Options Pricing using Monte Carlo Simulations

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Contents

Introduction

Motivation

Why MC?

- MC has proved to be a robust way to price options
- The advantage of MC over other techniques increases as the sources of uncertainty of the problem increase
- Essential for exotic options pricing where there are no analytical solutions (e.g. Asian options)
- Compare the results of the simulation with the Black-Scholes theory

What is an Option?

An option is a type of security which gives the holder the right (NOT the obligation) to buy or sell the underlying asset at a predefined price.

- A **call** option gives the holder the right to buy
- A **put** option gives the holder the right to sell

Types of Options

The two most popular types of options are

- European Options
- American Options

These are often referred to as vanilla options because of their simplicity. More non-standard options are called exotic options.

European Option Payoff

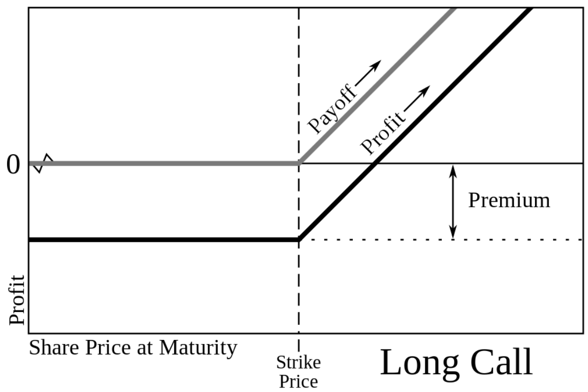

Figure: Payoff from buying a call option

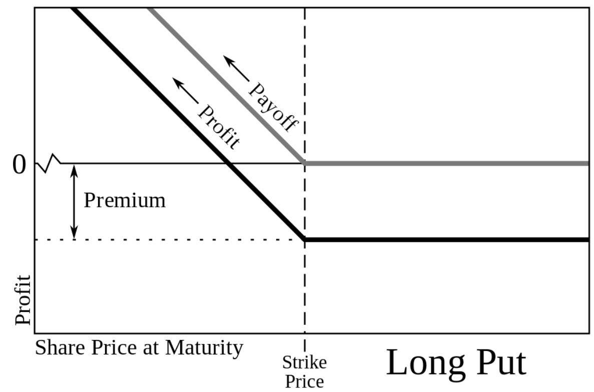

Figure: Payoff from buying a put option

Why Use Options?

Options are really risky assets to have in your portfolio!
Then why bother???

- Speculation
  Big money if you can predict the magnitude and the timing of the movement of the underlying security.
- Hedging
  Insurance against a risky investment

Brownian Motion

Wiener process

- $B(0) = x$
- $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_1) - B(0)$ are independent random variables
- For all $t \geq 0$ and $\Delta t > 0$, $B(t + \Delta t) - B(t)$ are normally distributed with expectation 0 and standard deviation $\sqrt{\Delta t}$

Geometric Brownian Motion
A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{1}$$

where W_t is a Wiener process, μ is the drift used to model deterministic trends and σ is the volatility used to model unpredictable events.
For an arbitrary initial value S_0, the analytical solution of equation (1) is given by

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{2}$$

Ito's Lemma

If the variable x follows the Ito process

$$dx = a(x,t)\,dt + b(x,t)\,dz \tag{3}$$

then a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right)dt + \frac{\partial G}{\partial x}b\,dz \tag{4}$$

Hence, for $\quad dS = \mu S\,dt + \sigma S\,dz \quad$ we get

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz \tag{5}$$

Lognormal Property of Stock Prices

Using Ito's Lemma with $G = \ln S$ produces an interesting result!
Since

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

it follows from equation (5) that

$$\mathrm{d}G = \left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}t + \sigma \, \mathrm{d}z$$

The change of $\ln S$ between time 0 and a future time T is therefore normally distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$.
Hence,

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right] \tag{6}$$

The Expected Return

If Y is $LN[m, s^2]$, then $E(Y) = e^{m+\frac{s^2}{2}}$

In our case $m = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T$ and $s = \sigma\sqrt{T}$

Hence,

$$E(S_T) = S_0 e^{\mu T} \tag{7}$$

The expected return μ is driven by

- The riskiness of the stock
- Interest rates in the economy

Black-Scholes Assumptions

Assumptions

- The stock price follows geometric Brownian motion
- The short selling of securities with full use of proceeds is permitted
- No transaction fees or taxes. All securities are perfectly divisible
- No dividends
- No arbitrage
- Continuous trading
- The risk-free rate r is constant and the same for all maturities

Black-Scholes-Merton derivation

The stock price process is assumed to follow

$$dS = \mu S\,dt + \sigma S\,dz \tag{8}$$

Suppose that f is the price of a derivative of S, the variable f must be some function of S and t. Hence from Ito's Lemma we get,

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S\,dz \tag{9}$$

Black-Scholes-Merton derivation

Consider the portfolio

$$\Pi = -f + \frac{\partial f}{\partial S} S \tag{10}$$

The change $\Delta \Pi$ in the value of the portfolio in the time interval Δt is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \tag{11}$$

Using equations (8) and (9) into equation (11) yields

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{12}$$

Notice that this is a riskless portfolio!
For a risk-free portfolio we have $\Delta \Pi = r \Pi \Delta t$. Hence, substituting in equation (12) we get the Black-Scholes-Merton differential equation

$$\boxed{\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf} \tag{13}$$

Black-Scholes pricing of European options

For European options we have

$$f_{call} = \max(S - K, 0) \quad \text{and} \quad f_{put} = \max(K - S, 0)$$

For a non-dividend-paying stock, the prices at time 0 are

$$\boxed{c = S_0 N(d_1) - K e^{-rT} N(d_2)} \tag{14}$$

and

$$\boxed{p = -S_0 N(-d_1) + K e^{-rT} N(-d_2)} \tag{15}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

$N(x)$ is cumulative probability distribution function for a standardized normal distribution $\phi(0, 1)$.

Risk-Neutral Valuation

Notice that the variables that appear in the equation (13) are all independent of the risk preferences of the investors.

No $\mu \longrightarrow$ No risk dependence $\longrightarrow$ Any rate can be used when evaluating f

We assume that all investors are risk neutral!

Risk-neutral valuation of a derivative

- Assume $\mu = r$
- Calculate the expected payoff from the derivative
- Discount the expected payoff using r

Calculations

The Monte Carlo method

The Monte Carlo simulation can be divided in three main steps

- Calculation of the potential future price using GBM
- Calculation of the pay-off for this price
- Discount the pay-off back to today's price

Repeating the above procedure for a reasonable number of times, gives a good estimate of the average pay-off and the price of the option.

Input parameters of the Monte-Carlo simulation

- The initial price of the underlying stock 100
- The strike price at maturity 102
- The expected annual return 1%
- The risk-free annual rate 1%
- The expected annual volatility 20%
- Number of steps 252
- Years to maturity 1
- Number of trials 2500

Typical output

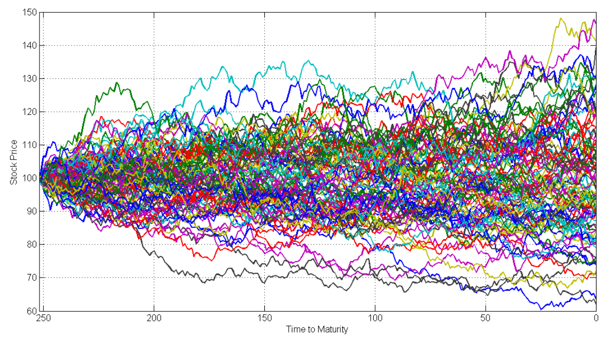

Figure: Typical output of the simulation for 1% drift

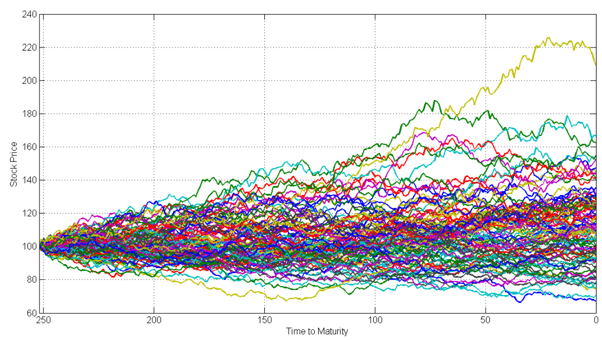

Figure: Typical output of the simulation for 10% drift

Stock Price at Maturity

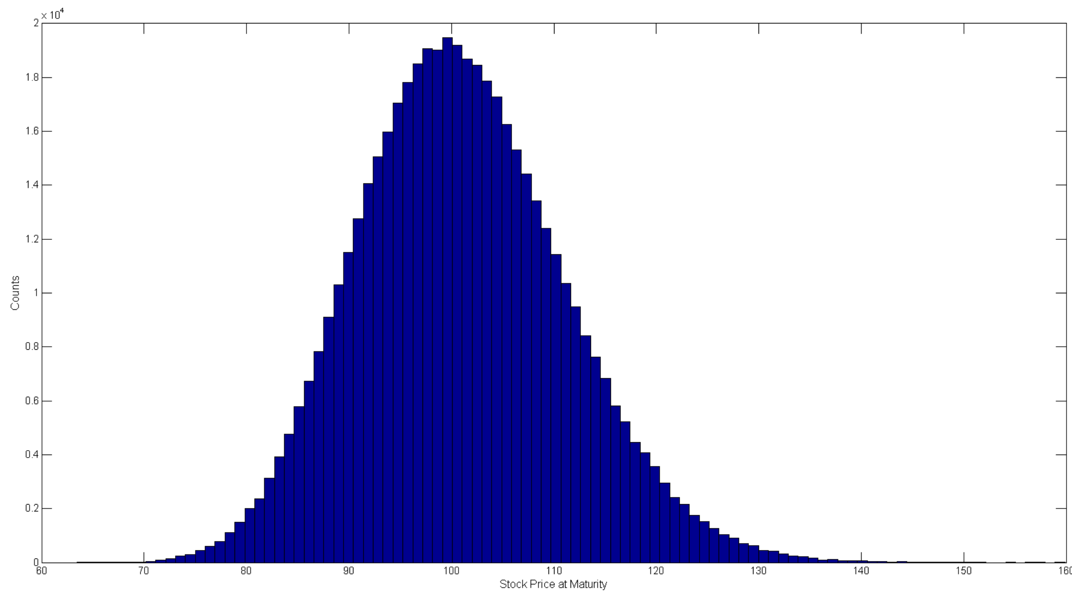

Figure: Lognormal distribution of the stock price at maturity

Option Price versus Initial Price

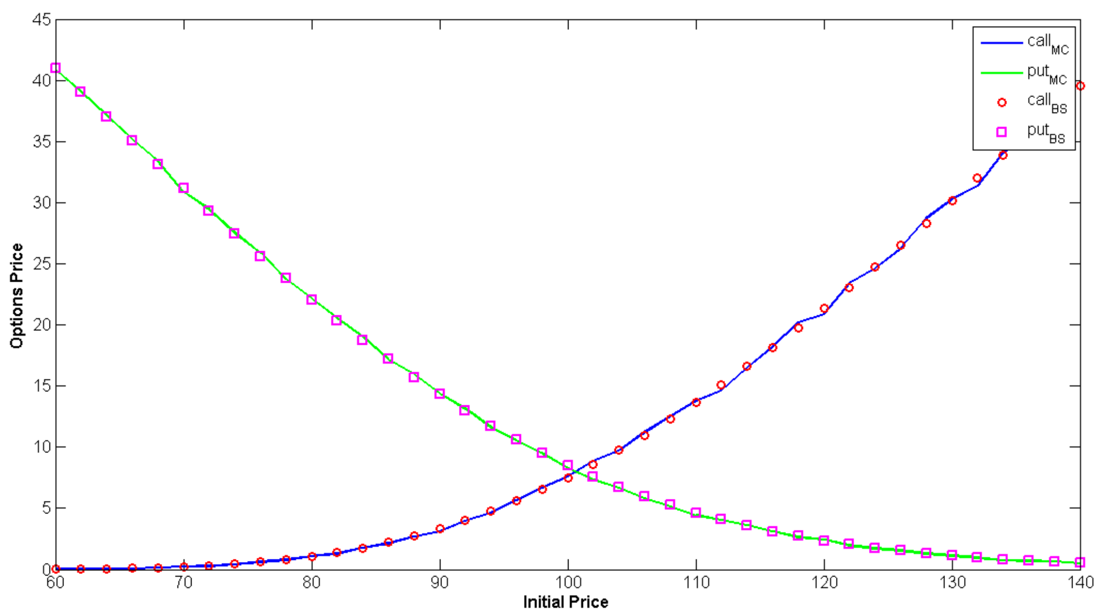

Figure: Options Price versus Initial Price

Option Price versus Strike Price

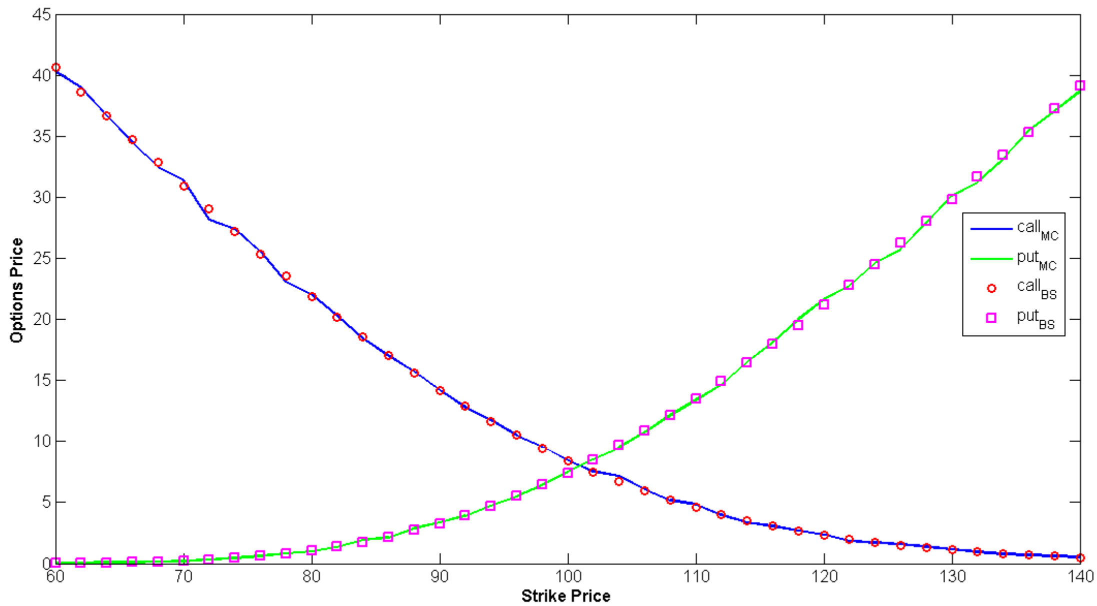

Figure: Options Price versus Strike Price

Option Price versus Return Rate

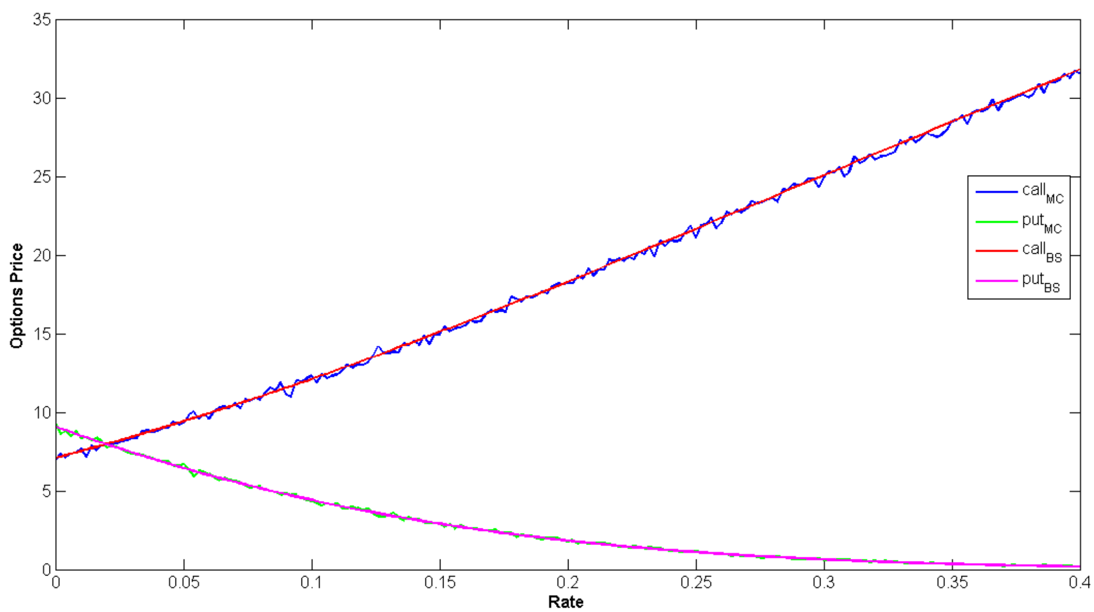

Figure: Options Price versus Return Rate

Option Price versus Volatility

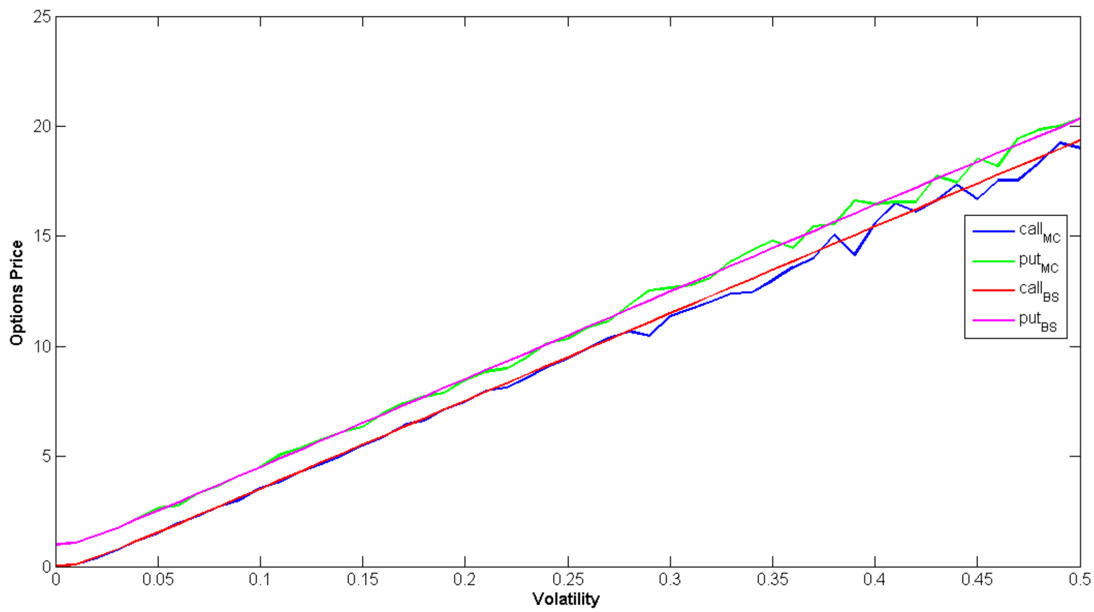

Figure: Options Price versus Volatility

Convergence Tests versus the Expectation Value $\bar{S}_T/E(S_T)$

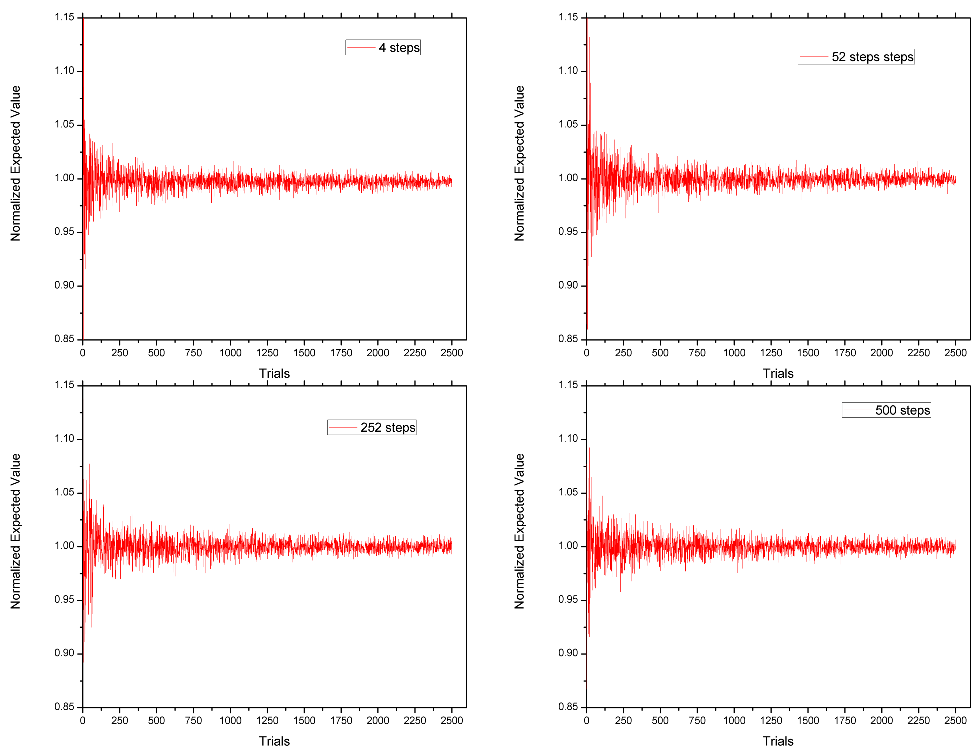

Figure: Time steps versus number of trials compared with $E(S_T) = S_0 e^{rT}$

Convergence Tests

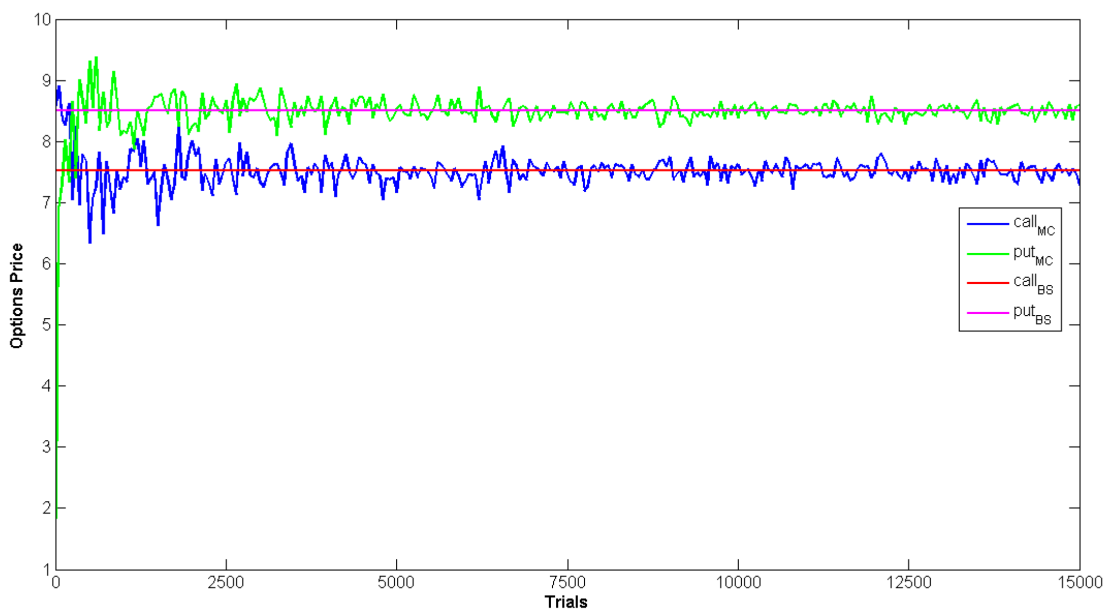

Figure: Price versus number of trials

Convergence Tests versus the Call Price c_{MC}/c_{BS}

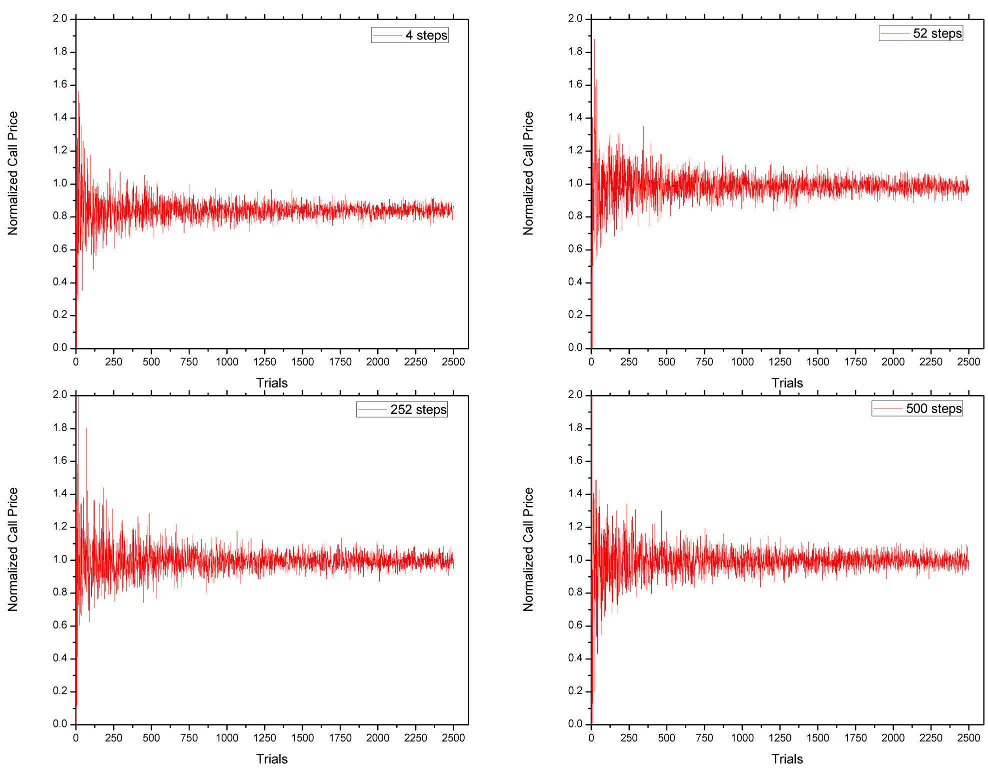

Figure: Time steps versus number of trials compared with the normalized call price

Convergence Tests

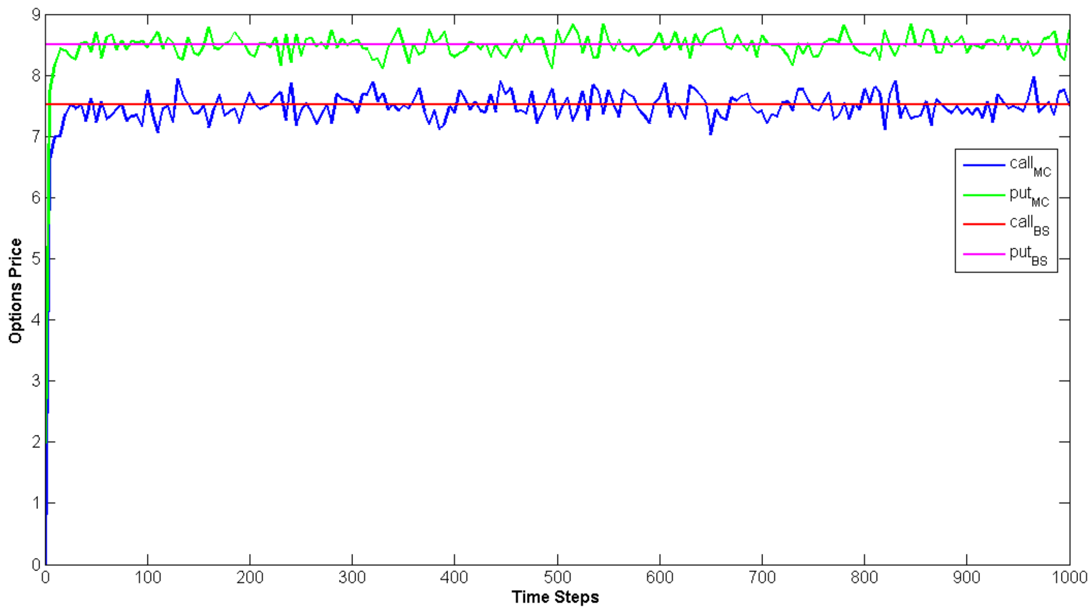

Figure: Options price versus time steps

Effects of the Strike Price on Convergence

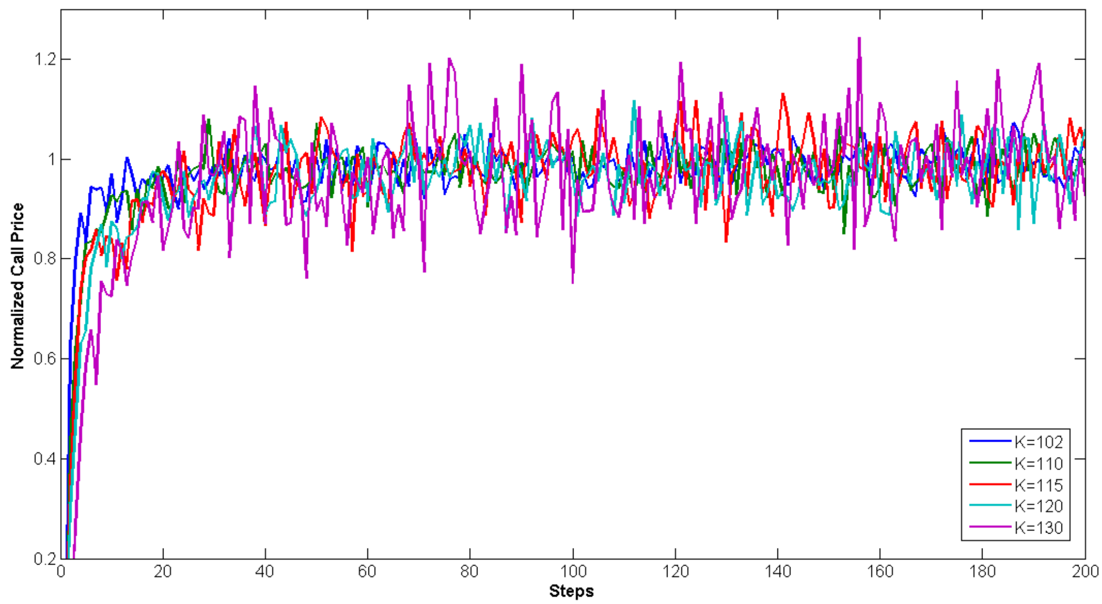

Figure: Normalized call price versus time steps for different K-values

Effects of the Strike Price for different Volatilities

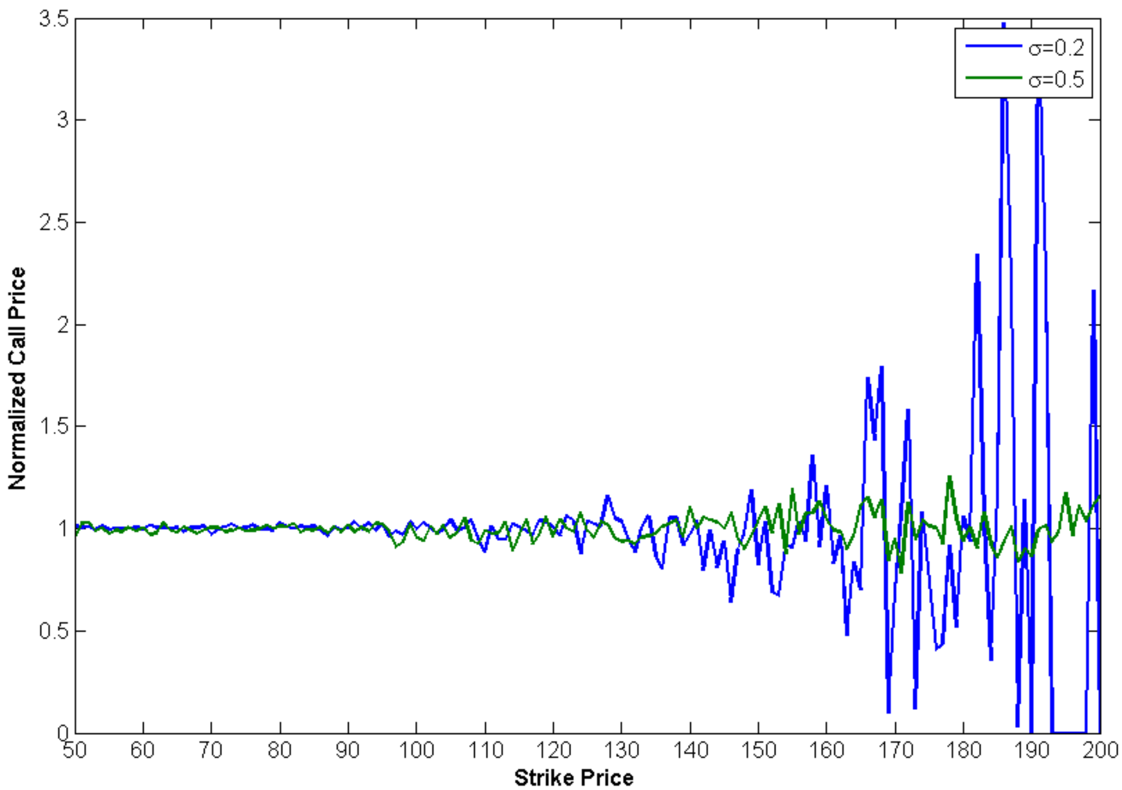

Figure: Normalized call price versus strike price

Effects of the Strike Price

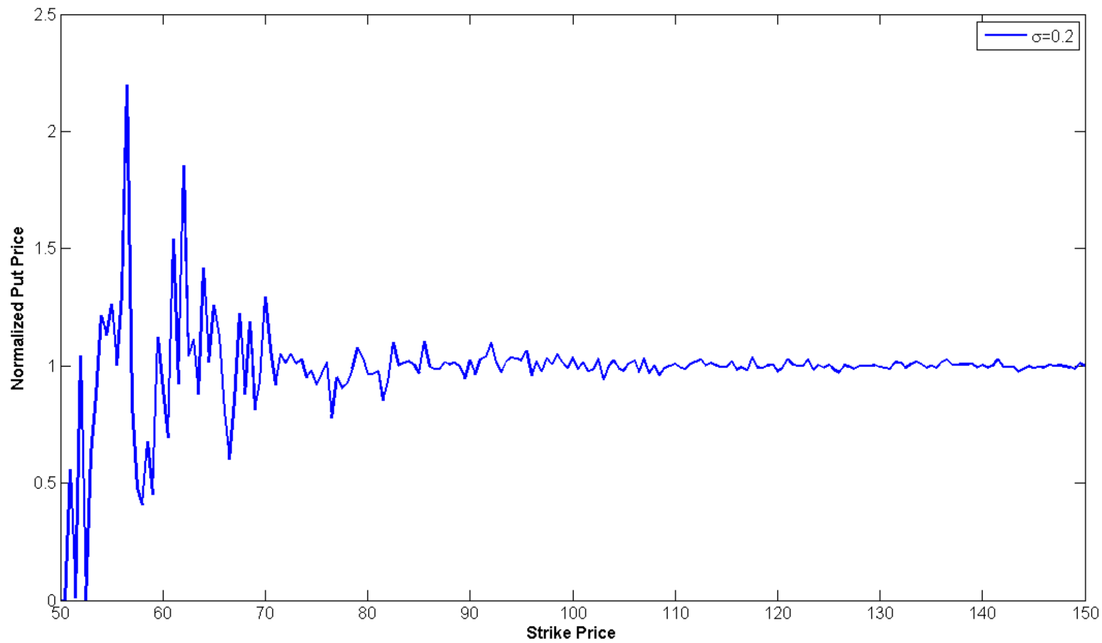

Figure: Normalized put price versus strike price

Trials and Effects of the Strike Price

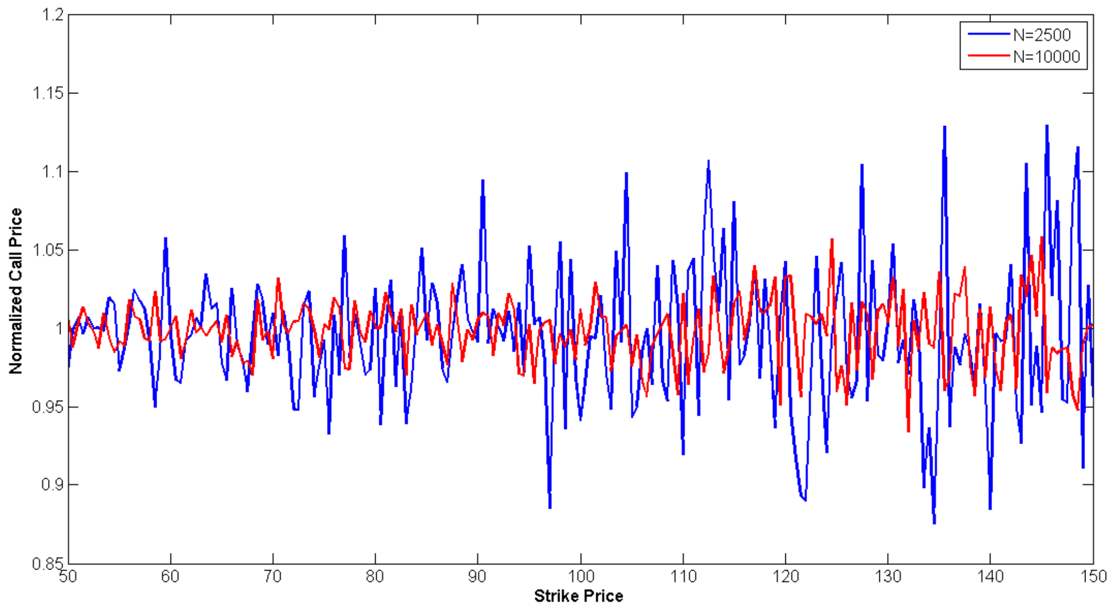

Figure: Normalized call price versus number of trials for $K = 150$ for different number of trials

Steps and Number of Trials

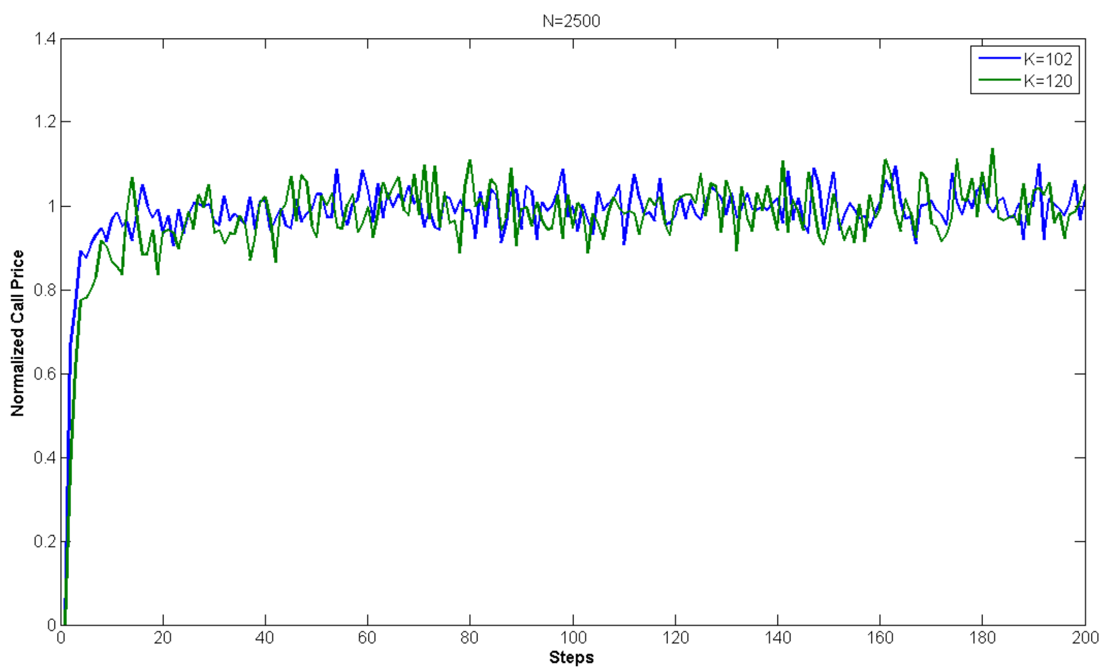

Figure: Normalized call price versus steps for different K values

Steps and Number of Trials

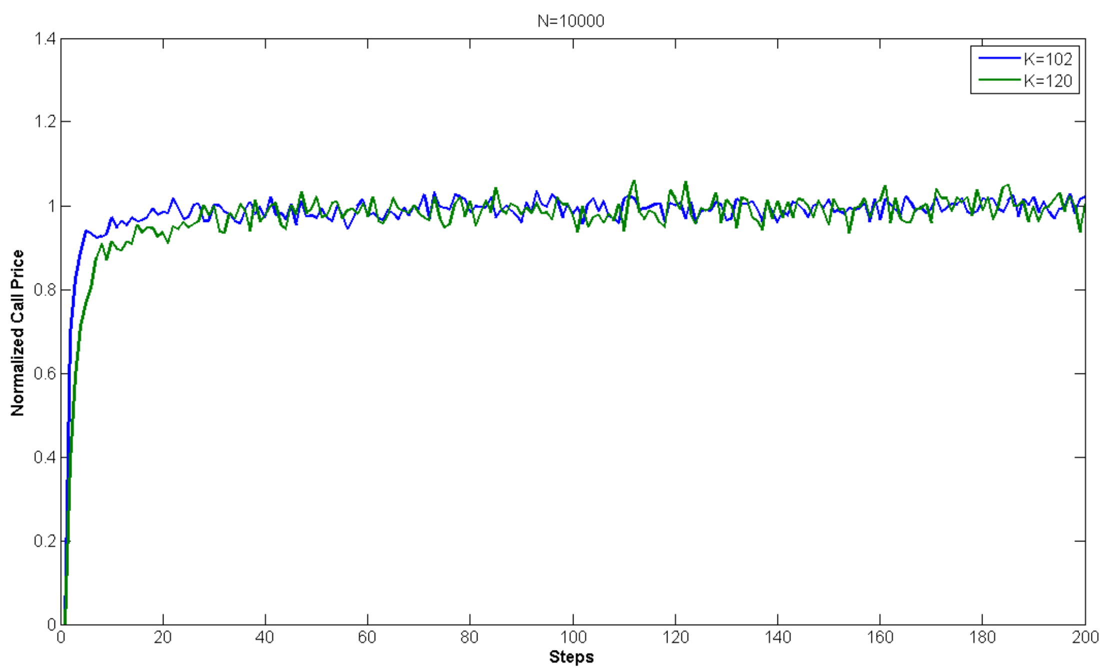

Figure: Normalized call price versus steps for different K values

Expected Return versus Risk-Free Rate

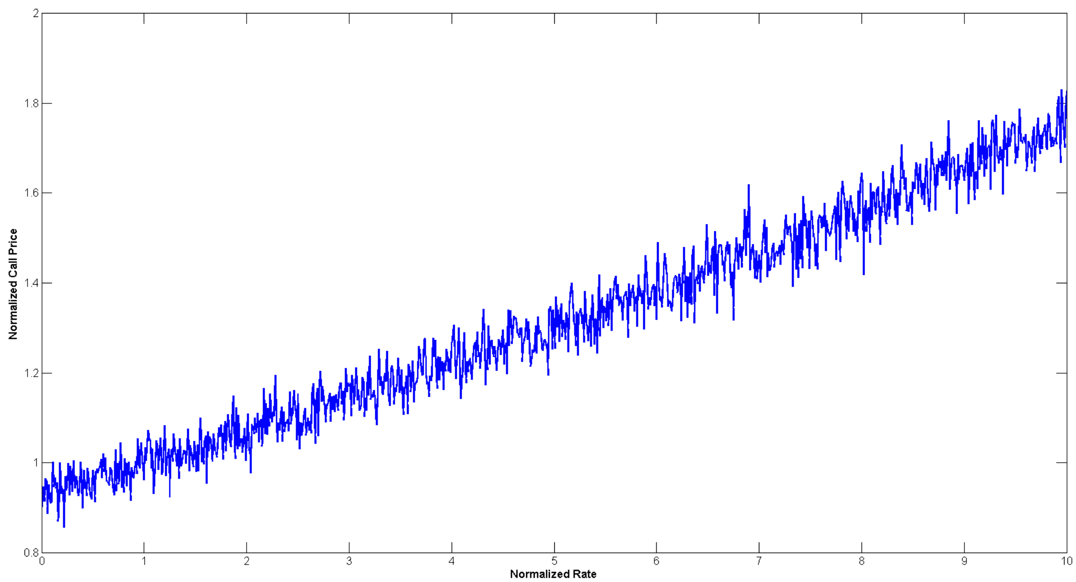

Figure: Normalized expected return rate versus risk-free rate (μ/r)

Girsanov's Theorem

A powerful theorem that allows us to translate any result at any expected rate to the risk-free world.
In other words, the rate we use to price the derivative is irrelevant.

Conclusions

Summary

- Very few time steps may not give converged results. They become irrelevant after some point.
- More trials produce results with less uncertainty but the computational cost increases.
- Important to use the same rate both for Monte Carlo and Black-Scholes
- The results of the Monte Carlo approach are very accurate compared to the Black-Scholes for reasonable parameters

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Thank you for your attention!