# ON THE GROWTH OF BUSINESS FIRMS

# A GENERALIZED MODEL OF PROPORTIONAL GROWTH



# Innovation, Business Firms Growth, and Aggregate Fluctuations

Growth and Volatility: Micro data availability and statistical regularities across domains

Innovation, turnover, and the composition of the economy: Micro level shocks diluted by aggregation or key elements to understand aggregate volatility?

Unevenness of Human Know How, Capabilities, Variety, and Development: Institutions and Firms

# Innovation, Instability, and Growth: Preliminary Discussion

*Human Know How, Capabilities, and the Wealth of Nations*. Entry of new business opportunities/new technologies/new firms is a key element of known growth distributions.

*Animal Spirits', Firms, Market Structure, and 'the State of the Economy'*: Firm level and sector specific shocks are an important part of business cycle fluctuations.

*Diversification and Country Size reduce Volatility*, but less than expected. *Rare and extreme events are not outliers*.

*Entry, Growth, and Unevenness* as instantiations of the same process. Human capital accumulation, industrial and social mobility.

Scarce capabilities, clustering, low frequency processes, vs. global, high frequency, amplifying factors: *A Supranational Dimension for Investment, Stabilization, and Redistribution*.

# Firm growth models

The distribution of firms size is the outcome of underlying dynamics such as: entry of new firms, entry of new products, merger, acquisitions, firm exit, etc.

Several models have been proposed to account for these dynamics, most referred to the Gibrat's law or Simon model as useful benchmarks.

In 1931 Gibrat proposed a model where the expected value of a firm's growth rate is independent form its size.

The predictions of the Gibrat's stochastic model can be summarized as follow:

1. The size distribution of firms is lognormal
2. The growth distribution of firms is lognormal
3. The relationship between size and average growth rate is flat
4. The relationship between size and variance of growth rate is flat

# Firm growth models

The model developed by Simon and co-authors ([Simon and Bonini 1958], [Ijiri and Simon1964]) allows for the entry of new firms and new business opportunities into the market.

In Simon's framework, the market consists of a sequence of many independent "opportunities" which arise over time, each of size unity. At time t a new firm is born with probability *a* or an existing firm, randomly selected with probability ß proportional to its size, increases its size with probability *1 -a*.

For this stochastic process can be proved that:

1. the size distribution of firms is Pareto
2. the relationship between size and average growth rates is flat
3. the relationship between size and variance of growth rates has a power law behavior = $\sigma(S) \sim S \exp(-\beta(S))$ with $\beta=1/2$

# Firm growth models

The Bose-Einstein growth process is a system where:

-all the new business opportunities are captured by existing firms;

-all units grow uniformly;

-the size of entering units is equal to the size of existing units.

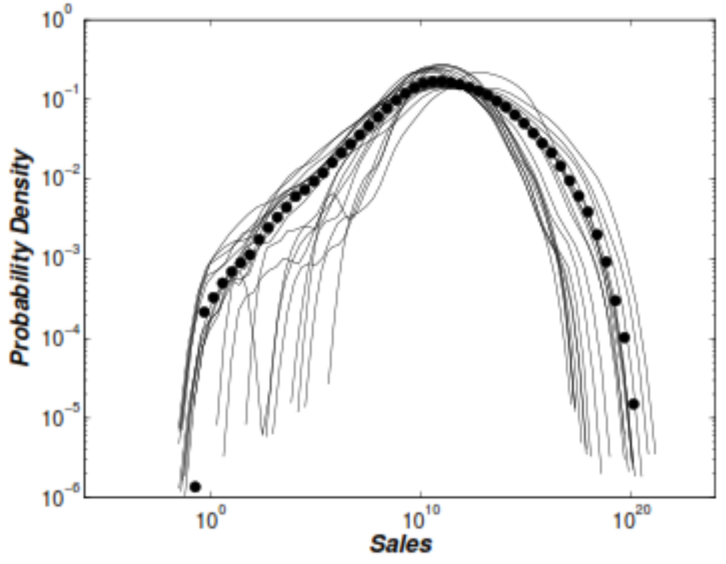
-In this scheme, can be proved that:

-

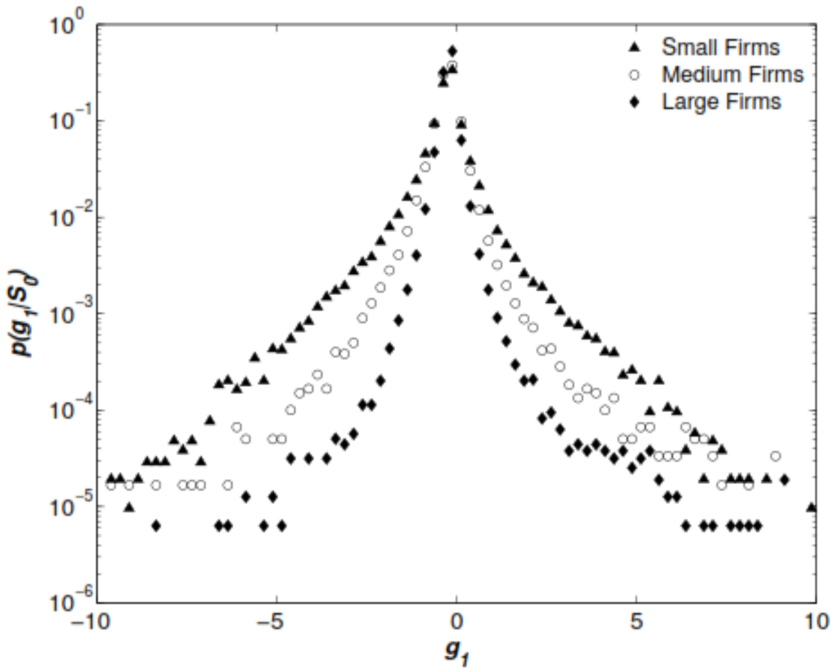
1. the size distribution of firms is exponential
2. the growth distribution of firms is tent shape $\sim r \uparrow -3$
3. the relationship between size and average growth rates is flat
4. the relationship between size and variance of growth rates has a power law behavior = σ(S)˜S exp(−β(S)) with β=1/2

# Empirical Evidence I

The size distribution of firm is skewed

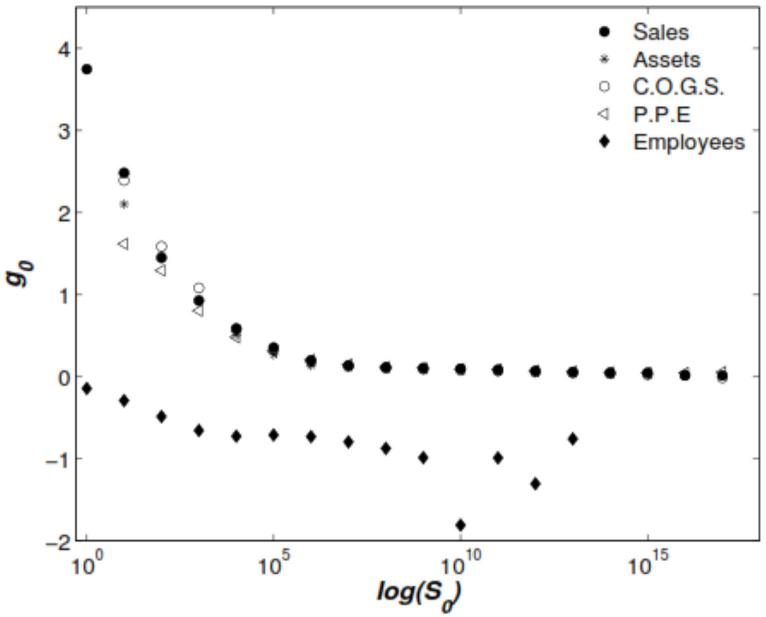



The growth rate distribution is not gaussian but «tent shape»

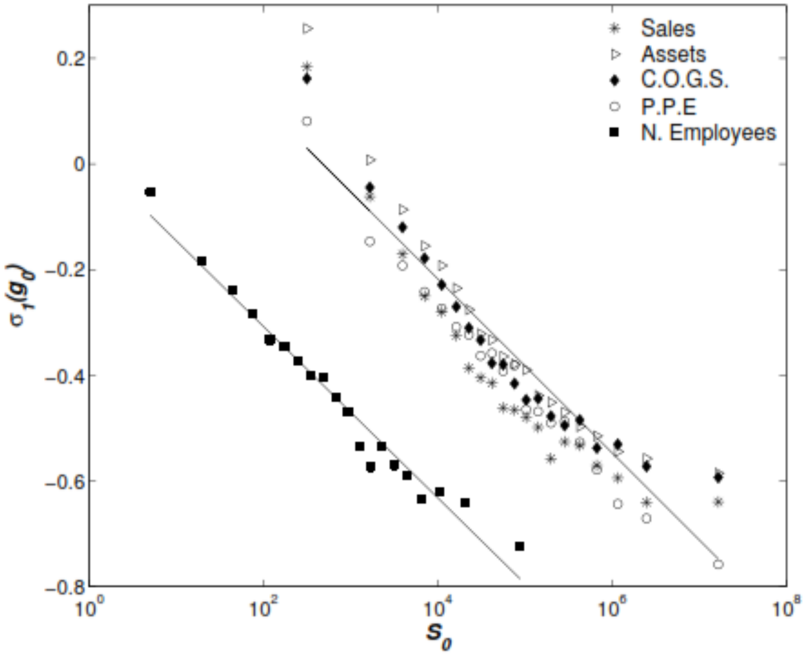


# Empirical Evidence II

The relationship between size and average growth rate is negative




The variance of growth rates is systematically higher for smaller firms



# The Generalized Proportional Growth Model (GPGM)

The GPGM model includes the Gibrat proportional growth and the Simon preferential attachment:

— the number of units in a firm grows in proportion to its existing number of units (the Simon growth process);

— the size of each unit grows in proportion to its size, independently of other units (the Gibrat growth process).

Two key sets of assumption in the model are that the number of units in a class grows in proportion to the existing number of units (1-4) and the size of each unit fluctuates in proportion to its size (5-7)

# Innovation and Growth: A Stochastic Benchmark

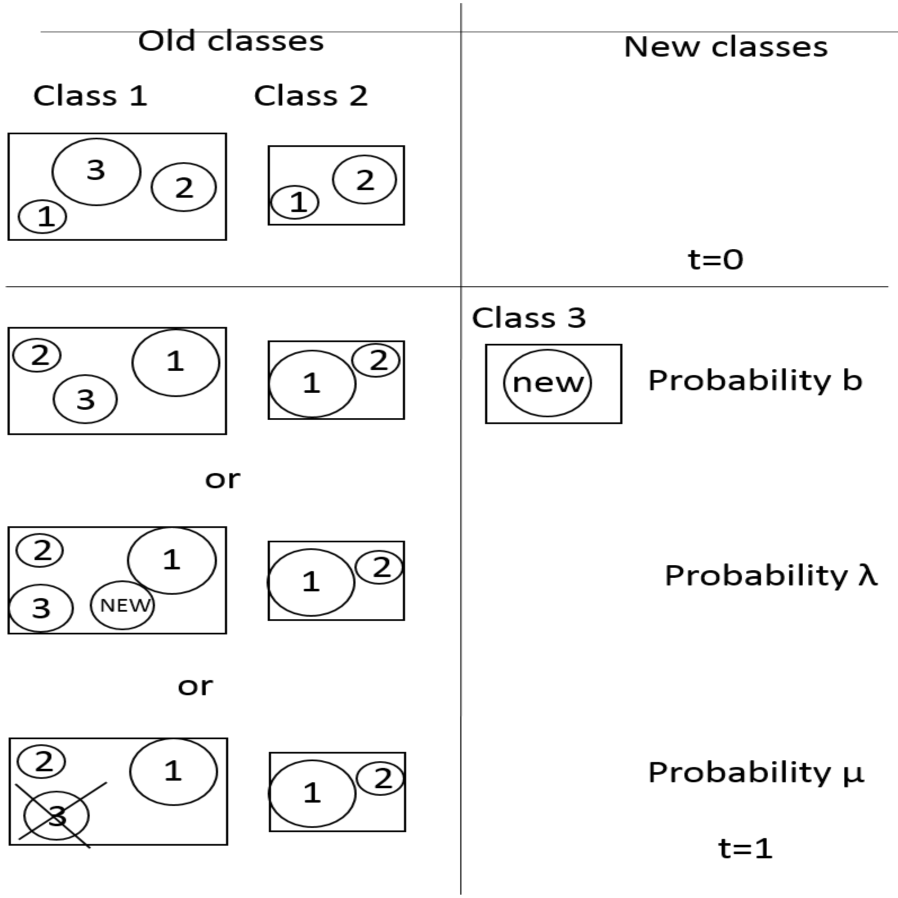



Fig 1. Schematic representation of the stochastic benchmark. At time t=0, there are N(0)=2 classes (□) and n(0)=5 units (○) Assumption (1). The area of each circle is proportional to the size ξ of the unit, and the size of each class is the sum of the areas of its constituent units (assumption 4) At time t=1 a new unit is created or deleted. With probability b the new unit is assigned to a new class (Class 3 in the figure) (Assumption 2). The size of the unit is taken from the distribution of the existing units (assumption 5). With probability λ the new unit is assigned to an existing class with probability proportional to the number of units in the class (assumption 2). In this example, a new unit is assigned to class 1 with probability 3/5 or to class 2 with probability 2/5. With probability μ, a randomly selected unit is deleted.

# The Generalized Proportional Growth Model (GPGM)

(1) At time $t$ the system consists of $N(t)$ firms. Each firm $i$ consists of $K_i(t)$ units. We characterize the system by the number of firms, $N_k(t)$, consisting of exactly $k$ units. By definition

$$N(t) = \sum_{k=0}^{\infty} N_k(t).$$

The total number of units in the system $n(t)$ is

$$n(t) = \sum_{k=0}^{\infty} k N_k(t) \equiv \langle K(t) \rangle N(t),$$

where $\langle K(t) \rangle$ is the average number of units in the firm. We assume that at time $t = 0$ there are $N_k(0)$ firms consisting of $k$ units. We denote the initial number of firms and units as $N(0) \equiv N_0$ and $n(0) \equiv n_0$, respectively. Accordingly, we introduce initial average number of units in the firm

$$\langle k \rangle = n_0 / N_0 = \langle K(0) \rangle.$$

We introduce $N_0(t)$ to account for the currently inactive firms, those that have lost all their units. We define the initial distribution of firm sizes as $P_k^o = N_k(0)/N_0$

# The Generalized Proportional Growth Model (GPGM)

(2) At each time interval $\Delta t$, a number of new units $\Delta_\lambda n$ is created in proportion to the current size of the economy measured in the total number of units: $\Delta_\lambda n = \lambda n(t)\Delta t$, where $\lambda$ is the growth rate. These units are distributed among existing firms with probability $p_i$, which is proportional to the size of firm $i$: $p_i = K_i(t)/n(t)$.

(3) At each time step, any unit can be deleted with probability $\mu$. Thus the number of units deleted during time interval $\Delta t$ is $\Delta_\mu n = \mu n(t)\Delta t$. The probability that a deleted unit belongs to the firm $i$ is $p_i = K_i(t)/n(t)$.

# The Generalized Proportional Growth Model (GPGM)

(4) At each time interval $\Delta t$, a number of new firms $\Delta_\nu N = \nu' n(t)\Delta t$ is created, where $\nu'$ is the birth rate of new firms. We assume that a new firm has $k$ units with probability $P'_k$. Thus the total number of units added to new firms is $\Delta_\nu n = \nu n(t)\Delta t$, where

$$\nu \equiv \nu' \sum_k P'_k k = \nu' \langle k \rangle'$$

and $\langle k \rangle'$ is the average number of units in the new firms.

# The Generalized Proportional Growth Model (GPGM)

Based on Assumptions (1-4) the number of units $n(t)$ obeys a differential equation

$$\frac{dn}{dt} = (\lambda - \mu + \nu)n(t)$$

from where

$$n(t) = n_0 e^{(\nu+\lambda-\mu)t}$$

and

$$N(t) = \frac{\nu'}{\nu + \lambda - \mu}(n(t) - n_0) + N_0$$

# The Generalized Proportional Growth Model (GPGM)

(5) At time $t$, each firm $i$ has $K_i(t)$ units of size $\xi_j(t)$, $j = 1, 2, ...K_i(t)$ where $\xi_j > 0$ are independent random variables taken from the distribution $P_\xi$. We assume that $\mathrm{E}[\ln \xi_i(t)] \equiv m_\xi$ and $\mathrm{Var}[\ln \xi_i(t)] = \mathrm{E}[(\ln \xi_i)^2] - m_\xi^2 \equiv V_\xi$, where $\mathrm{E}[x]$ and $\mathrm{Var}[x]$ are respectively mathematical expectation and variance of a random variable $x$. The size of a firm is defined to be $S_i(t) \equiv \sum_{j=1}^{K_i(t)} \xi_j(t)$.

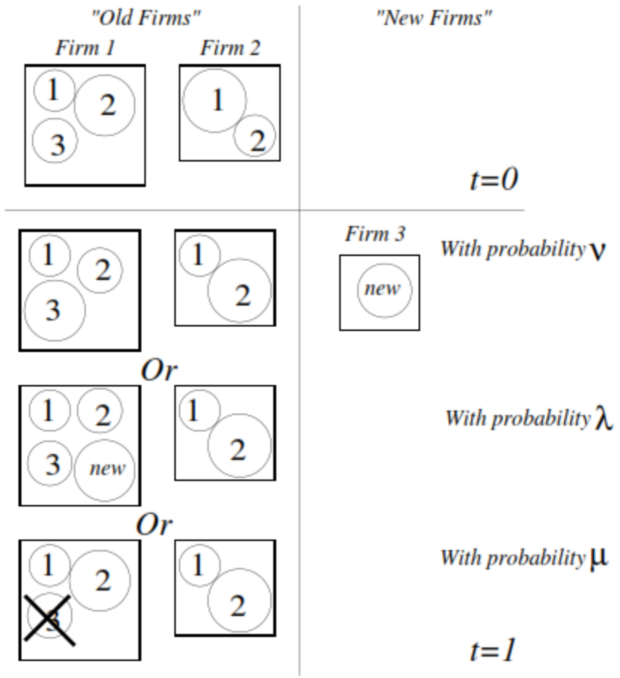
(6) At time $t+1$, the size of each unit is decreased or increased by a random factor $\eta_j(t) > 0$ so that

$$\xi_j(t+1) = \xi_i(t)\, \eta_i(t).$$

We assume that $\eta_j(t)$, the growth factor of unit $j$, is a random variable taken from a given probability distribution $P_\eta$. It is assumed that $\mathrm{E} \ln \eta_i(t) \equiv m_\eta$ and $\mathrm{Var}[\ln \eta_i(t)] = \mathrm{E}[(\ln \eta_i)^2] - m_\eta^2 \equiv V_\eta$. Note that $\eta_j$ is independent of $\xi_j, K_i$ and all other random variables characterizing the firm.

# The Generalized Proportional Growth Model (GPGM)

(7) The size of each new unit arriving at time $t$ is drawn randomly from the distribution of unit sizes $P_\xi$. Its expected size is denoted $\bar{\xi}(t)$.



# The Generalized Proportional Growth Model (GPGM)

Based on these assumption we derive the predictions for:

- the size distribution of firms;
- the distribution of firm growth rates;
- the size-mean growth rate relationship
- the size-variance relationship.

# The Generalized Proportional Growth Model (GPGM)

In general for a given distribution $P_\xi(\xi)$ the distribution of the firm size is given by: $$P(S) = \sum_{K=1}^{\infty} P(S|K) P_K ,$$

Where: $P(S|K) = P_\xi^{(K)}(S),$ is the sum of K independent variables (i.e. convolution of K distributions $P_\xi(\xi)$) .

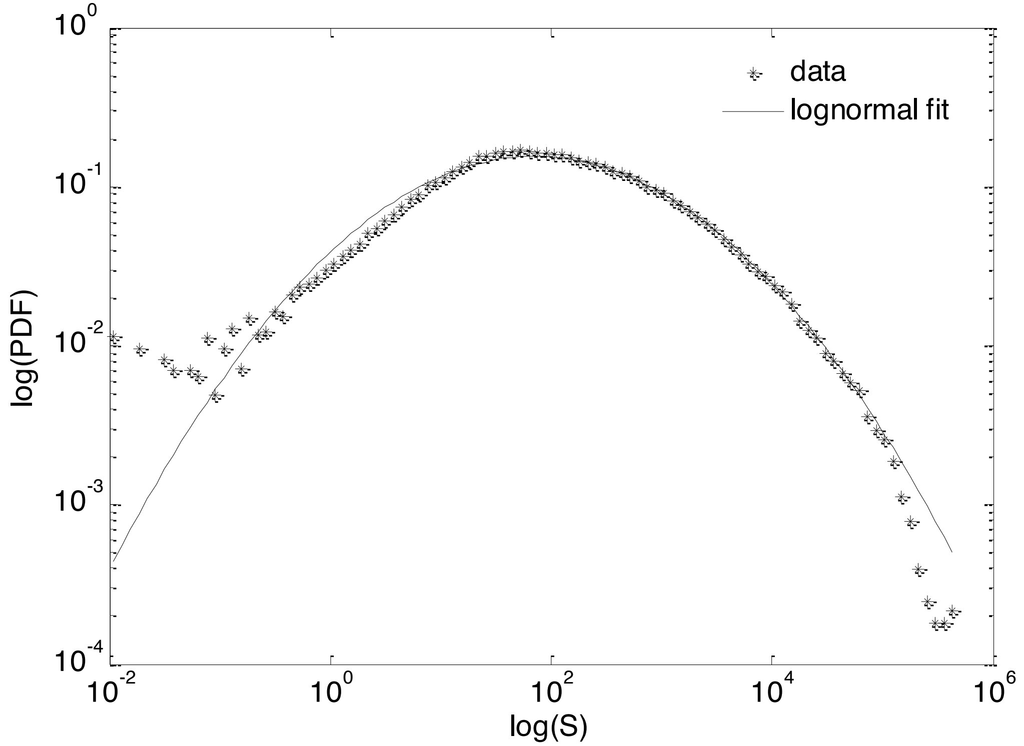
we have convolution of lognormals → the convergence to a Gaussian depends on $V_\xi$ and K.

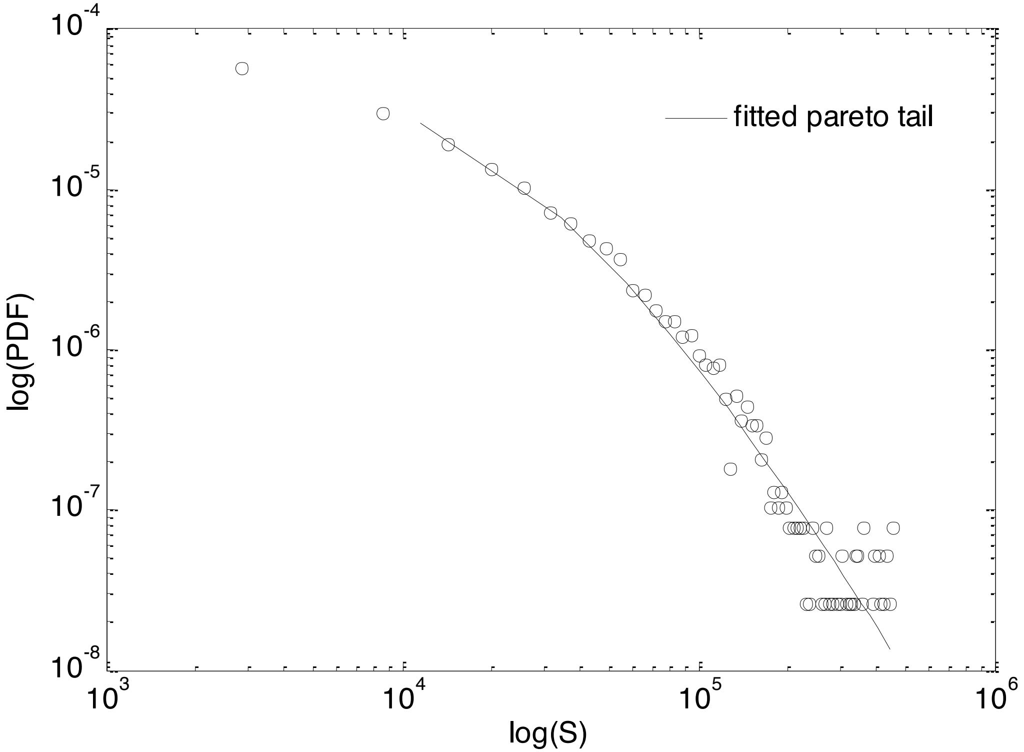
$P_k$→ distribution of number of units within classes → with our assumption, con be proved that the $P_k$ is Pareto with exponential cut off

P(S)→ is a lognormal distribution with a Pareto tail.

# The Generalized Proportional Growth Model (GPGM)

if $\nu$>0, i.e. system with new entries, the distribution of the firm sizes converge to a lognormal with a Pareto right tail.



# The Generalized Proportional Growth Model (GPGM)

The distribution of growth rates is given by:

$$P_r(r) \equiv \sum_{K=1}^{\infty} P_K P_r(r|K),$$

where $P_k$ is the distribution of the number of the units in each firm, while $P_r(r|K)$ is the conditional distribution of growth rates of firms with a given number of units determined by the distribution $P_\xi(\xi)$ and $P_\eta(\eta)$.

We will assume that both $P_\xi(\xi)$ and $P_\eta(\eta)$ are lognormal distributions (as it follows from the Gibrat growth process):

$$P_\xi(\xi) = \frac{1}{\xi\sqrt{2\pi V_\xi}} e^{-\frac{(\ln \xi - m_\xi)^2}{2MV_\xi}},$$

$$P_\eta(\eta) = \frac{1}{\eta\sqrt{2\pi V_\eta}} e^{-\frac{(\ln \eta - m_\eta)^2}{2MV_\eta}},$$

# The Generalized Proportional Growth Model (GPGM)

In this case is not possible to obtain a closed form for P(r|K) and for its mean and variance.
The exact solution exists only in the limiting cases K→0 and K→∞. In the limit of very large K, P(r|K) converge to a Gaussian:

$$P_r(r|K) = \frac{\sqrt{K}}{\sqrt{2\pi V_r}} \exp\left(-\frac{(r-m_r)^2 K}{2V_r}\right)$$

Where the mean growth rate is: $m_r = m_\eta + V_\eta/2 + \ln(1+\lambda-\mu),$

and the normalized variance is:

$$V_r = \frac{(1+\lambda-\mu)\exp(V_\xi)[\exp(V_\eta)-1] + (\lambda+\mu)\exp(V_\xi)}{(1+\lambda-\mu)^2} + \frac{(\lambda^2-\mu^2)[\exp(V_\xi)-1]}{(1+\lambda-\mu)^2}.$$

# The Generalized Proportional Growth Model (GPGM)

Assuming that $P_k$ is exponential with <K>= k and replacing the summation with the integration (to find a close form), we obtain:

$$P_r(r) \approx \frac{1}{\sqrt{2\pi V_r}} \int_0^{\infty} \frac{1}{\kappa(t)} \exp\left(\frac{-K}{\kappa(t)}\right) \exp\left(-\frac{(r-m_r)^2 K}{2\,V_r}\right) \sqrt{K}\, dK,$$

$$= \frac{\sqrt{\kappa(t)}}{2\sqrt{2\,V_r}} \left(1 + \frac{\kappa(t)}{2V_r}(r-m_r)^2\right)^{-\frac{3}{2}},$$

A tent shape distribution that asymptotically decays at $1/r^3$.

# The Generalized Proportional Growth Model (GPGM)

PROBLEM: Assuming that the growth rates ln(η) for individual units is lognormal, the GPGM predicts a growth rate distribution at the firm level that is tent shape only when the distribution of number of units in the classes is not strongly dominated by classes with few units (i.e exponential distribution).

Otherwise, the growth distribution does not develop tent-shape wings, since the behavior for large *r* is dominated by the distribution of the growth rates ln(η), which is assumed to be gaussian.

Empirical investigations suggest that products growth rate distribution is already tent-shaped. Elementary units.

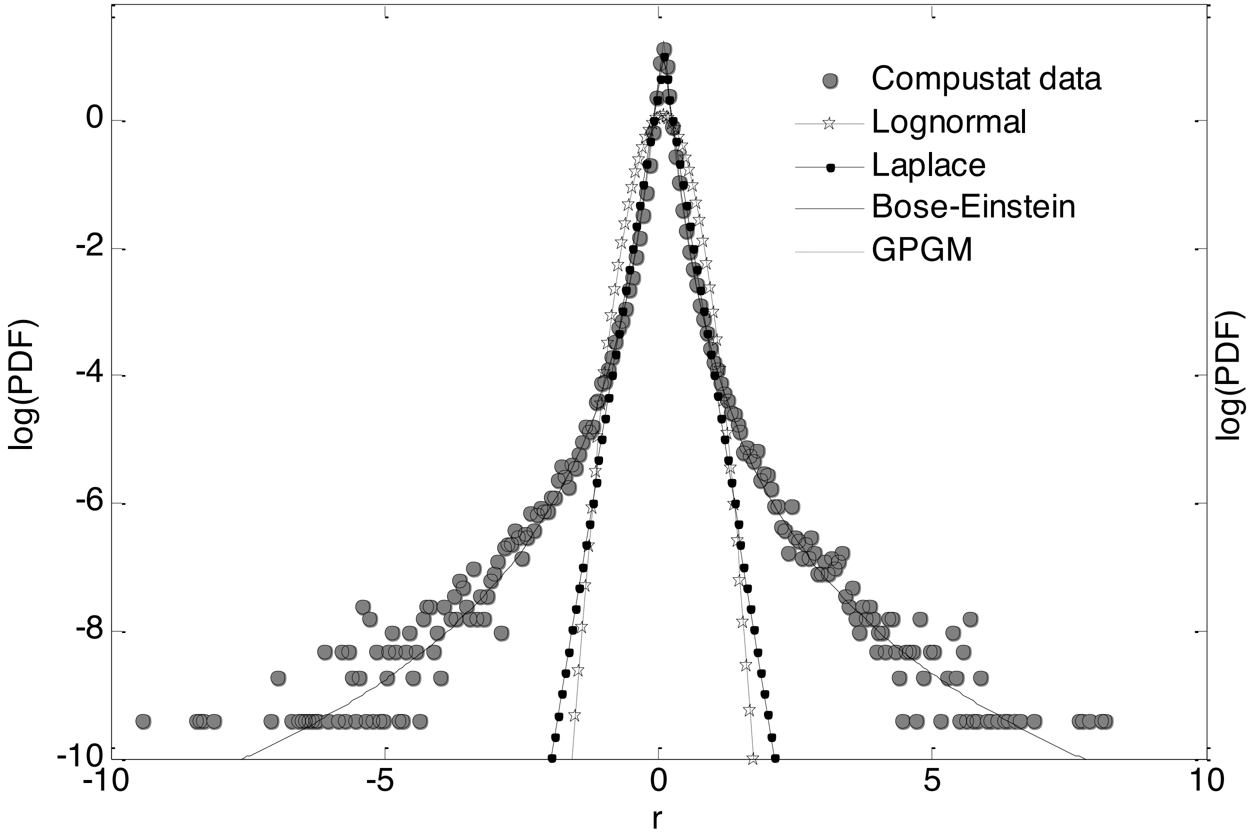
We derived a two-step GPGM model:

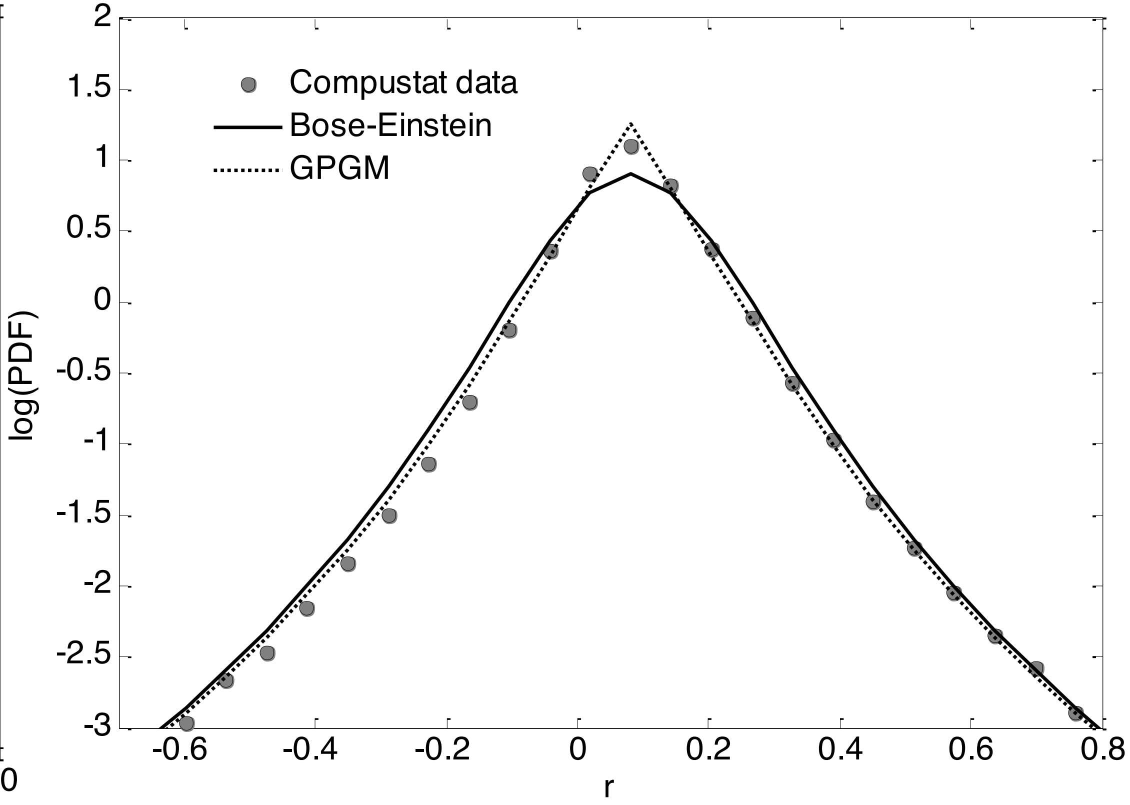
-the most elementary units (that are not observable) have lognormal distribution of $P_\xi(\xi)$ and $P_\eta(\eta)$;

-the first level of aggregation (i.e.products of firms) consist of L elementary units with geometric distribution $P_1(L)$;

-the second level of aggregation (i.e. firms) consist of M of these compounds units with Pareto distribution of $P_2(M)$.

# The Generalized Proportional Growth Model (GPGM)



# The Generalized Proportional Growth Model (GPGM)

Statistical tests of goodness of fitting (Kolmogorov-Smirnov and Anderson-Darling) confirm that the GPGM model outperforms other models.

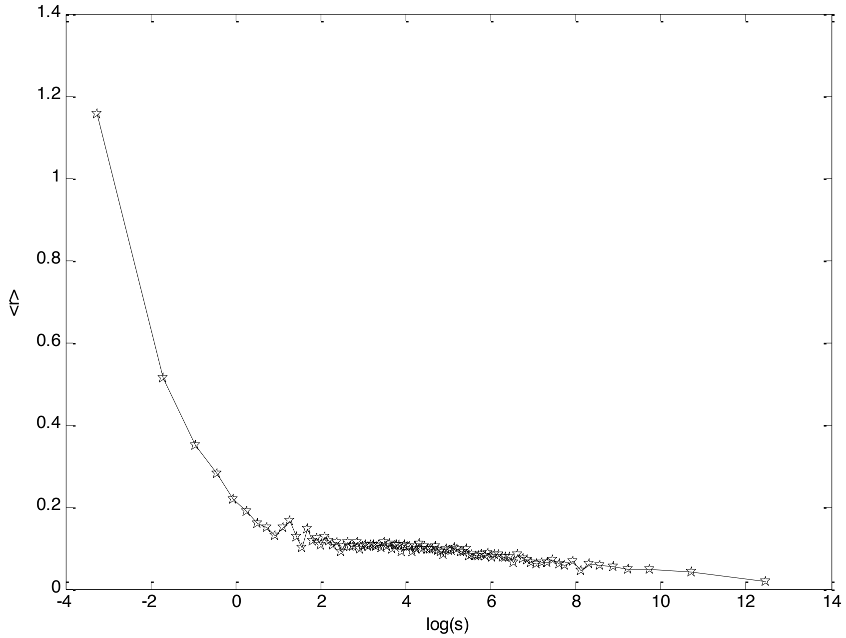
Maximun Likelihood Estimates (MLE) of the growth rate distribution.

| Distribution | $\mu$ | $\sigma$ | $\frac{k}{2V_r}$ | KS | AD |
|---|---|---|---|---|---|
| Gaussian | 0.0844 | 0.3702 | – | 17.1173 | 2.29E+79 |
| Laplace | 0.0844 | 0.1854 | – | 5.0453 | 1.10E+07 |
| Bose-Einstein | – | – | 24.5 | 3.7989 | 0.0837 |
| GPGM | – | – | 12.25 | 1.6734 | 0.0343 |

# The Generalized Proportional Growth Model (GPGM)

As for the relation between size and average growth rate the GPGM model:

$$\langle r \rangle = \langle \ln(S_i(t+1)/S_i(t)) \rangle.$$

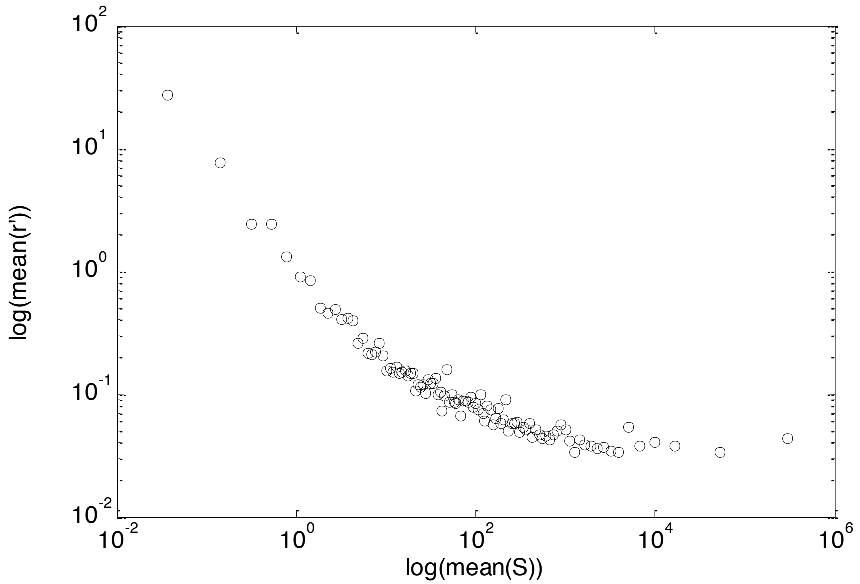



This definition excludes firms that lost all their products and hence Accordingly, it creates a positive bias for small firms consisting of one unit. For firms consisting of few units it creates a negative bias due to the asymmetry of the *ln(x)* function (which diverge to →∞ for x→ 0, but slowly increase for x→1. Only for very large firms, consisting of many independent units, and the asymmetry of *ln(x)* becomes negligible.

# The Generalized Proportional Growth Model (GPGM)

As regards to the relation between size and average growth rate the GPGM model we must stressed that:

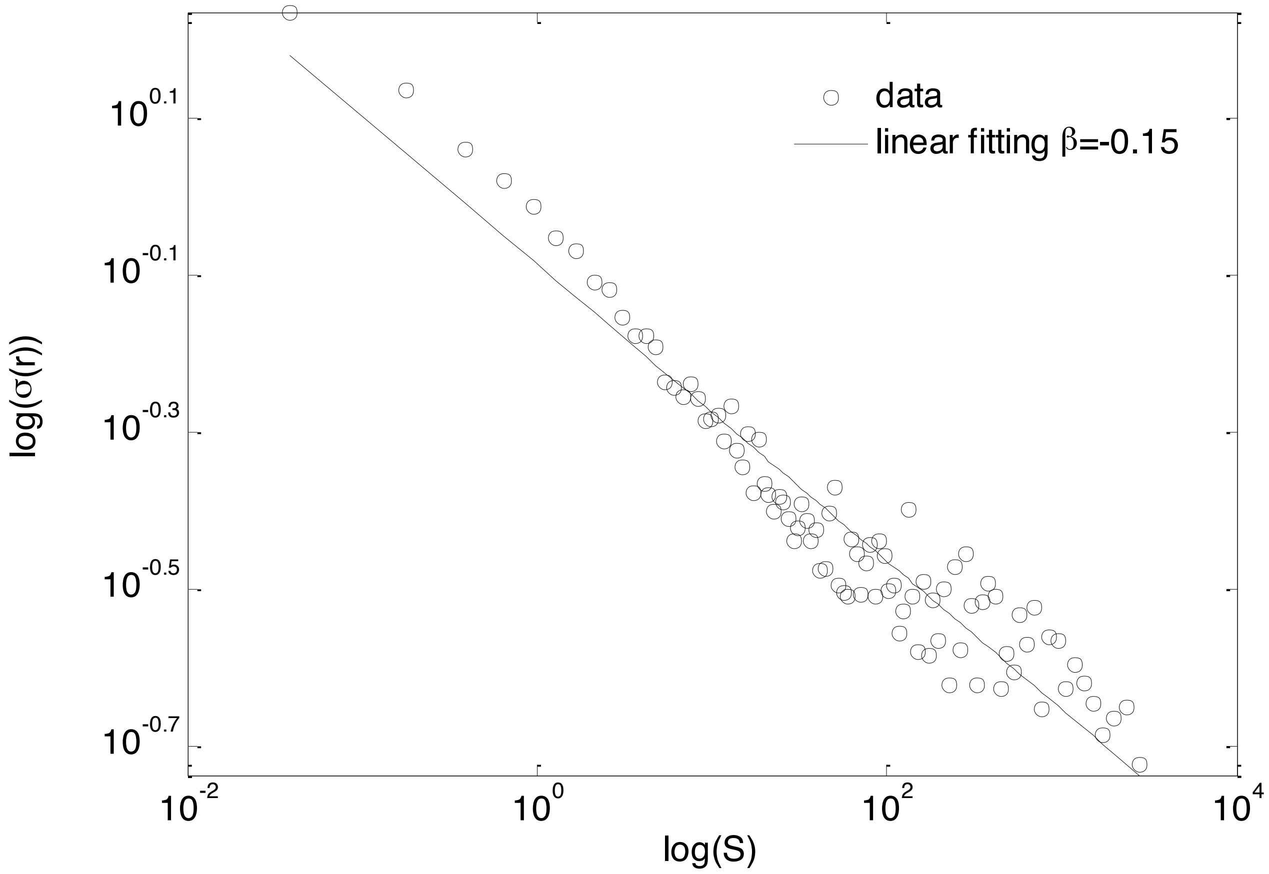
$$\langle r' \rangle = \langle S_i(t+1)/S_i(t) - 1 \rangle.$$




In the light of our assumptions the average growth rate is:

$$< r' >= \lambda[exp(m_\xi + V_\xi/2]/S - \mu$$

# The Generalized Proportional Growth Model (GPGM)



# Innovation and Growth at Different Levels of Aggregation

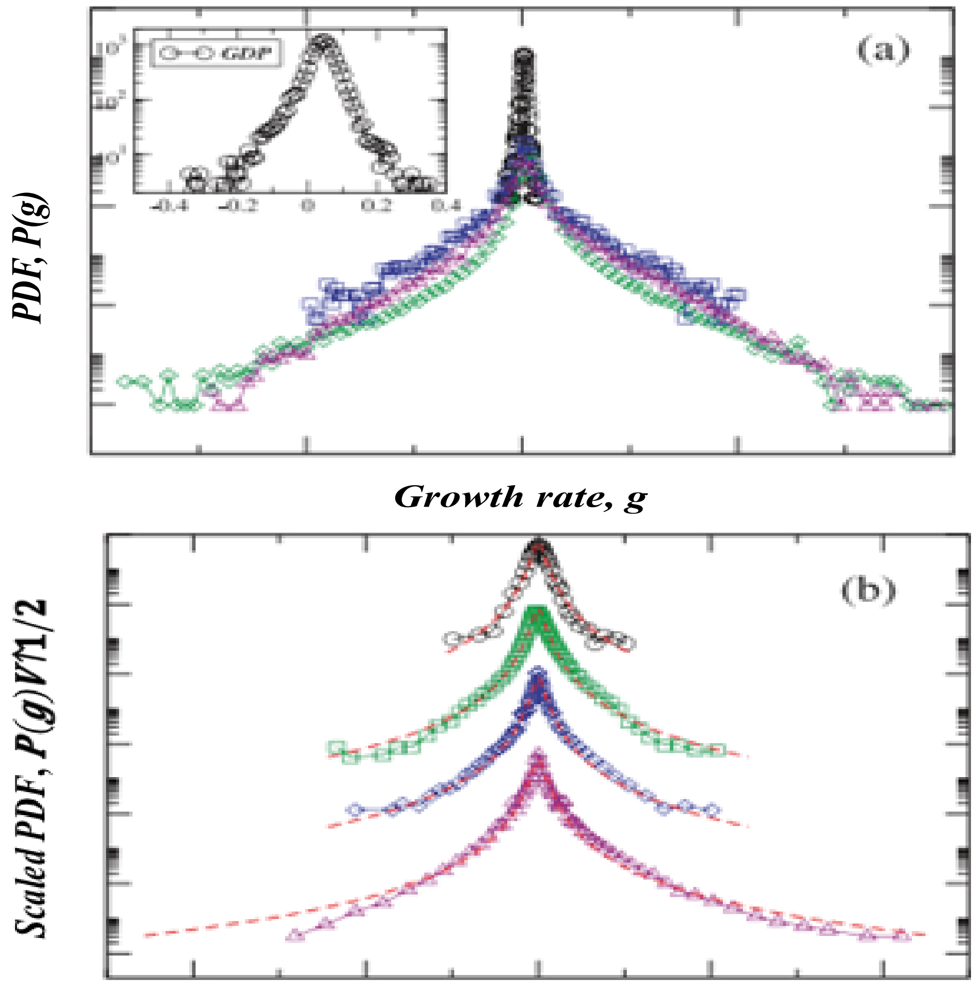



Fig. 2 (a) Empirical results of the probability density function (PDF) p(g) of growth rates: Country GDP, all firms within one industrial sector, firms monitored in Compustat and all the products sold by firms in sector mentioned above. (b) Empirical test of equation (6) for the probability density function P(g) of the growth rates rescaled by the square root of V. Dashed lines are obtained based on equation (6) with $V=4x10^{-4}$ for GDP, V=0.014 for firms, V=0.019 for firms in Compustat, and V=0.01 for products. After rescaling, the four PDFs can be fit by the same function. Firms are offset by a factor of $10^2$, Compustat firms by a factor of $10^4$ and the products by a factor of $10^6$.

GDP
Compustat
Firms
Products

# A Pervasive Feature. From Scientific Output, Trade Flows…

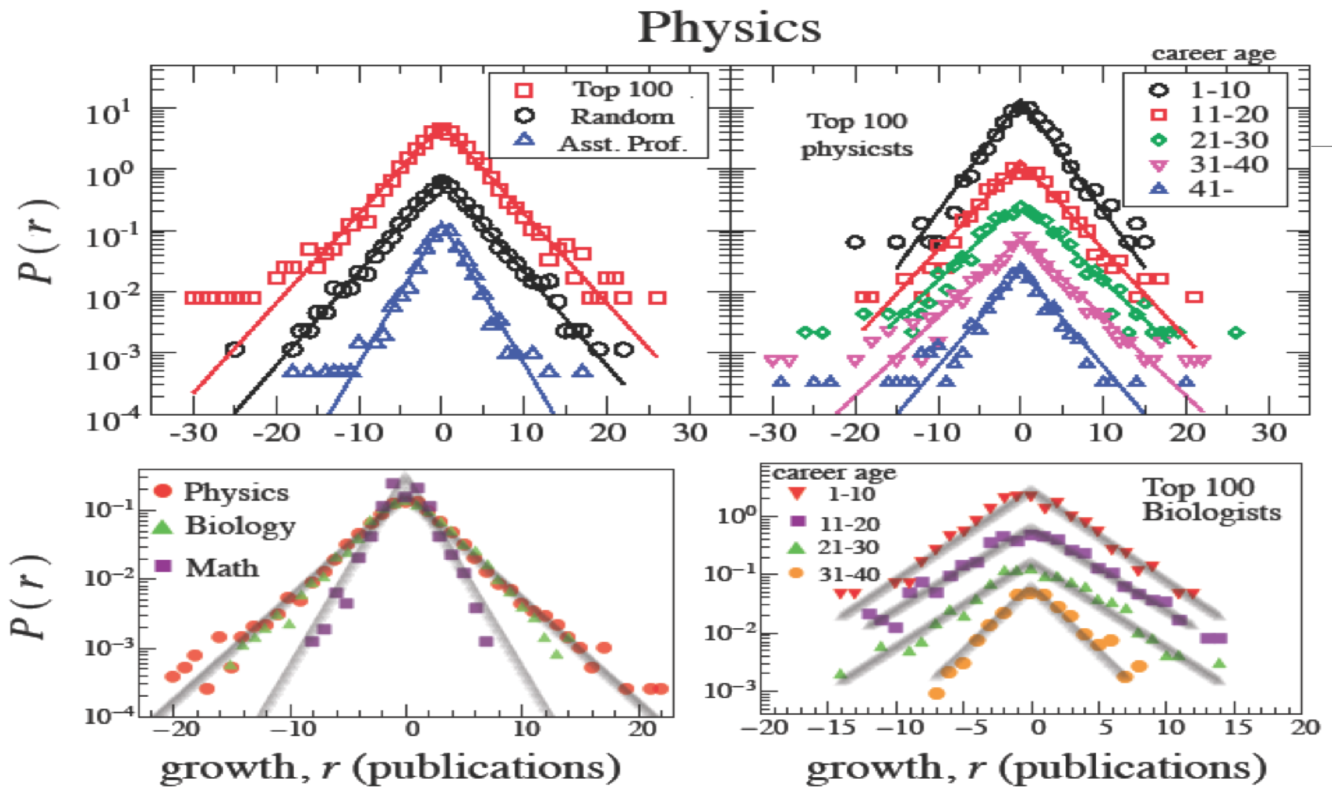



Fig 6. Distributions of the publication growth, defined as the change in the number of publications per year for 3 cohorts of top scientists, also disaggregated at career age.

A. M. Petersen, M. Riccaboni, H. E. Stanley, F. Pammolli (2012) “Persistence and Uncertainty in the Academic Career”. *Proceedings of the National Academy of Sciences USA,* 109: 5213 - 5218

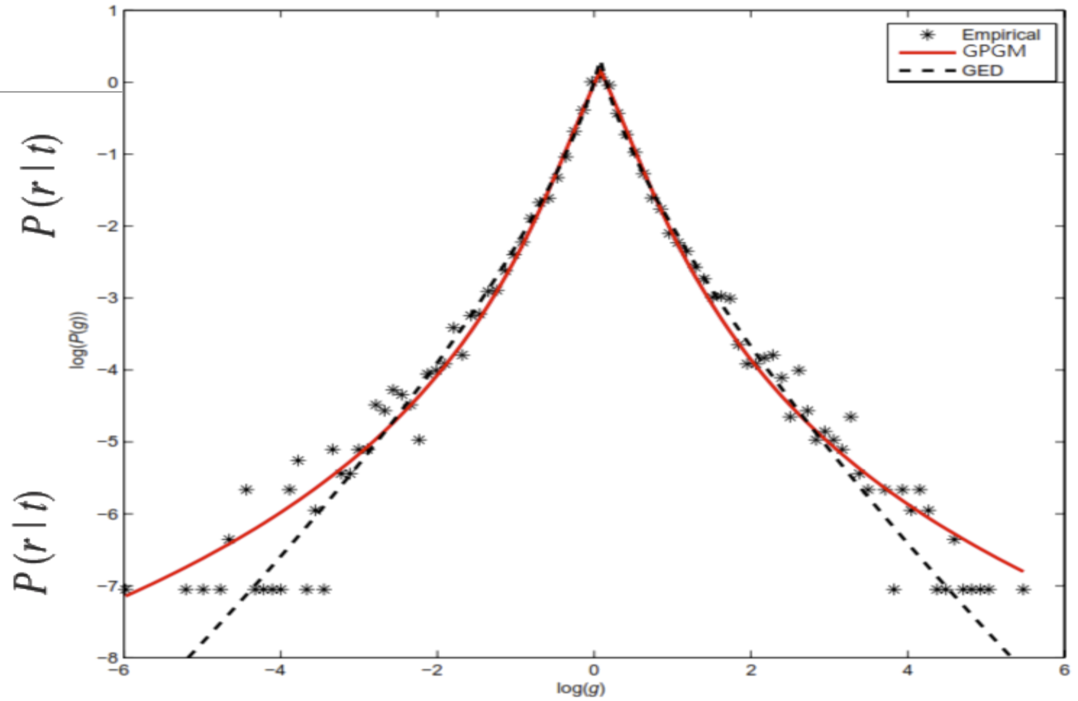
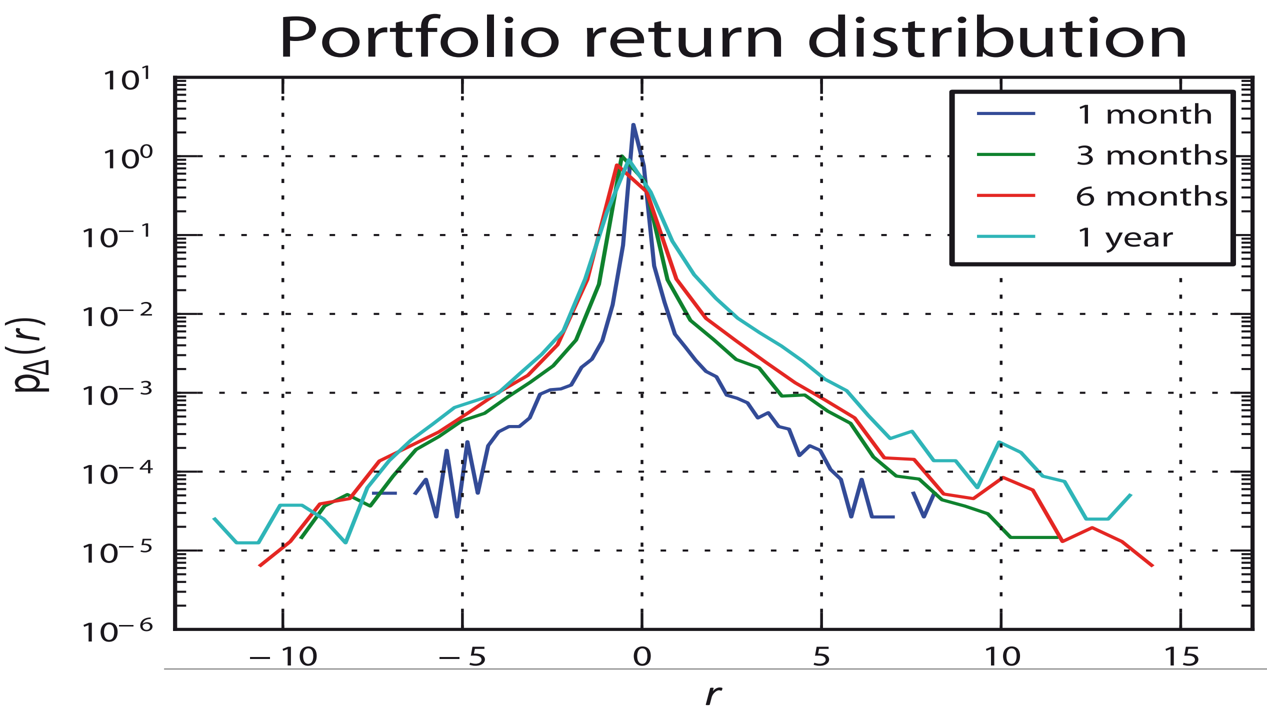



Fig 7. Distribution P(g) of the growth rate g of aggregate trade flows

Riccaboni M, Schiavo S. (2010), ‘Structure and growth of weighted networks’, *New Journal of Physics. 12-023003*.

# …. To Financial Diversification




**Probability density function p(*r*) of the logarithmic growth of funds portfolios: variations computed for time horizons ranging from 30 days to one year.**
Data: CRSP on US mutual funds' portfolio holdings, November 2011 - September 2012, for a total of 19493 portfolios investing in a total of 576598 different assets and total investments value ranging from 371 billions to 81 trillions.

Source: Pammolli, Delpini, et al., forthcoming

# Size, Diversification, and Instability: From Products to GDP

*In case of skewed size distributions, the size-variance relationship scales with the share of the largest unit*

The size-variance relationship crucially depends on the partition of firm (country) size across constituent components. If firms have P(K) units and $V_\eta$=0, for the Law of Large Numbers, $\sigma(K) \approx K^\beta$, where β =1/2. On the contrary, if each firms consists of a single unit only and $V_\eta$=>0, =0. When both mechanisms are at work, the speed of the crossover depends on the skewness of P(K). At one extreme, if all entities have the same number of units, β=0 and there is no crossover. On the contrary, if P(K) is power-law distributed, for a wide range of empirically plausible $V_\eta$, β is far from 1/2 and statistically different from zero. The size-variance relationship is not a true power law with a single well-defined exponent β, but undergoes a slow cross over from β=0 for $S \rightarrow 0$, to β=1/2 for $S \rightarrow \infty$.

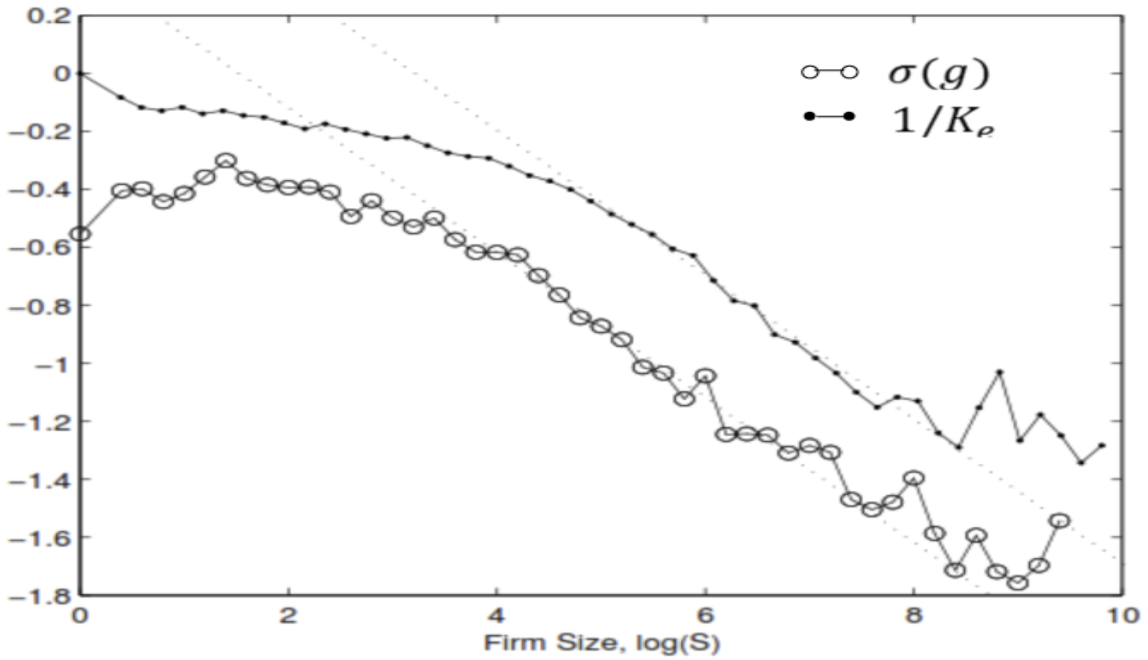



*Fig. 4* Size-Variance relationship. The standard deviation of firm growth rates (σ) (circles), and the share of the largest products ($1/K_e$) (squares) versus the size of the firms (*S*). For $S < S1 = \mu_\xi \approx 3.44$, $\beta \approx 0$. For $S > S1$ $\beta$ increases but never reaches 1/2 because of the slow growth of the number of products ($K_e$). The flattening of the upper tail is due to some large companies with unusually large products.