to solve for the rotational inertia gives:

$$I = \frac{\Sigma\vec{\tau}}{\vec{\alpha}}.$$

The torque and the angular acceleration are both clockwise, allowing us to divide the magnitude of the torque by the magnitude of the angular acceleration to determine the ball's rotational inertia about an axis through the pin.

$$I = \frac{L\,Mg}{g / L} = M\,L^2.$$

Thus the rotational inertia of an object of mass M in which all the mass is at a particular distance L from the axis of rotation is $I = M\,L^2$.

Key ideas for rotational inertia: An object's rotational inertia is determined by three factors: the object's mass; how the object's mass is distributed; and the axis the object is rotating around. **Related End-of-Chapter Exercises: 10, 27.**

Essential Question 10.6: Consider the three cases shown in Figure 10.18. In each case, a ball of a particular mass is placed on a light rod of a particular length. Each rod can rotate without friction about an axis through the left end. Rank the cases based on their rotational inertias, from largest to smallest.

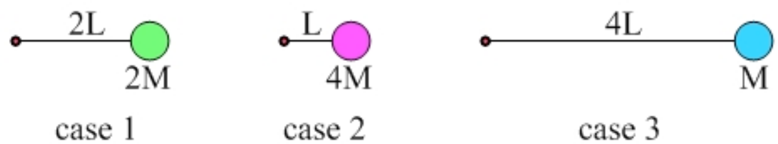


Figure 10.18: Three cases, each involving a ball on the end of a rod that can rotate about its left end.

Answer to Essential Question 10.6: The correct ranking is 3>1>2. In the rotational inertia equation, the distance from the axis to the ball (the length of the rod) is squared, while the mass is not. Thus, changing the length by a factor of 2 changes the rotational inertia by a factor of 4, whereas changing the mass by a factor of 2 changes the rotational inertia by only a factor of 2.

10-7 An Example Problem Involving Rotational Inertia

Our measure of inertia for rotational motion is somewhat more complicated than inertia for straight-line motion, which is just mass. Consider the following example.

EXAMPLE 10.7 – Spinning the system.

Three balls are connected by light rods. The mass and location of each ball are:
Ball 1 has a mass M and is located at $x = 0$, $y = 0$.
Ball 2 has a mass of $2M$ and is located at $x = +3.0$ m, $y = +3.0$ m.
Ball 3 has a mass of $3M$ and is located at $x = +2.0$ m, $y = -2.0$ m.
Assume the radius of each ball is much smaller than 1 meter.

(a) Find the location of the system's center-of-mass.
(b) Find the system's rotational inertia about an axis perpendicular to the page that passes through the system's center-of-mass.
(c) Find the system's rotational inertia about an axis parallel to, and 2.0 m from, the axis through the center-of-mass.

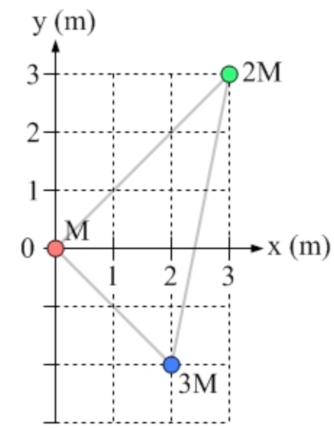


Figure 10.19: A diagram showing the location of the balls in the system described in Example 10.7.

SOLUTION

Let's begin, as usual, by drawing a diagram of the situation. The diagram is shown in Figure 10.19.

(a) To find the location of the system's center-of-mass, let's apply Equation 6.3. To find the x-coordinate of the system's center-of-mass:

$$X_{CM} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{(0)M + (+3.0\,\text{m})(2M) + (+2.0\,\text{m})(3M)}{M + 2M + 3M} = \frac{(+12.0\,\text{m})M}{6M} = +2.0\,\text{m}$$

The y-coordinate of the system's center-of-mass is given by:

$$Y_{CM} = \frac{y_1 m_1 + y_2 m_2 + y_3 m_3}{m_1 + m_2 + m_3} = \frac{(0)M + (+3.0\,\text{m})(2M) + (-2.0\,\text{m})(3M)}{M + 2M + 3M} = \frac{(0)M}{6M} = 0\,.$$

(b) To find the system's rotational inertia about an axis through the center-of-mass we can find the rotational inertia for each ball separately, using $I = M\,L^2$, and then simply add them to find the total rotational inertia. Figure 10.20 is helpful for seeing where the different L values come from.

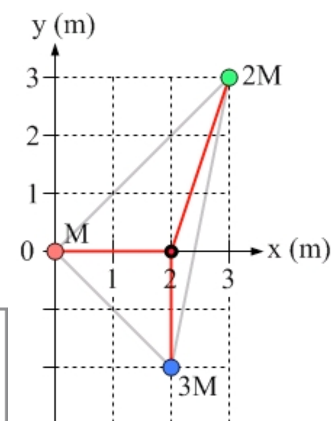


Figure 10.20: The center-of-mass of the system is marked at (+2 m, 0). The axis of rotation passes through that point. The dark lines show how far each ball is from the axis of rotation.

For ball 1, $L^2 = (2.0\,\text{m})^2 = 4.0\,\text{m}^2$ so $I_1 = M\,L^2 = \left(4.0\,\text{m}^2\right)M$.

For ball 2, $L^2 = 10\,\text{m}^2$ so $I_2 = 2M\,L^2 = \left(20\,\text{m}^2\right)M$.

For ball 3, $L^2 = 4.0\,\text{m}^2$ so $I_3 = 3M\,L^2 = \left(12\,\text{m}^2\right)M$.

The total rotational inertia is the sum of these three values, $(36\,\text{m}^2)M$.

(c) To find the rotational inertia through an axis parallel to the first axis and 2.0 m away from it, let's choose a point for this second axis to pass through. A convenient point is the origin, $x = 0$, $y = 0$. Figure 10.21 shows where the L values come from in this case.

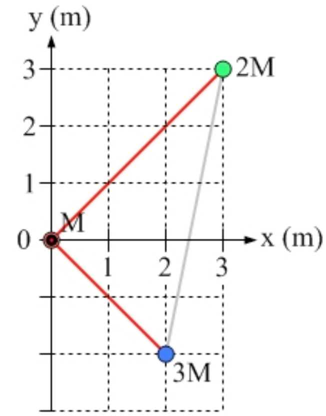


Figure 10.21: The axis of rotation now passes through the ball of mass M at the origin. The red lines show how far the other two balls are from the axis of rotation.

Repeating the process we followed in part (b) gives:

For ball 1, $L^2 = 0$ so $I_1' = 0$.

For ball 2, $L^2 = 18\,\text{m}^2$ so $I_2' = 2M\,L^2 = \left(36\,\text{m}^2\right)M$.

For ball 3, $L^2 = 8.0\,\text{m}^2$ so $I_3' = 3M\,L^2 = \left(24\,\text{m}^2\right)M$.

The total rotational inertia is the sum of these three values, $(60\,\text{m}^2)M$.

Related End-of-Chapter Exercises: 29, 31.

Does it matter which point the second axis passes through? What if we had used a different point, such as $x = +2.0$ m, $y = -2.0$ m, or any other point 2.0 m from the center-of-mass? Amazingly, it turns out that it doesn't matter. Any axis parallel to the axis through the center-of-mass and 2.0 m from it gives a rotational inertia of $(60\,\text{m}^2)M$. It turns out that the rotational inertia of a system is minimized when the axis goes through the center-of-mass, and the rotational inertia of the system about any parallel axis a distance h from the axis through the center-of-mass can be found from

$I = I_{CM} + mh^2$, (Equation 10.11: **The parallel-axis theorem**)

where m is the total mass of the system.

Let's check the parallel-axis theorem using our results from (b) and (c). In part (b) we found that the rotational inertia about the axis through the center-of-mass is $I_{CM} = (36\,\text{m}^2)M$. The mass of the system is $m = 6M$ and the second axis is $h = 2.0$ m from the axis through the center-of-mass. This gives $I = (36\,\text{m}^2)M + 6M(2.0\,\text{m})^2 = (60\,\text{m}^2)M$, as we found above.

Essential Question 10.7: To find the total mass of a system of objects, we simply add up the masses of the individual objects. To find the total rotational inertia of a system of objects, can we follow a similar process, adding up the rotational inertias of the individual objects.

Answer to Essential Question 10.7: Yes, the rotational inertia of a system of objects can be found be adding up the rotational inertias of the various objects making up the system. This is precisely the process we followed in Example 10.7.

10-8 A Table of Rotational Inertias

Consider now what happens if we take an object that has its mass distributed over a length, area, or volume, rather then being concentrated in one place. Generally, the rotational inertia in such a case is calculated by breaking up an object into tiny pieces, finding the rotational inertia of each piece, and adding up the individual rotational inertias to determine the total rotational inertia.

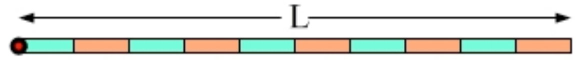


Figure 10.22: A uniform rod of length L and mass M, divided into 10 equal pieces. The axis of rotation passes through the left end of the rod and is perpendicular to the page.

We can get a feel for the process by considering how we would find the rotational inertia of a uniform rod of length L and mass M, rotating about an axis through the end of the rod that is perpendicular to the rod itself. If all the mass were concentrated at the far end of the rod, a distance L from the axis, then the rotational inertia would be ML^2. Because most of the mass is closer than L to the axis of rotation, the rod's rotational inertia turns out to be less than ML^2. If we broke up the rod into ten equal pieces, with centers at 5%, 15%, 25%, 35%,…,95% of the length of the rod (see Figure 10.22), we would calculate a rotational inertia of 0.3325 ML^2. This is very close to the value we would get by doing the integration, $I_{rod,end} = ML^2/3$. The rotational inertia's of various shapes, and for various axes of rotation, are shown in Figure 10.23.

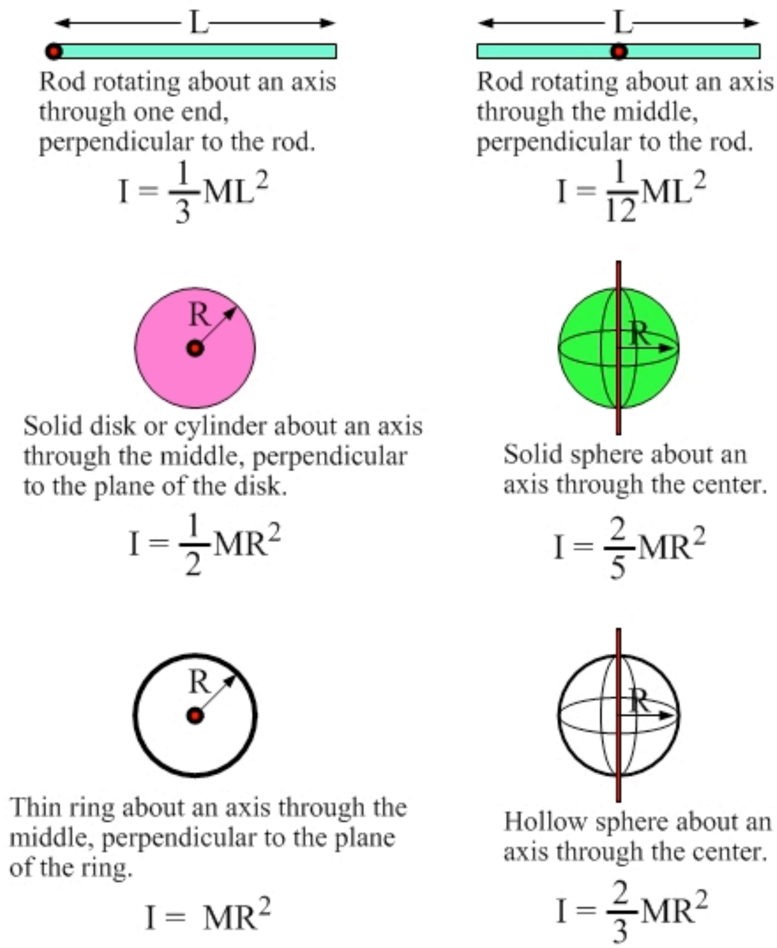


Figure 10.23: Expressions for the rotational inertia of various objects about a particular axis. In each case, the object has a mass M.

Essential Question 10.8: In Figure 10.23, all the values for rotational inertia are of the form $I = cMR^2$, or $I = cML^2$, where c is generally less than 1. The exception is the rotational inertia of a ring rotating about an axis through the center of the ring and perpendicular to the plane of the ring, where $c = 1$. Why do we expect to get $I = MR^2$ for the ring rotating about that central axis?

Answer to Essential Question 10.8: The expression for the rotational inertia of the ring has no factor less than 1 in front of the MR^2 because every bit of mass in the ring is a distance *R* from the center of the ring. In all the other cases shown in Figure 10.23, most of the mass of the given object is at a distance less than *R* (or less than *L*) from the axis in question.

10-9 Newton's Laws for Rotation

In Chapter 3 we considered Newton's three laws of motion. The first two of these laws have analogous statements for rotational motion.

Newton's First Law for Rotation: an object at rest tends to remain at rest, and an object that is spinning tends to spin with a constant angular velocity, unless it is acted on by a nonzero net torque ***or there is a change in the way the object's mass is distributed***.

Recall that the net torque is the sum of all the forces acting on an object. Always remember to add torques as vectors. The net torque can be symbolized by $\sum \vec{\tau}$.

The first part of the statement of Newton's first law for rotation parallels Newton's first law for straight-line motion, but the phrase about how spinning motion can be affected by a change in mass distribution is something that only applies to rotation.

Newton's second law for rotation, on the other hand, is completely analogous to Newton's second law for straight-line motion, $\sum \vec{F} = m\,\vec{a}$. Replacing force by torque, mass by rotational inertia, and acceleration by angular acceleration, we get:

$\sum \vec{\tau} = I\vec{\alpha}$. (Equation 10.12: **Newton's Second Law for Rotation**)

We'll spend the rest of this chapter, and a good part of the next chapter, looking at how to apply Newton's second law in various situations. In Chapter 11, we will deal with rotational dynamics, involving motion and acceleration. For the remainder of this chapter, however, we will focus on situations involving static equilibrium.

Conditions for static equilibrium

An object is in static equilibrium when it remains at rest. Two conditions apply to objects in static equilibrium. These are:

$$\sum \vec{F} = 0 \qquad \text{and} \qquad \sum \vec{\tau} = 0 .$$

Expressed in words, an object in static equilibrium experiences no net force and no net torque. Using these conditions, we will be able to analyze a variety of situations. Many excellent examples of static equilibrium involve the human body, such as when you hold your arm out; when you bend over; and when you stand on your toes. In each case, forces associated with muscles, bones, and tendons maintain the equilibrium situation.

Essential Question 10.9: Newton's first law for rotation includes a phrase that says spinning motion can be affected by a change in the way an object's mass is distributed. Can you think of a real-life example of this?

Answer to Essential Question 10.9: A familiar example is a figure skater who spins relatively slowly with her arms held out from her body, but then pulls her arms in and spins much faster.

10-10 Static Equilibrium

Let's first apply Newton's second law for rotation in a **static equilibrium** situation, in which an object remains at rest. Conditions for static equilibrium are given in section 10-9.

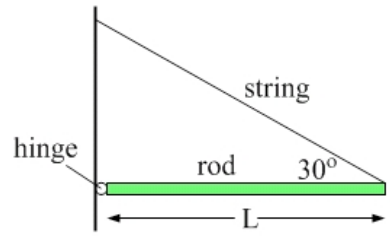


Figure 10.24: A diagram of the rod, connected to the wall by a hinge and held horizontal by a string tied to the end of the rod.

EXPLORATION 10.10 – A hinged rod.

Return to the hinged rod we looked at in Exploration 10.5. The rod's mass of 2.0 kg is uniformly distributed along its length *L*. The rod is attached to a wall by a hinge at one end. As shown in Figure 10.24, the angle between the rod and the string, which holds the rod in a horizontal position, is 30˚. Use g = 10 N/kg to simplify the calculations.

Step 1 – ***Sketch a free-body diagram of the rod.*** The free-body diagram is in Figure 10.25. Start by drawing the force of tension applied to the rod by the string, which goes away from the rod along the string. Where should we draw the force of gravity? Until now, all we had to do was to show the direction of a force correctly on a free-body diagram. Now that we're dealing with torques, it is also critical to locate the force accurately. The force of gravity should be drawn at the **center-of-gravity** of the rod, which is at the rod's geometrical center because the rod is uniform. For now, we can assume that the center-of-mass and the center-of-gravity are the same point. We'll distinguish between the two later in the chapter.

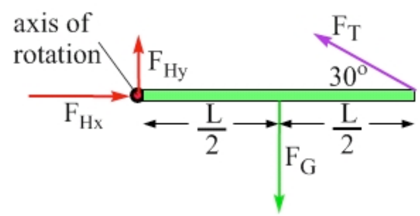


Figure 10.25: The free-body diagram of the rod.

What other forces act on the rod, in addition to gravity and tension? First, for the rod to remain in equilibrium, there must be a force directed right, to balance the component of the force of tension directed left. Second, because the hinge is in contact with the rod, the hinge very likely exerts a force on the rod. Generally, we draw this hinge force already split into components. The horizontal component of the hinge force, $\vec{F}_{Hx}$, is directed right to balance the horizontal component of the force of tension. The vertical component of the hinge force, $\vec{F}_{Hy}$, is shown directed up. This vertical component could, however, be directed down in some cases, or even be equal to zero. If you're not sure which direction a force is in, simply choose a direction. If the analysis gives a negative sign for a force, the force is opposite to the direction shown.

Step 2 – ***Apply Newton's second law twice, once for the horizontal direction and once for the vertical direction, to come up with two force equations for this situation.*** Figure 10.26 shows the *x*-*y* coordinate system, with positive *x* to the right and positive *y* up. The force of tension has been split into components parallel to the coordinate axes.

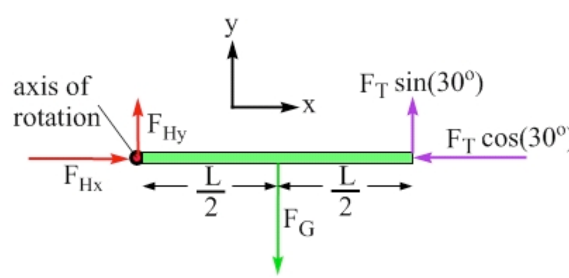


Figure 10.26: A free-body diagram showing the *x*-*y* coordinate system, and with all forces split into components.

Applying Newton's second law in the x-direction means:

1. Writing out Newton's second law: $\sum \vec{F}_x = m\,\vec{a}_x$.
2. Recognizing that the right-hand side equals zero, because the rod stays at rest.
3. Looking at the free-body diagram to evaluate the left-hand side of the equation: $+F_{H\,x} - F_T \cos(30°) = 0$.

Using a similar process for the y-direction, we start with $\sum \vec{F}_y = m\,\vec{a}_y$, and end up with:

$+F_{H\,y} - mg + F_T \sin(30°) = 0$

What can we solve for with these two force equations? We can't solve for anything! There are simply too many unknowns. If all we knew about were forces, we would be stuck.

Step 3 – ***Choose an appropriate axis to take torques about. Then, apply Newton's second law for rotation to write a torque equation to solve for the tension.*** What is an appropriate axis to use? Any axis can be used, but choosing an axis carefully can make a problem significantly easier to solve. ***The key is to choose an axis that one or more of the unknown forces pass through, because forces passing through an axis do not give any torque about that axis.*** In this case, we're trying to solve for the tension in the string, so we should pick an axis that eliminates the other unknown forces (the hinge forces), if possible. The most appropriate axis here is the axis perpendicular to the page that passes through the hinge. An axis through the hinge eliminates the two hinge forces, and the horizontal component of the force of tension, from the torque equation.

As with forces, we are free to choose a positive direction for torque. Let's use clockwise in this particular situation (although counterclockwise would be just as good). Applying Newton's second law for rotation means:

1. Writing out the equation: $\sum \vec{\tau} = I\,\vec{\alpha}$.
2. Recognizing that the right-hand side equals zero, because the rod stays at rest.
3. Looking at the free-body diagram to evaluate the left-hand side of the equation, and applying $\tau = r\,F\sin\theta$ to find the magnitude of the torque from each force. Recognizing that the torque due to the force of gravity is clockwise, while the torque due to the tension is counterclockwise, we get: $-L[F_T \sin(30°)]\sin(90°) + \frac{L}{2}(mg)\sin(90°) = 0$.

This equation demonstrates the power of using torque, because we can immediately solve for the tension. Note that the length of the rod, which is unknown, cancels out in the equation.

This gives: $F_T \sin(30°) = \frac{mg}{2}$.

Because $\sin(30°) = 0.5$, we find that $F_T = mg = (2.0\text{ kg})(10\text{ N/kg}) = 20\text{ N}$. Note that the fact that the force of tension has the same magnitude as the force of gravity in this case is highly coincidental, and happened only because the factor of two difference in the distances of these forces from the axis of rotation was exactly balanced by the factor of ½ we got from $\sin(30°)$.

Key ideas: By analyzing situations in terms of torque as well as force, we can solve problems that cannot be solved using force concepts alone. One of the keys to using torque is to choose an appropriate axis to take torques around. This is generally an axis that one or more of the unknown forces passes through. **Related End-of-Chapter Exercises: 8, 51 – 53.**

Essential Question 10.10: Return to Exploration 10.10, and solve for the x and y components of the force exerted on the rod by the hinge.

Answer to Essential Question 10.10: We solved for the force of tension in Exploration 10.10, so we can go on solve for the components of the hinge force. To find the *y*-component of the hinge force we could set up another torque equation, relative to an axis through the middle of the rod, for instance, or we could now make use of the force equation we worked out in Step 2. To find the *x*-component of the hinge force we can only use the force equation because, no matter which axis we choose, the torque from the *x*-component of the hinge force always exactly balances the torque from the *x*-component of the force of tension.

Making use of our *x*-component force equation, $+F_{Hx} - F_T\cos(30°) = 0$, we find that:

$$F_{Hx} = F_T\cos(30°) = (20\text{ N})\frac{\sqrt{3}}{2} = 10\sqrt{3}\text{ N}.$$

Using the *y*-component force equation, $+F_{Hy} - mg + F_T\sin(30°) = 0$, we get:

$$+F_{Hy} = mg - F_T\sin(30°) = 20\text{ N} - 20\text{ N}\left(\frac{1}{2}\right) = 10\text{ N}.$$

Note that, if we combine the two components of the hinge force, we find that the hinge force is 20 N, with an angle of 30˚ between the hinge force and the rod. In other words, the hinge force is a mirror image of the force of tension, because of the symmetry of the situation (both forces are applied at the ends of the rod, while the force of gravity is applied at the exact center).

10-11 A General Method for Solving Static Equilibrium Problems

Now that we have explored the idea of applying the concept of torque to solve a static equilibrium problem, let's list the basic steps in the process.

A General Method for Solving a Static Equilibrium Problem

Objects in static equilibrium remain at rest, so both the acceleration and the angular acceleration are zero. This allows us to use special-case of Newton's second law and Newton's second law for rotation.

1. Draw a diagram of the situation.
2. Draw a free-body diagram showing all the forces acting on the object.
3. Choose a rotational coordinate system. Pick an appropriate axis to take torques about, and then apply Newton's second law for rotation ($\sum\vec{\tau} = 0$) to obtain one or more torque equations.
4. If necessary, choose an appropriate *x*-*y* coordinate system for forces. Apply Newton's second law ($\sum\vec{F} = 0$) to obtain one or more force equations.
5. Combine the resulting equations to solve the problem.

Let's apply the method in the following example.

EXAMPLE 10.11 – Supporting the board

A uniform board with a weight of 240 N and a length of 2.0 m rests horizontally on two supports. Support A is under the left end of the board, while Support B is 50 cm from the right end (150 cm from the left end, in other words).

(a) Which support exerts more force on the board? Without doing the calculations to find the two support forces, come up with a conceptual argument to justify your answer.

(b) Find the two support forces.

SOLUTION

As usual, let's begin by drawing a diagram of the situation. We should also sketch a free-body diagram to show all the forces acting on the board. The diagram and free-body diagram are shown in Figure 10.27.

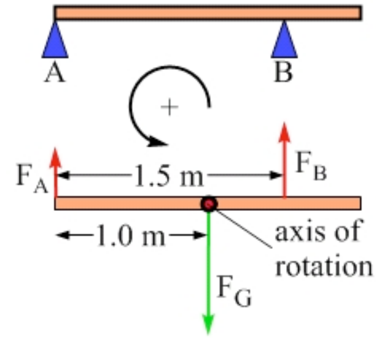


Figure 10.27: A diagram and free-body diagram for the board on two supports.

(a) Support B exerts a larger force on the board than support A. One way to see this is to sum torques about an axis through the center of the board, and perpendicular to the page. Taking counterclockwise to be the positive direction for torque, applying Newton's second law for rotation gives:

$$-(1.0 \text{ m}) F_A + (0.5 \text{ m}) F_B = 0 .$$

The force of gravity does not appear in this torque equation because the force of gravity passes through the axis, and thus does not give rise to a torque about that axis. Because the torques from the two support forces must balance one another, and the distance from support B to the axis is half that of the distance from support A to the axis, the force exerted on the board by support B must be twice as large as that exerted by support A.

(b) To solve for the support forces, we could combine the torque equation above with the force equation we get by applying Newton's second law, or we could set up another torque equation by taking an axis perpendicular to the page through one of the supports. Let's do the latter, using Figure 10.28 to help us set up the new torque equation, summing torques about an axis through support A.

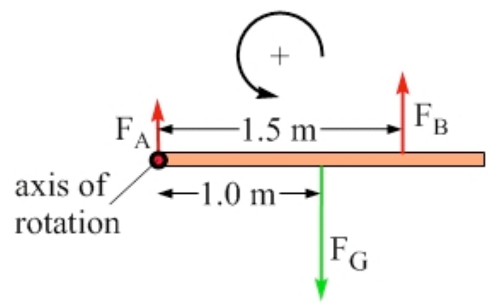


Figure 10.28: If we take torques about an axis through support A, the force applied to the board from support A does not give rise to a torque, because that force passes through the axis.

Applying Newton's second law for rotation, $\sum \vec{\tau} = I\vec{\alpha} = 0$, taking counterclockwise to be positive, gives:

$$-(1.0 \text{ m}) mg + (1.5 \text{ m}) F_B = 0 .$$

Thus, $F_B = \frac{2}{3} mg = \frac{2}{3}(240 \text{ N}) = +160 \text{ N}$.

There are several ways to solve for the force applied by support A. Let's apply Newton's second law, $\sum \vec{F} = m\vec{a} = 0$, taking up to be the positive direction:

$$+F_A - mg + F_B = 0 .$$

This gives: $+F_A = mg - F_B = mg - \frac{2}{3} mg = \frac{1}{3} mg = \frac{1}{3}(240 \text{ N}) = +80 \text{ N}$.

The fact that both support forces work out to be positive means they are in the direction shown in the diagrams, up.

Related End-of-Chapter Exercises: 33, 36.

Essential Question 10.11: In Example 10.11, would the support forces change if support B was moved a short distance to the right of its original position? If so, how would the forces change?

Answer to Essential Question 10.11: Moving a supporting force farther from the center-of-gravity generally reduces that support force. However, the two supports together support the weight of the board. Thus, moving support B to the right decreases the magnitude of force B, but support A's upward force increases to compensate. The simplest example is when support B is at the far right of the board. In that case, symmetry tells us that the supports share the load equally, with each support exerting an upward force of 120 N on the board.

10-12 Further Investigations of Static Equilibrium

Center-of-gravity

Previously, we discussed the importance of locating forces precisely on a free-body diagram. For instance, the force of gravity must be attached to the center-of-gravity of the system. If the acceleration due to gravity has the same direction at all points in a system, we can define the *x*-coordinate of the system's center-of-gravity as:

$$X_{CG} = \frac{x_1 m_1 g_1 + x_2 m_2 g_2 + x_3 m_3 g_3 + \ldots}{m_1 g_1 + m_2 g_2 + m_3 g_3 + \ldots}$$

(Eq. 10.13: ***X*-coordinate of the center-of-gravity**)

A similar equation gives the *y*-coordinate of the center-of-gravity. If the acceleration due to gravity is the same everywhere, *g* cancels out of the equation, giving the center-of-mass equation we used in Chapter 6. The center-of-gravity differs from the center-of-mass, therefore, only when the acceleration due to gravity is different for different parts of the object or system.

EXAMPLE 10.12 – Tipping the board

Let's continue from where we left off in Example 10.11, involving the 240 N board on two supports. Now you climb on the board and, starting at the left end of the board, you slowly walk along the board toward the right end. Your weight is 480 N.

(a) Defining up to be the positive direction, plot two graphs, on the same set of axes, of the support forces as a function of your distance *d* from the left end of the board. Use this graph to determine the value of *d* when the board tips over.

(b) Where is the center-of-gravity, of the system consisting of you and the board, when the board begins to tip?

SOLUTION

(a) Again, we should draw a diagram to help us analyze the situation. Let's place you on the board at a distance *d* from the left end, and sketch a free-body diagram of the system. These diagrams are shown in Figure 10.29.

Let's define counterclockwise to be the positive direction for torques, and take torques about an axis perpendicular to the page that passes through the left end of the board. Choosing this axis eliminates, from the torque equation, the force exerted on the system by support A, and allows us to solve for the force exerted by support B. Let's use *M* to represent your mass and *m* to represent the mass of the board. Applying Newton's second law for rotation in this situation, $\sum \vec{\tau} = I\vec{\alpha} = 0$, gives:

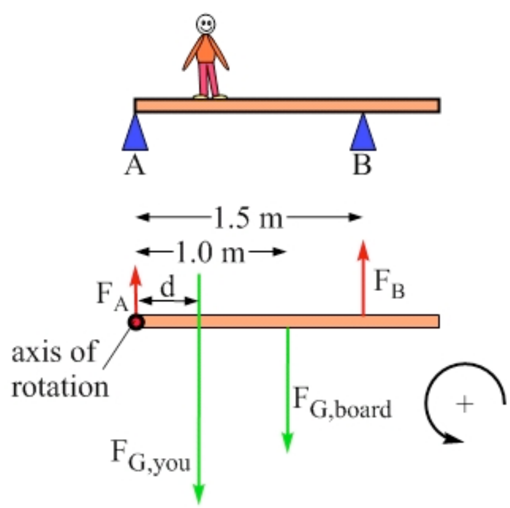


Figure 10.29: A diagram and free-body diagram of the system consisting of you and the board. You are a distance *d* from the left end of the board.

$$-d\,(Mg)\sin(90°)-(1.0\text{ m})(mg)\sin(90°)+(1.5\text{ m})\,F_B\sin(90°)=0\,.$$

Solving for the force exerted by support B gives:

$$F_B=\frac{2}{3}mg+\frac{2d}{3.0\text{ m}}Mg=160\text{ N}+320\,d\text{ N/m}\,.$$

We could follow a similar process to find an expression for the force exerted on the system by support A, taking torques about an axis through the board where support B is. In this case, however, it's probably easier to apply Newton's second law, $\sum\vec{F}=(M+m)\vec{a}=0$. Taking up to be positive gives: $+F_A-Mg-mg+F_B=0$.

Thus: $F_A=Mg+mg-F_B=480\text{ N}+240\text{ N}-160\text{ N}-320d\text{ N/m}=560\text{ N}-320d\text{ N/m}$.

Graphs of the two support forces, as a function of your position, are shown in Figure 10.30. Note that for values of $d > 1.75$ m, the force from support A must be negative (directed down) to maintain the system's equilibrium. A's force could be negative if the board was bolted to the support, and the support either had a significant mass or it was fastened firmly to the ground. In this case, however, the board simply rests on the support, so the support can only provide an upward force.

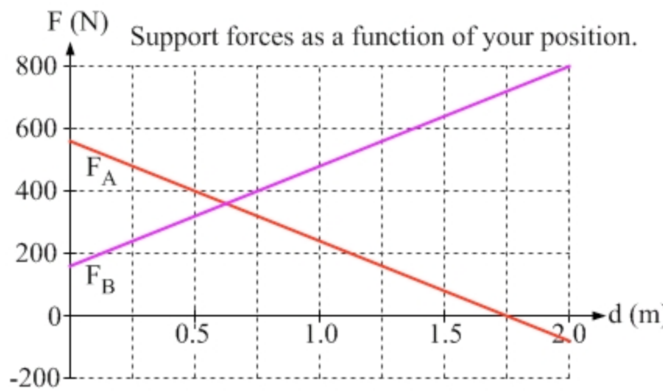


Figure 10.30: Graphs of the two support forces, as a function of your distance *d* from the left end of the board.

Thus, when $d = 1.75$ m, the board is on the verge of tipping, because the normal force between the board and support A goes to zero at that value of *d*. The board will tip if *d* exceeds 1.75 m. In this situation, then, Figure 10.30 shows the correct situation for $d \leq 1.75$ m.

(b) What happens to the center-of-gravity of the system, which consists of you and the board, as you walk to the right? Because your weight is shifting right, the center-of-gravity of the system shifts right, also. The *y*-coordinate of the center-of-gravity has no bearing on whether the system tips, so let's simply determine the *x*-coordinate of the center-of-gravity when $d = 1.75$ m:

$$X_{CG}=\frac{x_{board}\,mg+x_{you}\,Mg}{mg+Mg}=\frac{(1.0\text{m})(240\text{ N})+(1.75\text{m})(480\text{ N})}{240\text{ N}+480\text{ N}}=1.5\text{ m}\,.$$

It is no coincidence that the position of the center-of-gravity corresponds to the location of support B. If the center-of-gravity of a system is between its supports, the system is stable. If the center-of-gravity moves out from the region bounded by the supports, the system tips over.

Related End-of-Chapter Exercises: 12, 34.

Essential Question 10.12: Return to the expressions we found for the support forces in part (a) of the Example 10.12. Add the two expressions. What is the significance of this result?

Answer to Essential Question 10.12: Adding the two expressions for the support forces gives:

$$F_A + F_B = 560\text{ N} - 320d\text{ N/m} + 160\text{ N} + 320d\text{ N/m} = 720\text{ N}.$$

In other words, when the system is in equilibrium the sum of the support forces is always 720 N. This is expected because the supports combine to balance the weight of the system. Your weight of 480 N and the board's weight of 240 N add to 720 N.

Chapter Summary

Essential Idea for Rotational Motion

The methods we applied previously to solve straight-line motion problems, such as using constant-acceleration equations and Newton's Laws of Motion, can essentially be adapted to help us analyze situations involving rotational motion.

Rotational Kinematics

To help us understand how things move we defined the straight-line motion variables of position, displacement, velocity, and acceleration. The analogous rotational variables help us understand rotational motion.

Straight-line motion variable	Analogous rotational motion variable	Connection
Displacement, $\Delta\vec{s}$	Angular displacement, $\Delta\vec{\theta}$	$\Delta s = r\,\Delta\theta$
Velocity, $\vec{v} = \dfrac{\Delta\vec{s}}{\Delta t}$	Angular velocity, $\vec{\omega} = \dfrac{\Delta\vec{\theta}}{\Delta t}$	$v_T = r\omega$
Acceleration, $\vec{a} = \dfrac{\Delta\vec{v}}{\Delta t}$	Angular acceleration, $\vec{\alpha} = \dfrac{\Delta\vec{\omega}}{\Delta t}$	$a_T = r\alpha$

Table 10.2: Connecting straight-line motion variables to rotational variables. To prevent confusion with r, the radius, the variable $\vec{s}$ is used to represent position. The T subscripts denote tangential, for components that are tangential to the circular path.

In the special case of one-dimensional motion with constant acceleration, we derived a set of useful equations. An analogous set applies to rotation with constant angular acceleration.

Straight-line motion equation	Analogous rotational motion equation
$v = v_i + at$ (Equation 2.9)	$\omega = \omega_i + \alpha t$ (Equation 10.6)
$x = x_i + v_i t + \frac{1}{2}at^2$ (Equation 2.11)	$\theta = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$ (Equation 10.7)
$v^2 = v_i^2 + 2a\,\Delta x$ (Equation 2.12)	$\omega^2 = \omega_i^2 + 2\alpha\,\Delta\theta$ (Equation 10.8)

Table 10.1: Comparing the one-dimensional kinematics equations from chapter 2 to the rotational motion equations that can be applied to rotating objects.

Static Equilibrium

An object is in static equilibrium when it remains at rest. Two conditions apply to objects in static equilibrium. These are:

$$\sum \vec{F} = 0 \qquad \text{and} \qquad \sum \vec{\tau} = 0 .$$

Expressed in words, an object in static equilibrium experiences no net force and no net torque.

A General Method for Solving a Static Equilibrium Problem

1. Draw a diagram of the situation.
2. Draw a free-body diagram showing all the forces acting on the object.
3. Choose a rotational coordinate system. Pick an appropriate axis to take torques about, and then apply Newton's Second Law for Rotation ($\sum \vec{\tau} = 0$) to obtain one or more torque equations.
4. If necessary, choose an appropriate *x*-*y* coordinate system for forces. Apply Newton's Second Law ($\sum \vec{F} = 0$) to obtain one or more force equations.
5. Combine the resulting equations to solve the problem.

Rotational Dynamics

Mass is our measure of inertia for straight-line motion, while rotational inertia depends on the mass, the way the mass is distributed, and the axis about which rotation occurs. Torque is the rotational equivalent of force. The concepts of mass, force, and acceleration are linked by Newton's Second Law; an analogous law links the concepts of rotational inertia, torque, and angular acceleration.

Straight-line motion concept	Analogous rotational motion concept	Connection
Inertia: mass, m	Rotational Inertia, $I = cMR^2$ (c depends on axis and object's shape)	$I = \sum m_i r_i^2$
Can change motion: Force, $\vec{F}$	Can change rotation: Torque, $\vec{\tau}$	$\tau = r F \sin\theta$
Newton's Second Law, $\sum \vec{F} = m\vec{a}$	Second Law for Rotation, $\sum \vec{\tau} = I\vec{\alpha}$	Same form

Table 10.5: Rotational dynamics is governed by concepts that are similar to those that govern dynamics in straight-line motion.

End-of-Chapter Exercises

Exercises 1 – 12 are conceptual questions that are designed to see if you have understood the main concepts of the chapter.

1. As shown in the overhead view in Figure 10.31, four cylindrical objects (two red and two blue) are spinning with a turntable that is moving counterclockwise at a constant rate. The two red cylinders are the same distance from the center, and the two blue ones are also equally distant from the center, but farther from the center than the two red cylinders. Which cylinders have the same (a) speed? (b) velocity? (c) angular velocity? (d) acceleration? (e) angular acceleration?

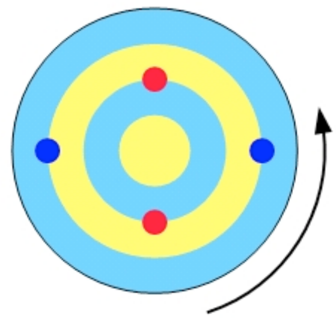


Figure 10.31: An overhead view of a turntable that is spinning counterclockwise at a constant rate. Four cylinders are moving with the turntable, two red ones that are equally distant from the center, and two blue ones that are the same distance from the center as one another but farther out than the red ones. For Exercises 1 and 2.

2. Return to the situation described in Exercise 2. (a) Draw a motion diagram for one of the red cylinders that corresponds to one complete rotation of the turntable. (b) Assuming the red cylinder is 2.0 m from the center of the turntable, construct a motion diagram that corresponds to the equivalent straight-line motion (as if you unrolled the motion diagram of the red cylinder you chose). Have we seen this kind of motion-diagram before? If so, what kind of motion did we classify it as?

3. A square sheet of plywood is subjected to four forces of equal magnitude, as shown in Figure 10.32. Relative to an axis that is perpendicular to the page and passes through the top left corner of the sheet, in which direction is the torque due to (a) $\vec{F}_1$; (b) $\vec{F}_2$; (c) $\vec{F}_3$; (d) $\vec{F}_4$?

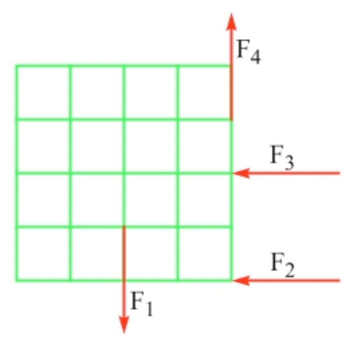


Figure 10.32: A square sheet of plywood subjected to four forces of equal magnitude, for Exercises 3 and 4.

4. Repeat Exercise 3, except this time use an axis that is perpendicular to the page and passes through the bottom left corner of the sheet.

5. A hockey puck is initially at rest on a frictionless ice rink. Two horizontal forces of equal magnitude are then simultaneously applied to the puck. For rotation, consider a vertical axis through the center of the puck. (a) Is it possible to apply the two forces so the puck has no acceleration and no angular acceleration? If so, sketch an example. (b) Is it possible to apply the forces so the puck's center-of-mass has no acceleration but the puck has a non-zero angular acceleration? If so, sketch an example. (c) Is it possible to apply the forces so the puck's center-of-mass has a non-zero acceleration but the puck has no angular acceleration? If so, sketch an example.

6. Many common household tools (hand tools, as opposed to power tools) enable us to make use of torque to make it easier to do something. A can opener is a good example of such a device. (a) Briefly describe how torque is involved in the operation of a human-powered can opener. (b) Name two other tools or devices you would find in a typical house that involve torque in their operation and briefly describe them.

7. Figure 10.33 shows a side view of a uniform rod of length L and mass M that is pinned at its left end by a frictionless hinge. The rod is held horizontal by means of a force F that is applied at a distance $3L/4$ along the rod. The angle between the rod and this force is θ. Fill in the two blanks in the following statement using either "increase", "decrease", or "stay the same." As the angle θ decreases, the torque associated with the force F must __________ while the magnitude of the force F must __________ so that the rod remains in equilibrium.

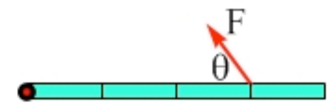


Figure 10.33: A side view of a rod that is hinged at its left end, and which is held in a horizontal position by an applied force *F*.

8. As shown in Figure 10.34, a rod, with a length of 80 cm and a mass of 6.0 kg, is attached to a wall by means of a hinge at the left end. The rod's mass is uniformly distributed along its length. A string will hold the rod in a horizontal position; the string can be tied to one of three points, lettered A-C, spaced at 20 cm intervals along the rod, starting with point A which is 20 cm from the left end of the rod. The other end of the string can be tied to one of three hooks, numbered 1-3, in the ceiling 30 cm above the rod. Hook 1 is directly above point A, hook 2 is directly above B, etc. For each case below, draw a line (and only one line) from one lettered point to one numbered hook representing the string you would use to achieve the desired objective. If you think it is impossible to achieve the objective, explain why. (a) How would you attach a string so the rod is held in a horizontal position with the hinge exerting no force at all on the rod? (b) How would you attach a string so the rod is held in a horizontal position while the force exerted on the rod by the hinge has no horizontal component, but has a non-zero vertical component directed straight up? (c) How would you attach a string so the rod is held in a horizontal position while the force exerted on the rod by the hinge has no vertical component, but has a non-zero horizontal component?

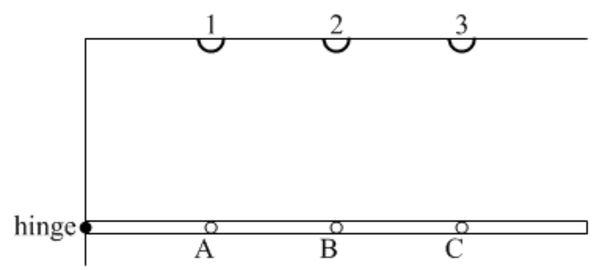


Figure 10.34: A hinged rod that you intend to hold horizontal by means of a string attached from one of the lettered points on the rod to one of the numbered hooks above the rod, for Exercise 8. This system represents a simple model of a broken arm you want to immobilize with a sling. The rod represents the lower arm, the hinge represents the elbow, and the string represents the sling.

9. You construct a mobile out of four objects, a sphere, a cube, a pyramid, and an ellipsoid. The mobile is in equilibrium in the configuration shown in Figure 10.35, where the vertical dashed lines are 20 cm apart. The mass of the strings (in blue) and rods (in red) can be neglected. If the pyramid has a mass of 400 g, what is the mass of each of the other objects?

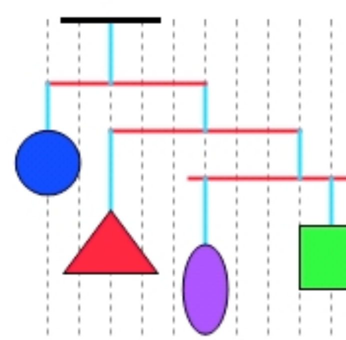


Figure 10.35: A mobile with four different objects, for Exercise 9.

10. Two cylinders have the same dimensions and mass, and the center-of-mass of each is at its geometric center. When you try to spin them, however, you notice that one cylinder is significantly more difficult to spin than the other. What is a good physical explanation for this? Assume you're trying to spin them about an axis through the center of the cylinder, perpendicular to the length of the cylinder, in each case.

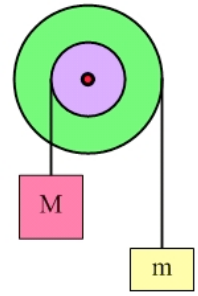


Figure 10.36: A dual-radius pulley system remains in equilibrium with two blocks hanging from it. For Exercise 11.

11. A pulley consists of a small uniform disk of radius $R/2$ mounted on a larger uniform disk of radius *R*. The pulley can rotate without friction about an axis through its center. As shown in Figure 10.36, a block with mass *m* hangs down from the larger disk while a block of mass *M* hangs down from the smaller disk. If the system remains in equilibrium, what is *M* in terms of *m*?

12. A particular type of leaning tower toy, as shown in Figure 10.37, remains upright until an extra piece is added to its top, at which point the tower falls over. Explain why this is.

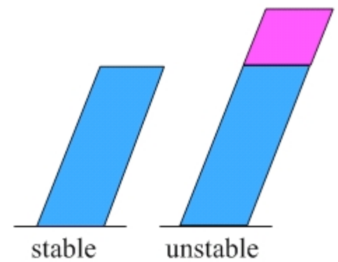


Figure 10.37: This leaning-tower toy tips over when the extra piece is added at the top. For Exercise 12.

Exercises 13 – 21 are designed to give you practice in solving a typical rotational kinematics problem. For each exercise, start with the following parts: (a) Draw a diagram of the situation. (b) Choose an origin to measure displacements from and mark that on the diagram. (c) Choose a positive direction and indicate that with an arrow on the diagram. (d) Create a table summarizing everything you know, as well as the unknowns you want to solve for. Try to solve all exercises using a similar systematic approach. Compare your approach to those you used for Exercises 33 – 42 in Chapter 2.

13. While repairing your bicycle, you have your bicycle upside down so the front wheel is free to spin. You grab the front wheel by the edge and smoothly accelerate it from rest, giving it an angular acceleration of 5.0 rad/s^2 clockwise. You let go after the wheel has moved through one-quarter of a revolution. Your goals in this exercise are to determine the wheel's angular velocity at the instant you let go and the time it took to reach that angular velocity. Parts (a) – (d) as described above. (e) Which equation(s) will you use to determine the wheel's final angular velocity? (f) Find that angular velocity. (g) Which equation(s) will you use to determine the time the wheel was accelerating? (h) Solve for that time.

14. You release a ball from rest at the top of a ramp, and it experiences a constant angular acceleration of 1.2 rad/s^2. At the bottom of the ramp, the ball is rotating at 4.0 revolutions per second. The goal here is to determine how long it took the ball to reach that speed. Parts (a) – (d) as above. (e) Which equation(s) will you use to determine the time it takes to reach 4.0 rev/s? (f) What is that time?

15. You spin a disk, giving it an initial angular velocity of 2.4 rad/s clockwise. The disk has an angular acceleration of 1.2 rad/s^2 counterclockwise. Your goal in this exercise is to solve for the maximum angular displacement of the disk from its initial position before reversing direction. Parts (a) – (d) as described above. (e) Which equation(s) will you use to determine the maximum angular displacement of the disk? (f) Solve for that angular displacement.

16. In this exercise, you will analyze the method you used in Exercise 15, so all these questions pertain to what you did to solve Exercise 15. (a) Is there only one correct choice for the origin? Why did you make the choice you made? (b) Is there only one correct choice for the positive direction? Would your answer to (f) above change if you chose the opposite direction to be positive? (c) Find an alternative method to determine the maximum angular displacement, and show that it gives the same answer as the method you used.

17. A cylinder is rolling down a ramp. When it passes a particular point, you determine that it is traveling at an angular speed of 30 rad/s, and in the next 2.0 seconds it experiences an angular displacement of 80 radians. The goal of this exercise is to determine the cylinder's angular acceleration, which we will assume to be constant. Parts (a) – (d) as described above. (e) Which equation(s) will you use to determine the angular acceleration? (f) What is the angular acceleration?

18. Repeat parts (e) and (f) of the previous exercise, but do not use the equation(s) you used in the previous exercise.

19. A pulley with a radius of 0.20 m is mounted so its axis is horizontal. A block hangs down from a string wrapped around the pulley, as shown in Figure 10.38. You give the pulley an initial angular velocity of 0.50 rad/s directed counterclockwise. The block takes a total of 6.00 s to return to the level it was at when you released it. Assuming the acceleration is constant through the entire motion, the goal of the exercise is to determine the maximum distance the block rises above its initial point. Parts (a) – (d) as described above. (e) Which equation(s) will you use to determine the maximum distance the block rose above its initial point? (f) What is that maximum distance?

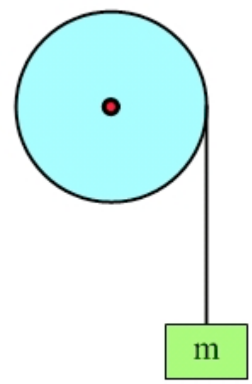


Figure 10.38: A block hanging down from a pulley, for Exercise 19.

20. An electric drill accelerates a drill bit, which has a radius of 3.0 mm, from rest to a maximum angular speed of 250 rpm (revolutions per minute) in 2.2 seconds. The goal of this exercise is to determine the drill bit's angular acceleration, assuming it to be constant. Parts (a) – (d) as described above. (e) What is the bit's angular acceleration?

21. With a quick flick of her wrist, an Ultimate Frisbee player can give a Frisbee an angular velocity of 8.0 revolutions per second. Assuming the player accelerates the Frisbee from rest through an angle of 75˚, the goal of this exercise is to determine the Frisbee's angular acceleration and the time over which this acceleration occurs. Parts (a) – (d) as described above. (e) What is the Frisbee's angular acceleration? (f) What is the time over which the acceleration occurs?

Exercises 22 – 26 involve calculating torque in various situations.

22. As shown in Figure 10.39, a disk mounted on an axle through its center is subjected to a 20 N force. The disk has a radius of 25 cm. What is the magnitude and direction of the torque associated with this force, measured with respect to an axis that is perpendicular to the page and (a) passes through the center of the disk? (b) passes through the point 50 cm above the center of the disk (i.e., move the axis up the page)? (c) 50 cm to the right of the center of the disk.

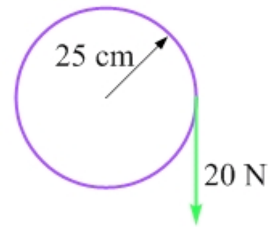


Figure 10.39: A disk of radius 25 cm that is subjected to a 20 N force, for Exercise 22.

23. A box measuring 3.0 m high by 4.0 m wide is subjected to a 10 N force, as shown in Figure 10.40. Consider an axis that is perpendicular to the page and which passes through the bottom left corner of the box. (a) Follow the procedures outlined in Exploration 10.6 to first determine the direction of the torque due to this force. Now determine the magnitude of the torque due to this force by (b) applying Equation 10.9 directly; (c) breaking the force into horizontal and vertical components before applying Equation 10.9; (d) using the lever arm method.

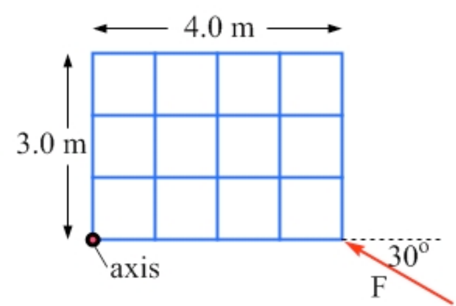


Figure 10.40: A box subjected to a 10 N force, for Exercise 23.

24. The plywood sheet shown in Figure 10.41 measures 2.0 m × 2.0 m, and each of the four forces the sheet is subjected to has a magnitude of 8.0 N. Relative to an axis that is perpendicular to the page and passes through the top left corner of the sheet, determine the magnitude of the torque due to (a) $\vec{F}_1$; (b) $\vec{F}_2$; (c) $\vec{F}_3$; (d) $\vec{F}_4$. (e) Find the magnitude and direction of the net torque, due to all four forces, about that axis.

25. Repeat Exercise 24, except this time use an axis that is perpendicular to the page and passes through the bottom left corner of the sheet.

26. Consider again the plywood sheet shown in Figure 10.41. Is there an axis that is perpendicular to the page about which the four forces give a net torque of zero? If so, where would such an axis be located? If there is at least one such axis, is there only one, or are there more than one? Explain.

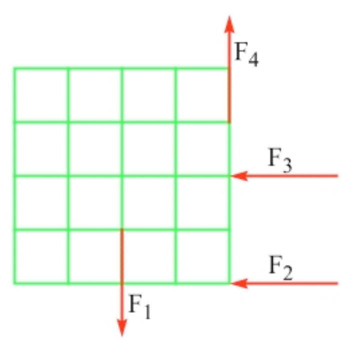


Figure 10.41: A square sheet of plywood subjected to four forces of equal magnitude, for Exercises 24 – 26.

Exercises 27 – 32 address issues associated with rotational inertia.

27. As shown in Figure 10.42, three identical balls, each with a mass $M = 1.0$ kg, are equally spaced along a rod of negligible mass. The distance between neighboring balls is 3.0 m, and you can assume that the radius of each ball is considerably less than the distance between them. If this system is spun about an axis that is perpendicular to the page, determine the system's total rotational inertia if the axis (a) passes through the center of the middle ball; (b) passes through the center of the ball at the left end.

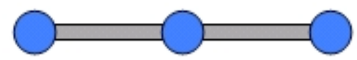


Figure 10.42: A system of three balls, for Exercise 27.

28. A solid sphere has a mass of $M = 8.0$ kg and a radius of $R = 20$ cm. Determine the sphere's rotational inertia about an axis (a) passing through the center of the sphere; (b) tangent to the outer surface of the sphere.

29. Two balls of negligible radius are connected by a rod with a length of 1.2 m and a negligible mass. One ball has a mass M, while the other has a mass of $2M$. (a) If you spin this system about an axis that is perpendicular to the rod, where should you place the axis to minimize the system's rotational inertia? (b) If $M = 1.0$ kg, what is this minimum rotational inertia?

30. Repeat Exercise 29, except now the balls are joined by a 1.2-meter uniform rod with a mass of $3M$.

31. Four balls of equal mass M are placed so that there is one ball at each corner of a square measuring $d \times d$. The balls are joined by rods of negligible mass that run along the sides of the square. Assume the radius of each ball is small compared to d. What is the rotational inertia of the system about an axis that is perpendicular to the plane of the square and passes through (a) the center of the square? (b) one of the corners of the square? (c) a distance d from the center of the square, in any direction.

32. The rotational inertia of a uniform rod of length L and mass M, about an axis through the end of the rod and perpendicular to the rod, is $I = ML^2/3$. Use this expression, and the parallel-axis theorem (Equation 10.10), to show that the rotational inertia of the rod about a parallel axis through the center of the rod is $I = ML^2/12$.

Exercises 33 – 38 are designed to give you practice in solving a typical static equilibrium problem. For each of these exercises begin with the following: (a) Draw a diagram of the situation. (b) Draw a free-body diagram to show each of the forces acting on the object.

33. A board with a weight of 40 N and a length of 2.0 m is placed horizontally on a flat roof with 75 cm of the board hanging over the edge of the roof. The goal of this exercise is to determine the magnitude and direction of the normal force exerted on the board by the roof, and the exact location the normal force can be considered to be applied. Parts (a) and (b) as described above. (c) Apply Newton's second law to your free-body diagram, and solve for the magnitude and direction of the normal force exerted on the board by the roof. (d) Choose an axis to take torques about. Why did you select the axis you did? (e) Apply Newton's second law for rotation to determine the location the normal force can be considered to act on the board.

34. Return to the situation described in Exercise 33, except now we will add a 20 N bucket of nails to the end of the board that is hanging out over the edge of the roof. Repeat parts (a) – (e). (f) What is the maximum weight that could be placed on the end of the board without the board tipping over? (g) Where would the normal force act in that situation?

35. A particular door consists of a uniform piece of wood, with a weight of 20 N, measuring 2.0 m high by 1.0 m wide. The door is mounted on two hinges, one 20 cm down from the top of the door and the other 20 cm up from the bottom. The door also has an ornate handle, with a weight of 10 N, located halfway down the door and 10 cm from the edge of the door farthest from the hinges. The goal of this exercise is to determine the horizontal components of each hinge force. Parts (a) – (b) as described above. (c) Apply Newton's second law in the horizontal direction to obtain a relationship between the horizontal components of the hinge forces. (d) Now, choose an axis to take torques about so you can solve for the horizontal component of the force exerted on the door by the bottom hinge. Explain why you chose the axis you did. (e) Apply Newton's second law for rotation and solve for the horizontal component of the force exerted on the door by the bottom hinge. (f) Solve for the horizontal component of the force exerted on the door by the upper hinge.

36. Consider the following design for a one-person seesaw. The seesaw consists of a uniform board with a length of 5.0 m and a mass of 40 kg balanced on a support that is 2.0 m from one end. Julie, with a mass of 16 kg, sits on the board so that the system is in balance with the board horizontal. The goal is to determine how far from the support Julie is. Parts (a) – (b) as described above. (c) Which side of the board is Julie on, the side with 2.0 m or the side with 3.0 m of the board extending beyond the support? (d) Choose an axis to take torques about. Justify your choice of axis. (e) Choose a positive direction for rotation and apply Newton's second law for rotation to find Julie's position.

37. A ladder with a length of 5.0 m and a weight of 600 N is placed so its base is on the ground 4.0 m from a vertical frictionless wall, and its tip rests 3.0 m up the wall. The ladder remains in this position only because of the static friction force between the ladder and the ground. The goal of this exercise is to determine the magnitude of the normal force exerted by the wall on the ladder, and the minimum possible value of the coefficient of static friction for the ladder-ground interaction. Assume the mass of the ladder is uniformly distributed. Parts (a) – (b) as described above. (c) Apply Newton's second law twice, once for the horizontal forces and once for the vertical forces, to find relationships between the various forces applied to the ladder. (d) Choose an axis to take torques about so that you can solve for the normal force exerted by the wall on the ladder. Justify why you chose the axis you did. (e) Choose a positive direction for rotation and apply Newton's second law for rotation to find the normal force exerted by the wall on the ladder. (f) Solve for the minimum possible coefficient of static friction so the ladder remains in equilibrium.

38. Repeat Exercise 37, with the addition of you, with a weight of 500 N, standing on the ladder so that you are a horizontal distance of 1.0 m from the wall.

General Problems and Conceptual Questions

39. You drop a ball from rest from the top of a tall building. Let's assume the ball has an acceleration of 10 m/s^2 directed straight down. At the same time you drop the ball, you flick a switch that starts a motor, giving a disk that was initially at rest an angular acceleration of 10 rad/s^2 directed clockwise. Write a paragraph or two comparing and contrasting these two motions.

40. Return to the situation described in Exercise 39. (a) For the first four seconds of the motion, plot graphs of the ball's acceleration, velocity, and position as a function of time, taking down to be positive. (b) Over the same time period, plot graphs of the disk's angular acceleration, angular velocity, and angular position as a function of time, taking clockwise to be positive. (c) Comment on the similarities between the two sets of graphs.

41. Consider again the situation described in Exercise 39. The disk rotates about an axis through its center. It turns out that all points on the disk at a particular distance from the disk have a speed that matches the speed of the ball at all times (at least until the ball hits the ground!). What is this distance?

42. In the "old days", long before CD's and MP3's, people listened to music using vinyl records. Long-playing vinyl records spin at a constant rate of 33⅓ rpm. The music is encoded into a continuous spiral track on the record that starts at a radius of about 30 cm from the center and ends at a radius of about 10 cm from the center. If a record plays for 24 minutes, estimate how far apart the tracks are on the record.

43. Let's say that, during your last summer vacation, you drove your car across the United States from Boston to Seattle, staying on interstate I-90 the entire time. Estimate how many revolutions each tire made during this trip.

44. You are driving your car at a constant speed of 20 m/s around a highway exit ramp that is in the form of a circular arc of radius 100 m. What is the magnitude of your (a) angular velocity? (b) centripetal acceleration? (c) tangential acceleration? (d) angular acceleration?

45. Repeat Exercise 44, except now your speed is decreasing. At the instant we are interested in, your speed is 20 m/s, and you are planning to come to a complete stop at a red light in 5.0 s. Assume your acceleration is constant.

46. The rotational inertia of a uniform disk of mass M and radius R is found to be MR^2 about an axis perpendicular to the plane of the disk. How far is this axis from the center of the disk?

47. Archimedes once made a famous statement about a lever, which relies very much on the principle of torque, using words to the effect of "*Give me a lever long enough, and a fulcrum on which to place it, and I shall move the world*." Briefly explain what Archimedes was talking about.

48. Figure 10.43 shows four different cases involving a uniform rod of length L and mass M that is subjected to two forces of equal magnitude. The rod is free to rotate about an axis that either passes through one end of the rod, as in (a) and (b), or passes through the middle of the rod, as in (c) and (d). The axis is marked by the red and black circle, and is perpendicular to the page in each case. This is an overhead view, and we can neglect any effect of the force of gravity acting on the rod. Rank these four situations based on the magnitude of the net torque about the axis, from largest to smallest.

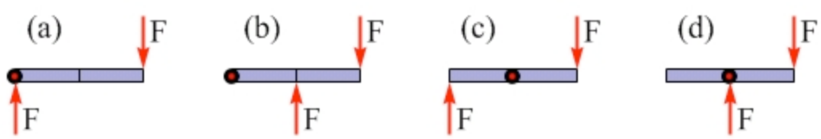


Figure 10.43: Four situations involving a uniform rod, which can rotate about an axis, being subjected to two forces of equal magnitude. For Exercises 48 and 49.

49. Return to the situation described in Exercise 48 and shown in Figure 10.43. If the rod has a length of 1.0 m, a mass of 3.0 kg, and each force has a magnitude of 5.0 N, determine the magnitude and direction of the net torque on the rod, relative to the axis in (a) Case (a); (b) Case (b); (c) Case (c); (d) Case (d).

50. Figure 10.44 shows a side view of a uniform rod of length L and mass M that is pinned at its left end by a frictionless hinge. The rod is held horizontal by means of a force F that is applied at a distance $3L/4$ along the rod. Determine the angle θ between the rod and the force F, such that the magnitude of F is exactly equal to the magnitude of the force of gravity acting on the rod.

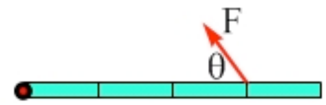


Figure 10.44: A side view of a rod that is hinged at its left end, and which is held in a horizontal position by an applied force F. For Exercise 50.

51. A rod, with a length of 80 cm and a mass of 6.0 kg, is attached to a wall by means of a hinge at the left end. The rod's mass is uniformly distributed along its length. A string will hold the rod in a horizontal position; the string can be tied to one of three points, lettered A-C, spaced at 20 cm intervals along the rod, starting with point A which is 20 cm from the left end of the rod. The other end of the string can be tied to one of three hooks, numbered 1-3, in the ceiling 30 cm above the rod. Hook 1 is directly above point A, hook 2 is directly above B, etc. **Use $g = 10$ m/s^2.** As shown in Figure 10.44, a string is attached from point A to hook 3. Remember that point B is 40 cm from the hinge. (a) Calculate the tension in the string. (b) Determine the magnitude and direction of the horizontal component of the hinge force. (c) Determine the magnitude and direction of the vertical component of the hinge force.

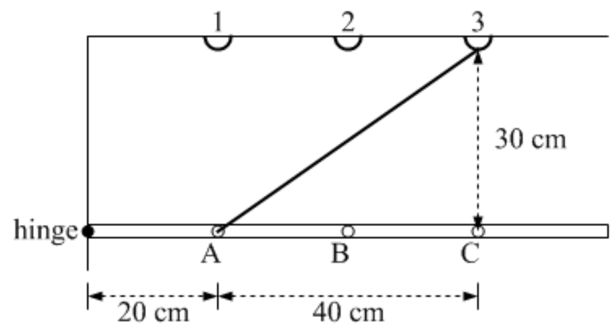


Figure 10.45: A diagram of a hinged rod, held horizontal by a string. Point B, in the middle of the uniform rod, is 40 cm from the hinge. For Exercises 51 – 53.

52. Repeat Exercise 51, with the string holding the rod horizontal attached from point A to hook 2 instead.

53. Repeat Exercise 51, with the string holding the rod horizontal attached from point C to hook 2 instead.

54. It is often useful to treat the lower arm as a uniform rod of length L and mass M that can rotate about the elbow. Let's say you are holding your arm so your upper arm is vertical (with your elbow below your shoulder), with a 90˚ bend at the elbow so the lower arm is horizontal. In this position we can say that three forces act on your lower arm: the force of gravity (Mg), the force exerted by the biceps, and the force exerted at the elbow joint by the humerus (the bone in the upper arm). Let's say the biceps muscle is attached to the lower arm at a distance of $L/10$ from the elbow, moving away from the elbow toward the hand. (a) Compare the force of gravity with the biceps force. Which has the larger magnitude? Briefly justify your answer. (b) Compare the force from the biceps with the force from the humerus. Which has the larger magnitude? Briefly justify your answer.

55. Return to the situation described in Exercise 54. If you now hold a 20 N object in your hand, at a distance L from the elbow joint, and your arm remains in the position described, the force from the biceps increases. (a) By how much does the force from the biceps increase? (b) Does the fact that you are holding a 20 N object in your hand change the force applied to your lower arm by the humerus at the elbow joint? If so, state both the magnitude and the direction of the change.

56. A uniform rod of mass M and length 1.0 m is attached to a wall by a hinge at one end. The rod is maintained in a horizontal position by a vertical string that can be attached to the rod at any point between 20 cm and 100 cm from the hinge. (a) Defining d to be the distance from the hinge to where the string is attached, plot a graph of the magnitude of the torque exerted on the rod by the string as a function of d, for $20\,\text{cm} \le d \le 100\,\text{cm}$. (b) Over the same range of d values, plot a graph of the magnitude of the tension in the string as a function of d.

57. Figure 10.46 shows an overhead view of a piece of wood, with a mass of 2.0 kg, that is on a slippery ice rink. Three horizontal forces are shown on the rod, and there is a fourth force of unknown magnitude, direction, and location that is not shown. Determine the magnitude, direction, and location of the mystery force if the piece of wood remains at rest.

58. Consider again the situation described in the previous exercise and shown in Figure 10.46. Is it possible to apply a fourth force so the piece of wood does not spin but accelerates at 3.0 m/s^2 to the right? Justify your answer.

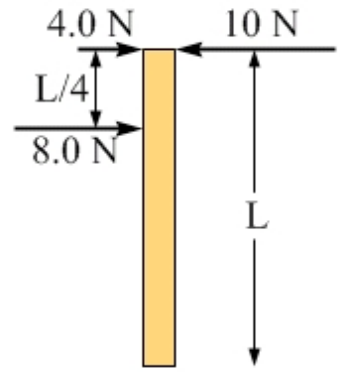


Figure 10.46: An overhead view of a piece of wood on a slippery ice rink. Four forces are applied to the wood but only three are shown. For Exercises 57 and 58.

59. Consider the mobile shown in Figure 10.47, in which all the balls have equal mass and in which the weight of the vertical strings and horizontal rods can be neglected. The vertical dashed lines in the figure are 10 cm apart. (a) Is this mobile in equilibrium in the configuration shown? How do you know? (b) If you add one additional ball to the configuration shown in the diagram can you get the mobile to be in equilibrium? If not, explain why not. If so, explain where you would place the additional ball. (c) If you have concluded that the mobile is not an equilibrium as shown, and that adding one additional ball will not achieve equilibrium, what is the minimum number of balls you can add to the system to achieve equilibrium, and where would you put them?

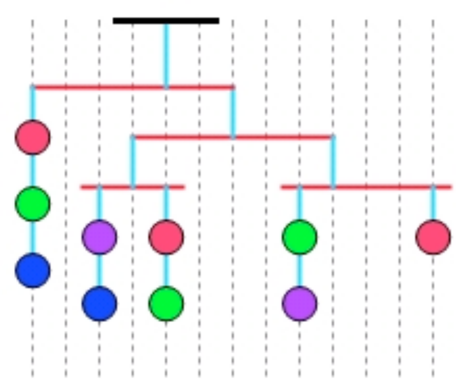


Figure 10.47: A mobile consisting of several balls of equal mass, for Exercise 59.

60. You are using a wheelbarrow to move a heavy rock, as shown in Figure 10.48. The diagram shows the location of the upward force you exert and the location of the force of gravity acting on the rock – wheelbarrow system. (a) Assuming the wheelbarrow is in equilibrium, how does the magnitude of the force you exert compare to the magnitude of the force of gravity acting on the system? (b) If the force of gravity has a magnitude of 420 N, solve for the magnitude of the force you exert on the wheelbarrow, and the magnitude and direction of any other force or forces necessary to maintain equilibrium.

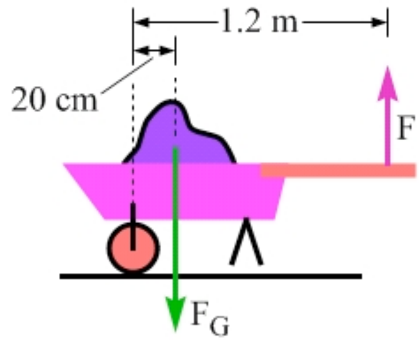


Figure 10.48: A side view of a wheelbarrow you are using to move a heavy rock, for Exercise 60.

61. Consider again the one-person seesaw described in Exercise 36. Would your answer for where Julie must sit to maintain equilibrium change if the system was on the Moon, as opposed to the Earth? Justify your answer.

62. Two of your friends, Latisha and Jorge, are carrying on a conversation about a physics problem. Comment on each of their statements, and state the answer to the problem the two of them are working on.

Latisha: In this problem, we have a uniform rod with a weight of 12 newtons, and it is supported at one end by a hinge. The question is, what is the smallest force that we can apply to keep the rod horizontal?Don't we need to apply a 12 newton force, at least, to hold it up?

Jorge: I think the idea is that the hinge can help support some of the weight, so, if we hold it in the right spot, we can apply a force that is less than 12 newtons.

Latisha: What if we start in the middle? If we hold it in the middle, then the rod is perfectly balanced, and we don't need the hinge at all. In that case, then we'd just need to apply a 12 newton force up to balance the 12 newton force of gravity, right?

Jorge: I think so. So, then, if we want to apply less force, should we move our force toward the hinge or away from the hinge? How do we figure that out if we don't know what the hinge force is?

Latisha: I guess this is why we're learning about torques. If we take torques about the hinge, then I think the hinge force cancels out – the distance is zero, for the torque from the hinge force. Then, if we apply our force farther from the hinge, can't we apply less force? But, how do we know how much less? We don't even know the length of the rod!

- Contents >
- Chapter 10: Additional Resources

Chapter 10: Additional Resources

Pre-session Movies on YouTube

- Rotational Kinematics
- Torque
- Static Equilibrium
- Newton's Second Law for Rotations

Examples

- Sample Questions

Solutions

- Answers to Selected End of Chapter Problems
- Sample Question Solutions

Additional Links

- *PhET* simulation: Ladybug Revolution