Application of Statistical Physics in Time Series Analysis

Duan Wang

Center for Polymer Studies, Boston University
wangduan@bu.edu

Advisor: H. Eugene Stanley

Outline

Basic Concepts in Financial Time Series

Random Matrix Theory

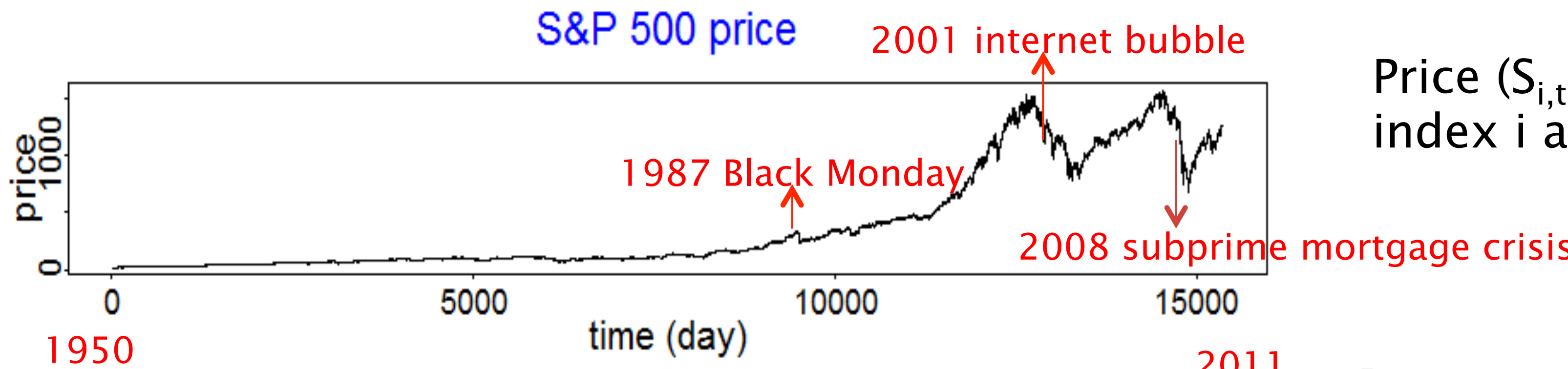
- Meaning of eigenvalues and eigenvectors
- Eigenvalue distribution for random correlation matrix
- Interpreting empirical eigenvalue distribution

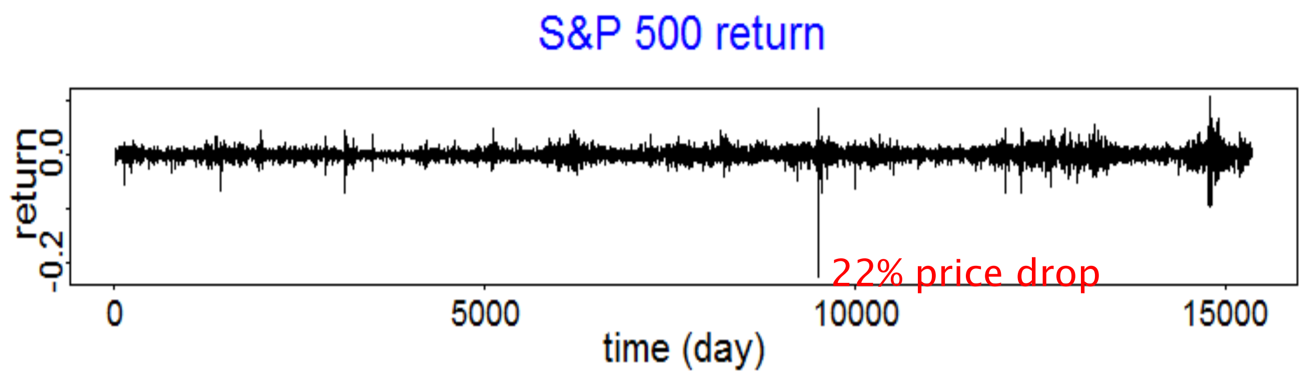
Extensions of Random Matrix Theory

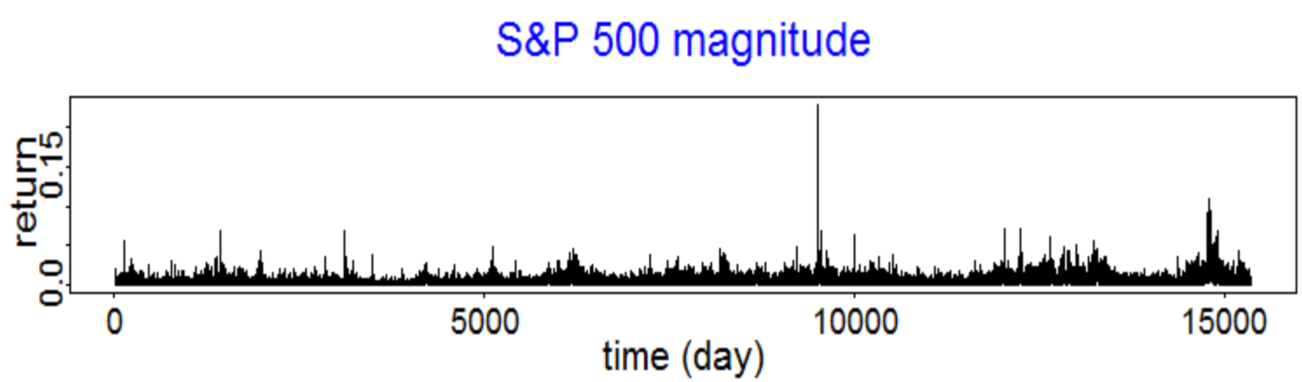
- Autoregressive random matrix theory (ARRMT)
- Time-lag random matrix theory (TLRMT)
- Global factor model (GFM)

(Application of RMT in portfolio optimization)

Price, Return and Return Magnitude







Price ($S_{i,t}$):
index i at time t

Return:

$$
\begin{aligned}
R_{i,t} &\equiv \log S_{i,t} - \log S_{i,t-1} \\
&= \log(S_t / S_{t-1}) \\
&= \log(\frac{S_{t-1} - S_t}{S_{t-1}}) \\
&\approx \frac{\Delta S_{t-1}}{S_{t-1}}
\end{aligned}
$$

Magnitude of return:

$$|r_{i,t}| \equiv |R_{i,t} - \langle R_{i,t} \rangle|$$

Measure of Dependence: Cross-Correlation, Autocorrelation, and Time-Lag Cross-Correlation

For time series $\{X_t\}$ and $\{Y_t\}$,

- Cross-correlation

$$C_{XY} \equiv \frac{\langle X_t Y_t \rangle - \langle X_t \rangle \langle Y_t \rangle}{\sigma_X \sigma_Y}$$

<> Denotes average

Standard deviations

- Autocorrelation

$$A_X(\Delta t) \equiv \frac{\langle X_t X_{t+\Delta t} \rangle - \langle X_t \rangle \langle X_{t+\Delta t} \rangle}{\sigma_X \sigma_X}$$

time lag

- Time-lag Cross-correlation

$$C_{XY}(\Delta t) \equiv \frac{\langle X_t Y_{t+\Delta t} \rangle - \langle X_t \rangle \langle Y_{t+\Delta t} \rangle}{\sigma_X \sigma_Y}$$

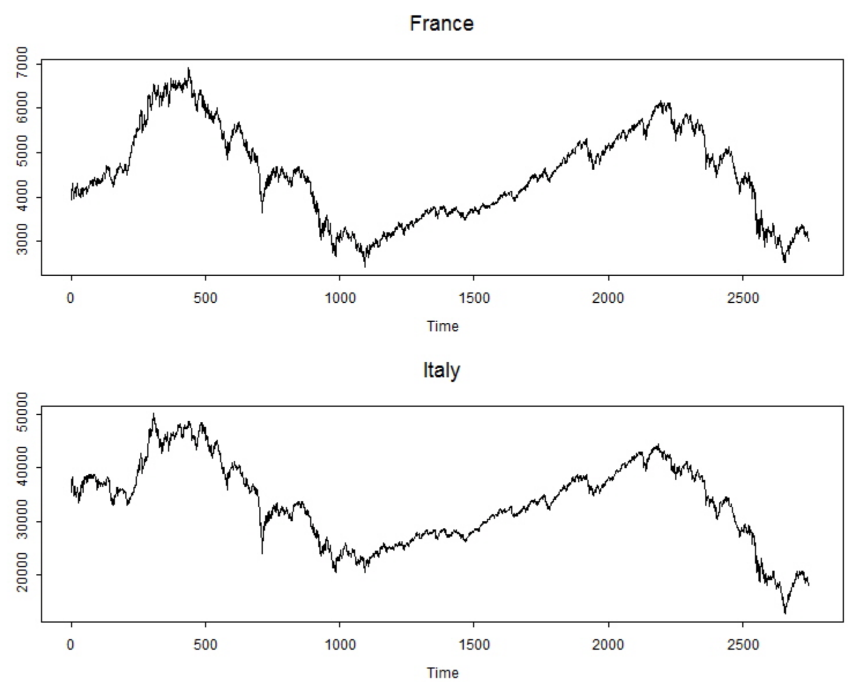
Properties:

$0<C<1$ correlated

$C=0$ uncorrelated

$-1<C<0$ anti-correlated

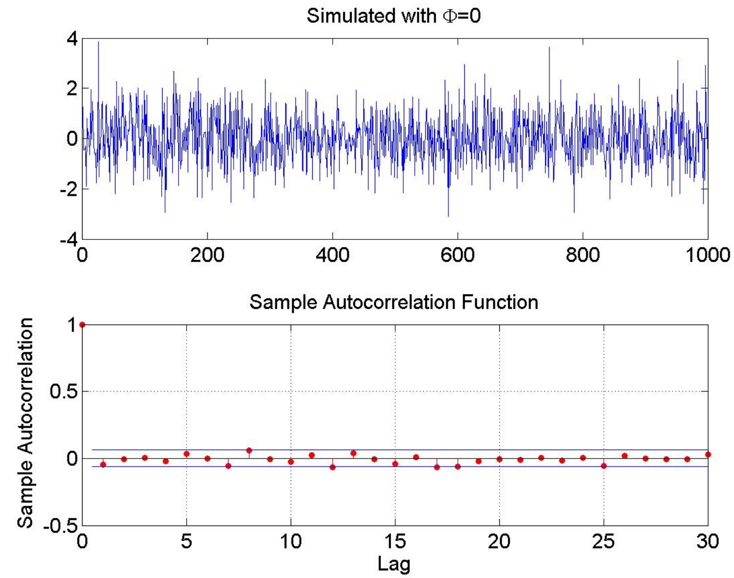
Example: highly correlated France and Italy indices



Examples of Return Autocorrelations and Magnitude Autocorrelations

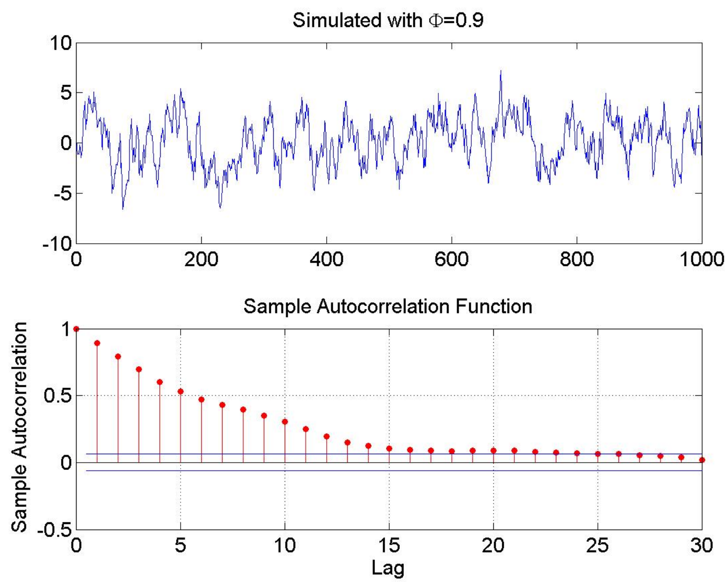
Simulate return autocorrelations: AR(1)

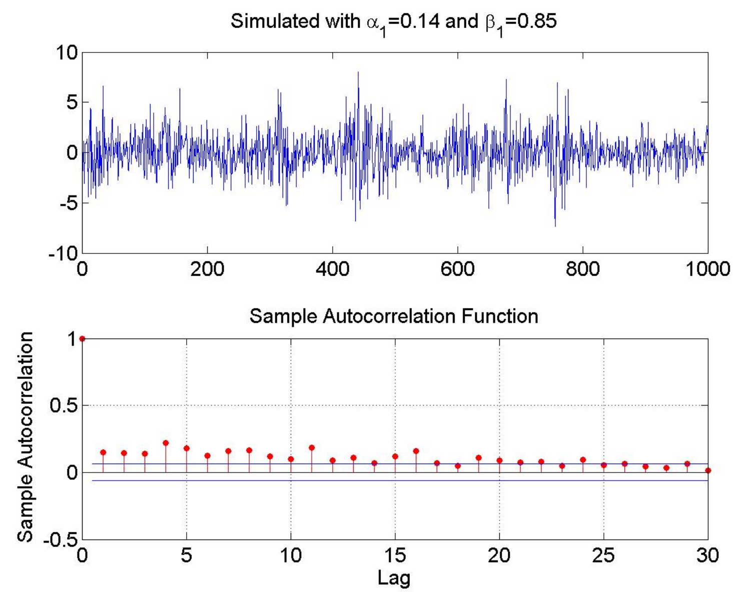
$$X_t = \phi X_{t-1} + \epsilon_t$$



Simulate magnitude autocorrelations: GARCH(1,1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$





Multiple Time Series

Annual Return of 6 US stocks

Date	GE	MSFT	JNJ	K	BA	IBM
3-Jan-94	56.44%	-1.50%	6.01%	-9.79%	58.73%	21.51%
3-Jan-95	18.23%	33.21%	41.56%	7.46%	-0.24%	6.04%
2-Jan-96	56.93%	44.28%	57.71%	37.76%	65.55%	27.33%
2-Jan-97	42.87%	79.12%	22.94%	-5.09%	54.34%	41.08%
2-Jan-98	47.11%	38.04%	17.62%	32.04%	37.11%	2.63%
4-Jan-99	34.55%	85.25%	26.62%	-10.74%	22.00%	-2.11%
3-Jan-00	28.15%	11.20%	3.41%	-48.93%	43.53%	23.76%
2-Jan-01	4.61%	-47.19%	10.69%	11.67%	28.29%	21.76%
2-Jan-02	-19.74%	4.27%	-7.00%	19.90%	-15.09%	4.55%
2-Jan-03	-44.78%	-29.47%	-5.67%	10.88%	-23.23%	15.54%
2-Jan-04	35.90%	18.01%	-1.27%	15.49%	39.82%	31.80%

6 Stocks
6x6 Correlation matrix

Symmetric matrix
Main diagonal=1

Correlation Matrix

	GE	MSFT	JNJ	K	BA	IBM
GE	1	0.5791	0.5383	-0.056	0.905	0.2819
MSFT	0.5791	1	0.56	-0.0532	0.3502	-0.0406
JNJ	0.5383	0.56	1	0.2838	0.3925	0.0002
K	-0.056	-0.0532	0.2838	1	-0.1234	-0.1427
BA	0.905	0.3502	0.3925	-0.1234	1	0.5821
IBM	0.2819	-0.0406	0.0002	-0.1427	0.5821	1

Eigenvalue decomposition

$$\mathrm{C} = \mathrm{Q}\Lambda\mathrm{Q}^{-1}$$

$$\Lambda = diag\{2.80, 1.45, 0.96, 0.43, 0.34, 0.02\}$$

Meaning of eigenvalues?

Explained by random matrix theory (RMT)

Original RMT

- Aim: find existence of collective behavior
- method: compare the eigenvalue distribution between

  (1) a symmetric random matrix and

  (2) a Hamiltonian matrix

RMT in Econophysics

- Aim: find existence of cross-correlation in multiple time series
- Method: compare the eigenvalue distribution of

  (1) a Wishart matrix (random correlation matrix) and

  (2) empirical cross-correlation matrix

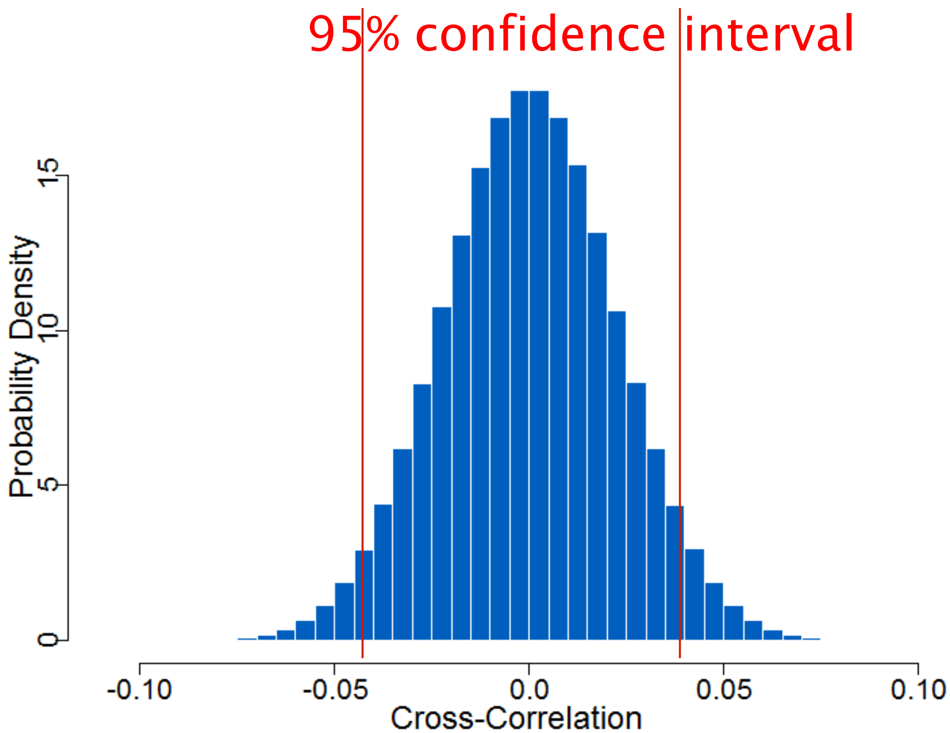
Wishart matrix: sample correlation matrix for uncorrelated time series

Cross-Correlation between Uncorrelated Finite Length Time Series

Sample cross-correlations between independent time series are not zero if time series are of finite length.

Mathematically: $C_{ij}=0$

Statistically: C_{ij} falls in $(-1.96\sqrt{\frac{1}{T}}, 1.96\sqrt{\frac{1}{T}})$, with 95% probability.



Cross-correlation distribution between 2 uncorrelated time series, each with length T=2000.

Sample correlation distribution:
$C_{ij} \sim N(0,1/T)$
Gaussian distribution with mean zero and variance 1/T

Meaning of Eigenvalues and Eigenvectors

Principal Components

- Linear combination of individual time series
- Orthogonal transformation of correlated time series

Eigenvectors (Factor Loadings)

- Weight of each individual time series in a principal component

Eigenvalues

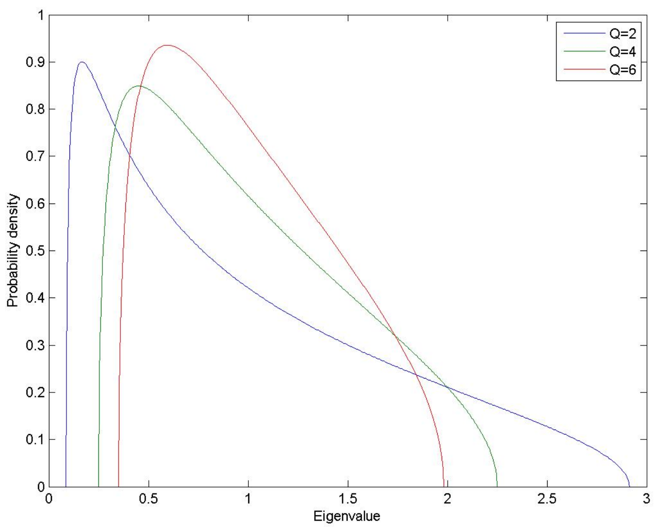
- Variance of each principal component
- Measures the significance of a factor

time series with length → principal components

Not all of them are significant

Compare them with the eigenvalues of Wishart matrix to find out the number of significant factors

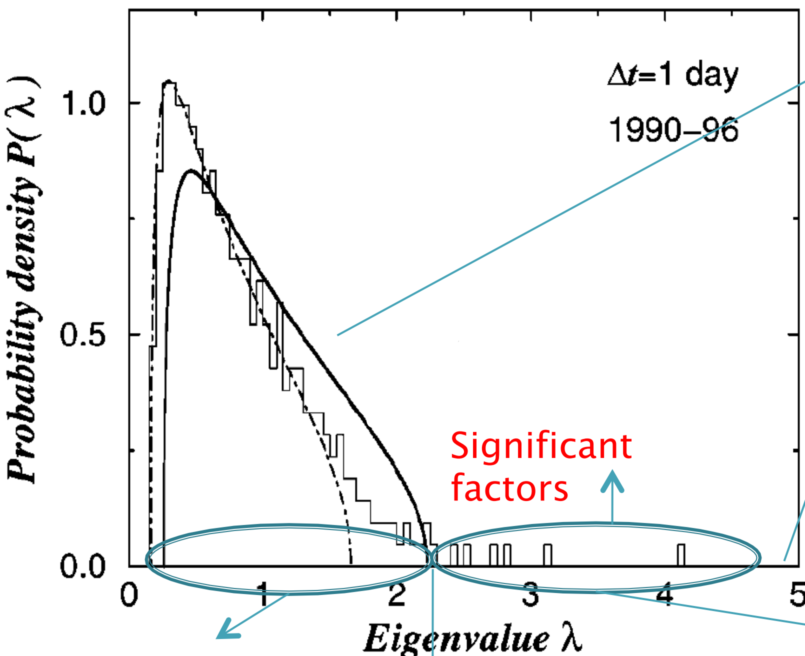
Eigenvalue Distribution for Wishart Matrix



- Eigenvalue distribution of a Wishart matrix when
- Only determine by and
- Has an upper and lower bound

Empirical Eigenvalue Distribution

From: Plerou *et. al.*, 1999, PRL., 2002, PRE. (422 stocks from S&P 500, 1737 daily returns)



Eigenvalue distribution of Wishart matrix

Largest eigenvalue (not shown):
Average behavior
(Bahavior of the whole market)

$$\lambda_{\pm} = 1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}}.$$

$$Q \equiv L/N \ (>1)$$

Significant factors

Other large eigenvalues:
Subgroups
(Sectors, industries, …)

Noise

Largest eigenvalue of uncorrelated time series

Random Matrix Theory: Summary

Aim

(1) Test significance of correlations in multiple time series

(2) Find number of significant factors

(3) Reduce noise in correlation matrix

Procedure

(1) Wishart matrix:

(2) Empirical eigenvalues:

(3) Significant factors:

(4) Noise:

Efficient Frontier

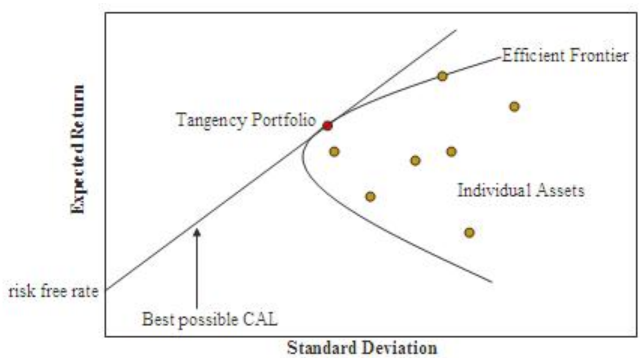
H. Markowitz (1952)

Given:

Individual return $\boldsymbol{\mu}$, Covariance matrix Σ, Portfolio return μ_*

Calculate:

Weigh $\boldsymbol{w}_{eff}$ that minimize the portfolio risk $\boldsymbol{w}^T \Sigma \boldsymbol{w}$



Calculate efficient frontier:

$$\boldsymbol{w}_{\text{eff}} = \operatorname{argmin}_{w} \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

Subject to

$$\boldsymbol{w}^T \boldsymbol{\mu} = \mu_*, \quad \boldsymbol{w}^T \mathbf{1} = 1$$

Solution:

$$A = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

$$B = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

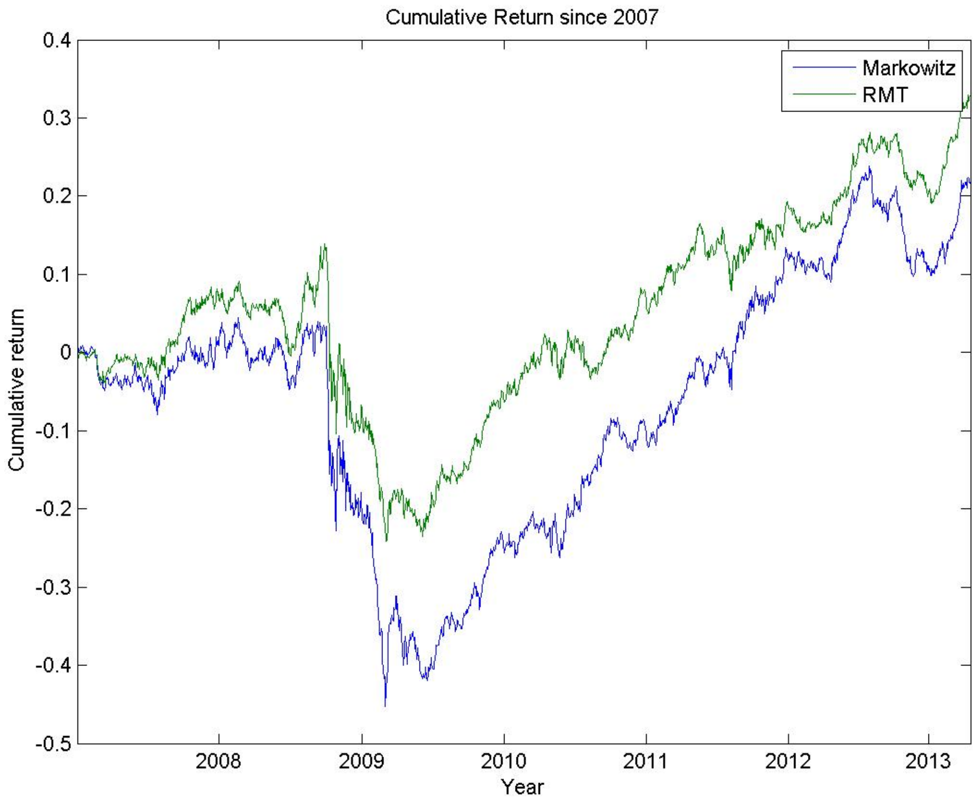
$$C = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

$$D = BC - A$$

$$\boldsymbol{w}_{\text{eff}} = \{B\boldsymbol{\Sigma}^{-1}\mathbf{1} - A\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mu_* (C\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - A\boldsymbol{\Sigma}^{-1}\mathbf{1})\}/D$$

Minimum Variance Portfolio: Cumulative Return

Portfolio:
404 stocks from S&P 500, rebalancing at the end of each year



	μ	σ
Markowitz	0.0001443	0.0086652
RMT	0.0002205	0.0080505

After Applying RMT in constructing the minimum variance portfolio, we increased returns and reduced risk.

Autogregressive Random Matrix Theory (ARRMT)

Problem

- Wishart matrix assumes no autocorrelation
- Autocorrelation can impact eigenvalue distribution
- We should not compare empirical eigenvalue distribution of autocorrelated time series with

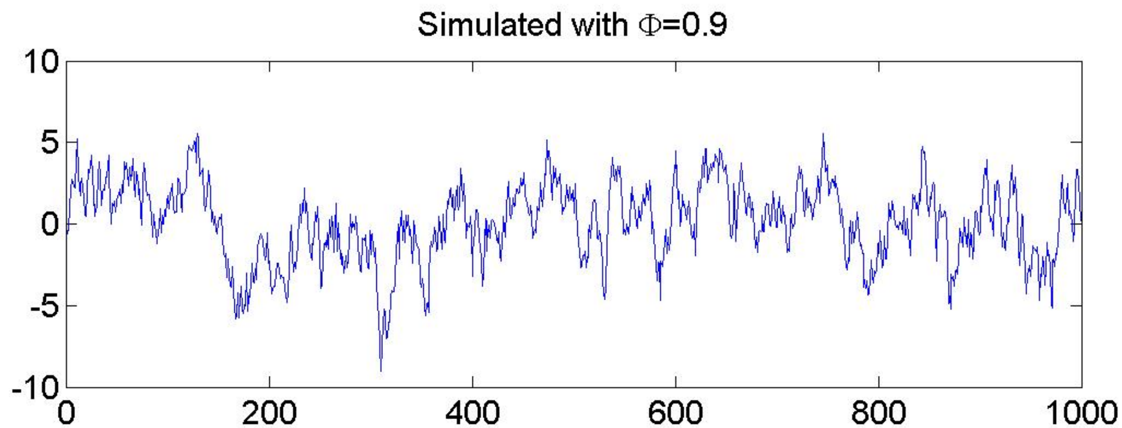
Action

- Quantify autocorrelation in empirical time series
- Compare its eigenvalues with the largest eigenvalue from simulated time series with no crosscorrelations but same autocorrelations as the empirical time series

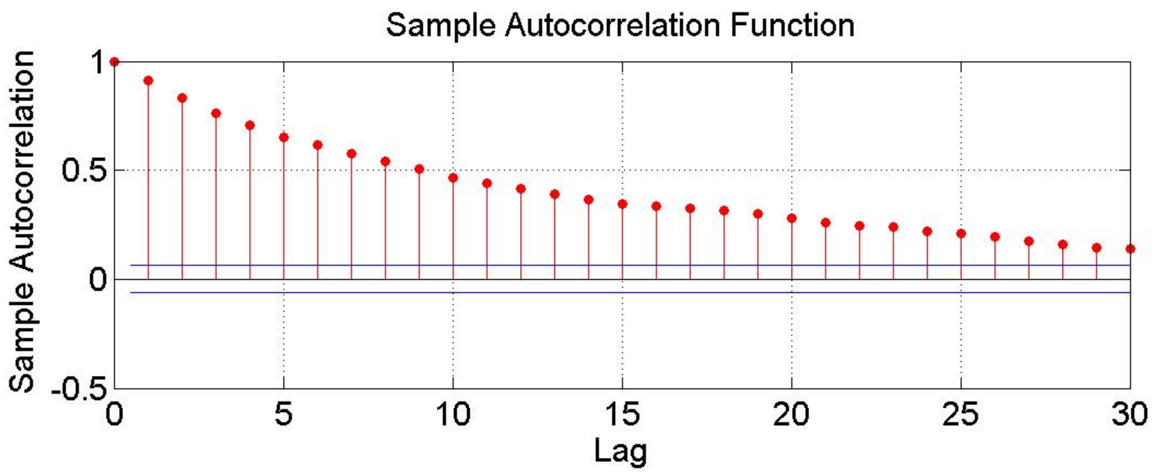
Quantify Autocorrelation:
First Order Autoregressive Model (AR(1))

AR(1) Model:

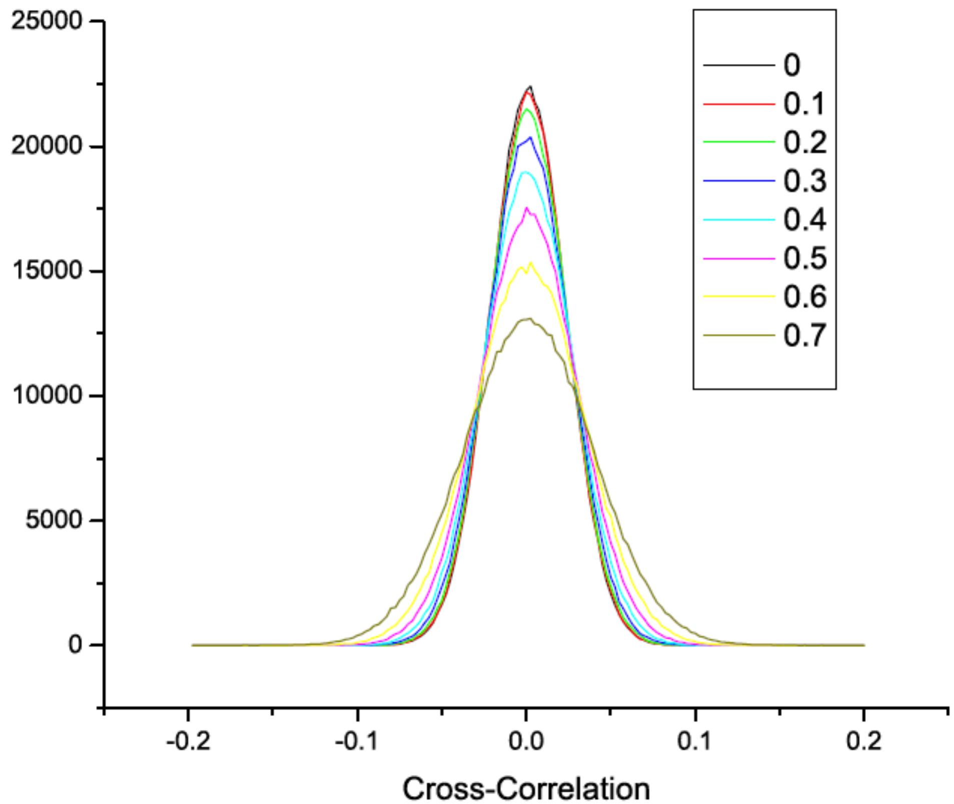
$$X_t = \phi X_{t-1} + \epsilon_t$$



Autocorrelation:



Correlation distribution for different AR(1) Coefficient

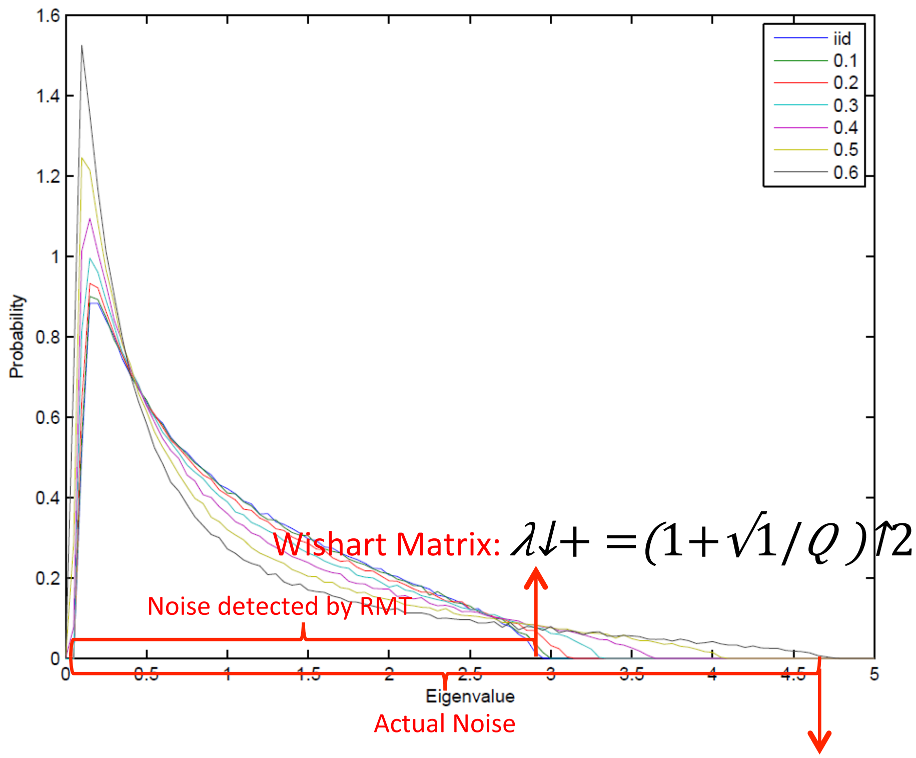


Variance of sample crosscorrelation

$$\frac{1}{T}[1 + 2\sum_{\Delta t=1}^{\Delta t_+} A(\Delta t)A'(\Delta t)].$$

Eigenvalue distribution for different AR(1) Coefficient

Autocorrelation in time series may influence the eigenvalue distribution for uncorrelated time series



Eigenvalue to compare with empirical eigenvalues

AR(1) Process

$$X_t = \phi X_{t-1} + \epsilon_t$$

Variance of correlation

$$\mathrm{Var}(r) = \frac{1}{T}\,\frac{1+\phi^2}{1-\phi^2}$$

Equivatlent length

$$T^* = T\,\frac{1-\phi^2}{1+\phi^2}$$

Equivalent Q

$$Q^* = T^*/N$$

ARRMT: Summary

Aim

Adjust RMT for autocorrelated time series

Procedure

(1) Fit AR models, get

(2) Simulate time series using the estimated

(3) Calculate largest eigenvalue

(4) Repeat (1)–(3)

(5) Distribution of largest eigenvalue

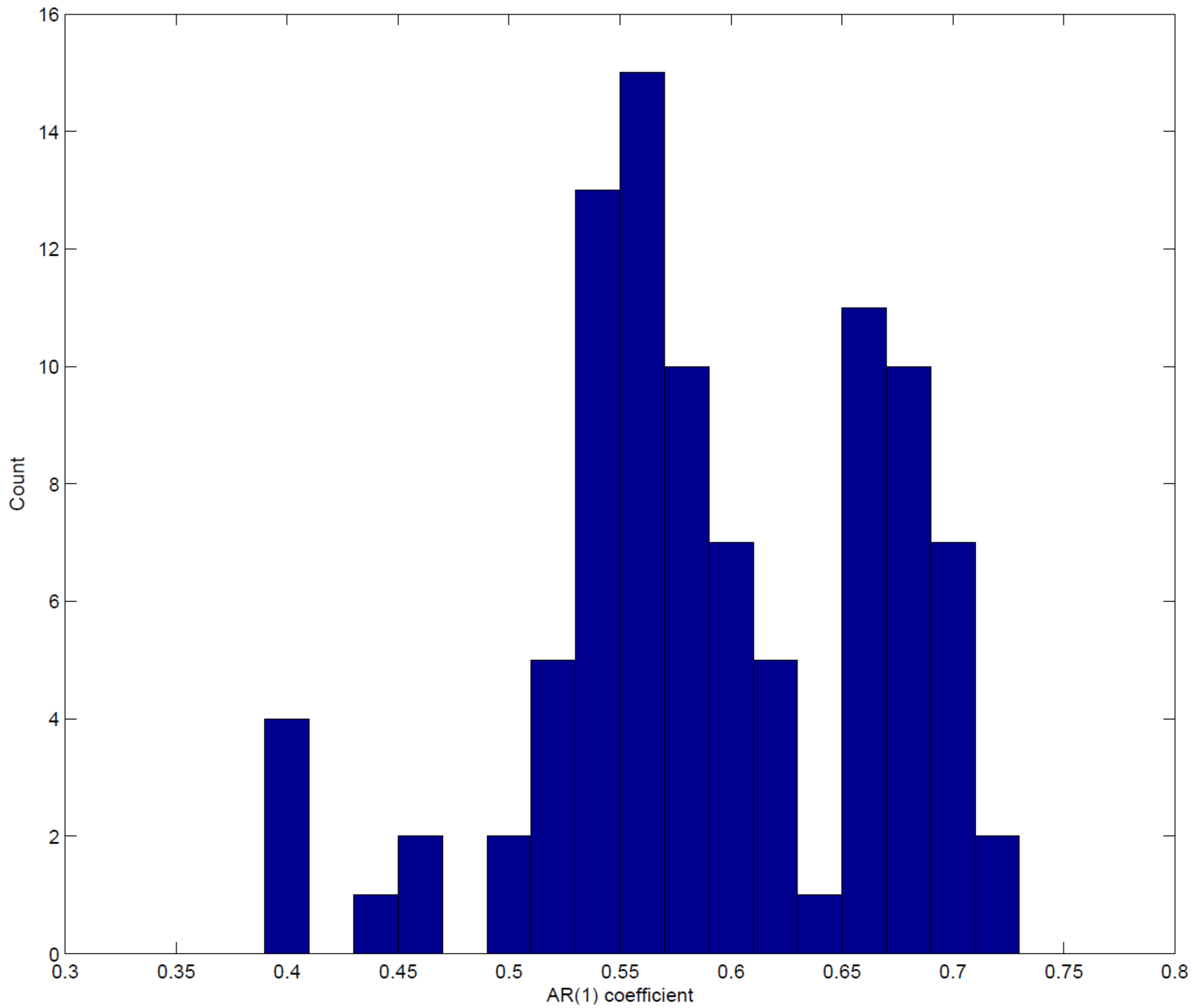
(6) For small data sets, choose 95th percentile

(7) Compare with empirical eigenvalues

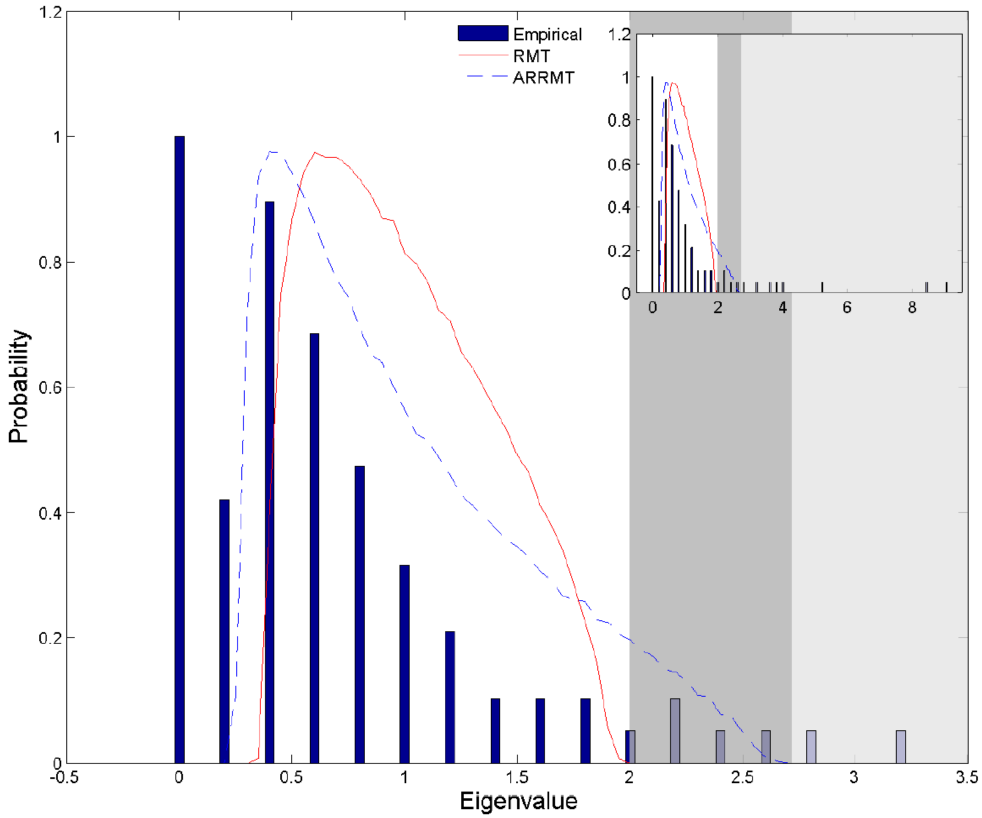
Empirical Data: Is Autocorrelation Significant?

Data: Change of air pressure in 95 US cities

Histogram of AR(1) Coefficients



RMT and ARRMT for Daily Air Pressure of 95 US Cities



RMT

- 11 significant factors

ARRMT

- 8 significant factors

Time Lag Random Matrix Theory (TLRMT)

(Podobnik, Wang et. al., 2010, EPL.)

- Aim: extend RMT to time lag correlation matrix
- Unsymmetrical matrices, eigenvalues are complex numbers
- Use singular value decomposition (SVD) instead of eigenvalue
- Largest singular value: strength of correlation

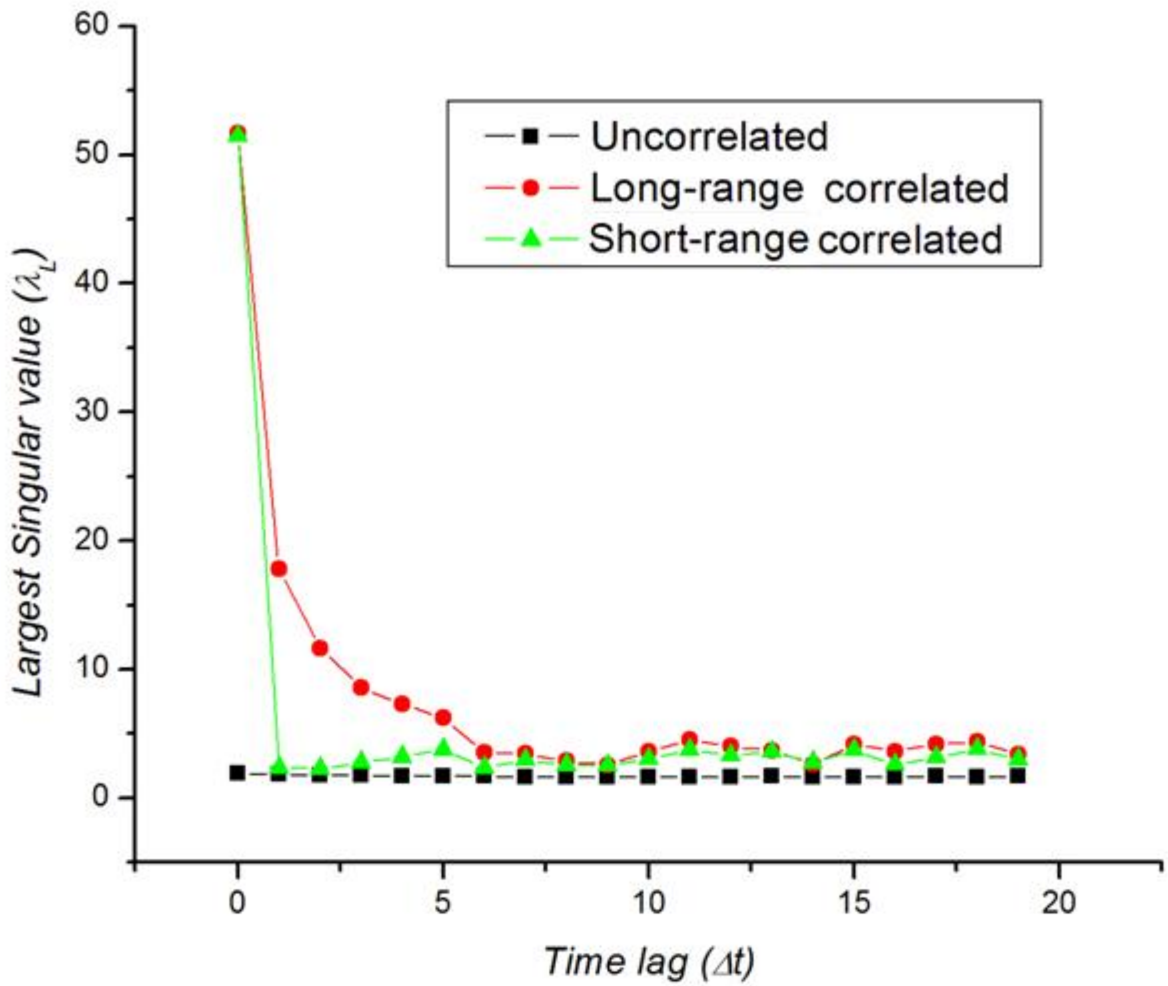
$$M = U\Sigma V^*$$

Correlation matrix for lag=0 and lag=1

Lag=0	GE	MSFT	JNJ
GE	1	0.579	0.538
MSFT	0.579	1	0.56
JNJ	0.538	0.56	1

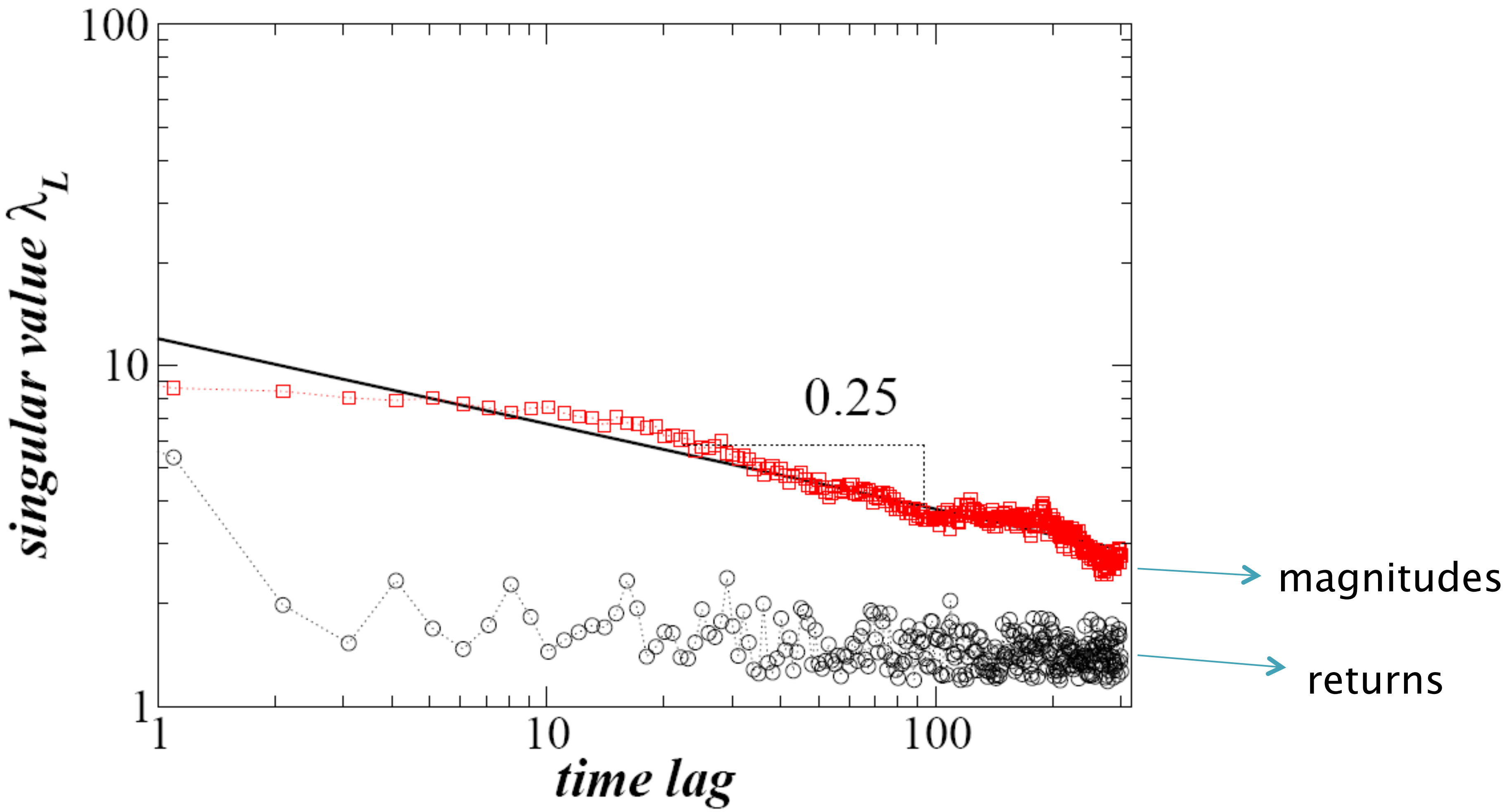
Lag=1	GE	MSFT	JNJ
GE	0.378	0.499	0.551
MSFT	0.528	0.339	0.295
JNJ	0.635	0.659	0.499

Time lag RMT:
Singular value vs time-lag cross-correlations



TLRMT Applied to 48 Stock Indices

We find:
(1)Short-range return cross-correlations (after time lag=2)
(2)Long-range magnitude cross-correlation (scaling exponent=0.25)



Global factor model (GFM)

Wang *et. al.*, 2011, PRE.

- Aim: explain cross-correlations using one single process.
- Assumption: Each individual index fluctuates in response to one common process, the “global factor” M_t.

Measures dependence of $R_{i,t}$ on M_t

$$R_{i,t} = \mu_i + b_i M_t + \epsilon_{i,t}.$$

individual return

mean return

Global factor

unsystematic noise

What is the global factor?

- A linear combination of all individual index returns

$$M_t = \omega_1 R_{1,t} + \omega_2 R_{2,t} + ... + \omega_N R_{N,t}$$

- The weight of each index return is calculated using statistical method.

GFM Properties:

- Variance of global factor---- Cross-correlation among individual indices (holds for both returns and magnitudes)

$$\mathrm{Cov}(r_{i,t}, r_{j,t}) = b_i b_j \mathrm{Var}(M_t).$$ → Return cross-correlation

$$\mathrm{Cov}(r_{i,t}^2, r_{j,t}^2) = b_i^2 b_j^2 \mathrm{Var}(M_t^2).$$ → Magnitude cross-correlation

- Autocorrelation of global factor---- Time lag cross-correlation among individual indices

$$\mathrm{Cov}(r_{i,t}, r_{j,t}, \Delta t) = b_i b_j A_M(\Delta t).$$

$$\mathrm{Cov}(r_{i,t}^2, r_{j,t}^2, \Delta t) = b_i^2 b_j^2 A_{M^2}(\Delta t).$$

A_M = autocorrelation of global factor

Conclusion:

Cross-correlation among individual indices decay in the same pattern as the autocorrelation of the global factor.

GFM explains the reason for the long range magnitude cross-correlations.

Estimate of the global factor:

Principal Component Analysis (PCA)

- Aim: Linear regression with unobservable M_t

$$R_{i,t} = \mu_i + b_i M_t + \epsilon_{i,t}.$$

Unknown, linear combinition of $R_{i,t}$

- Basis: Least total squared errors

Correlation matrix   Eigenvector matrix

- Calculation: Eigenvalue decomposition ($C=U^{+}DU$)

  (1) The principal components are related with the eigenvectors.

  (2) $M_t =$ the first principal component

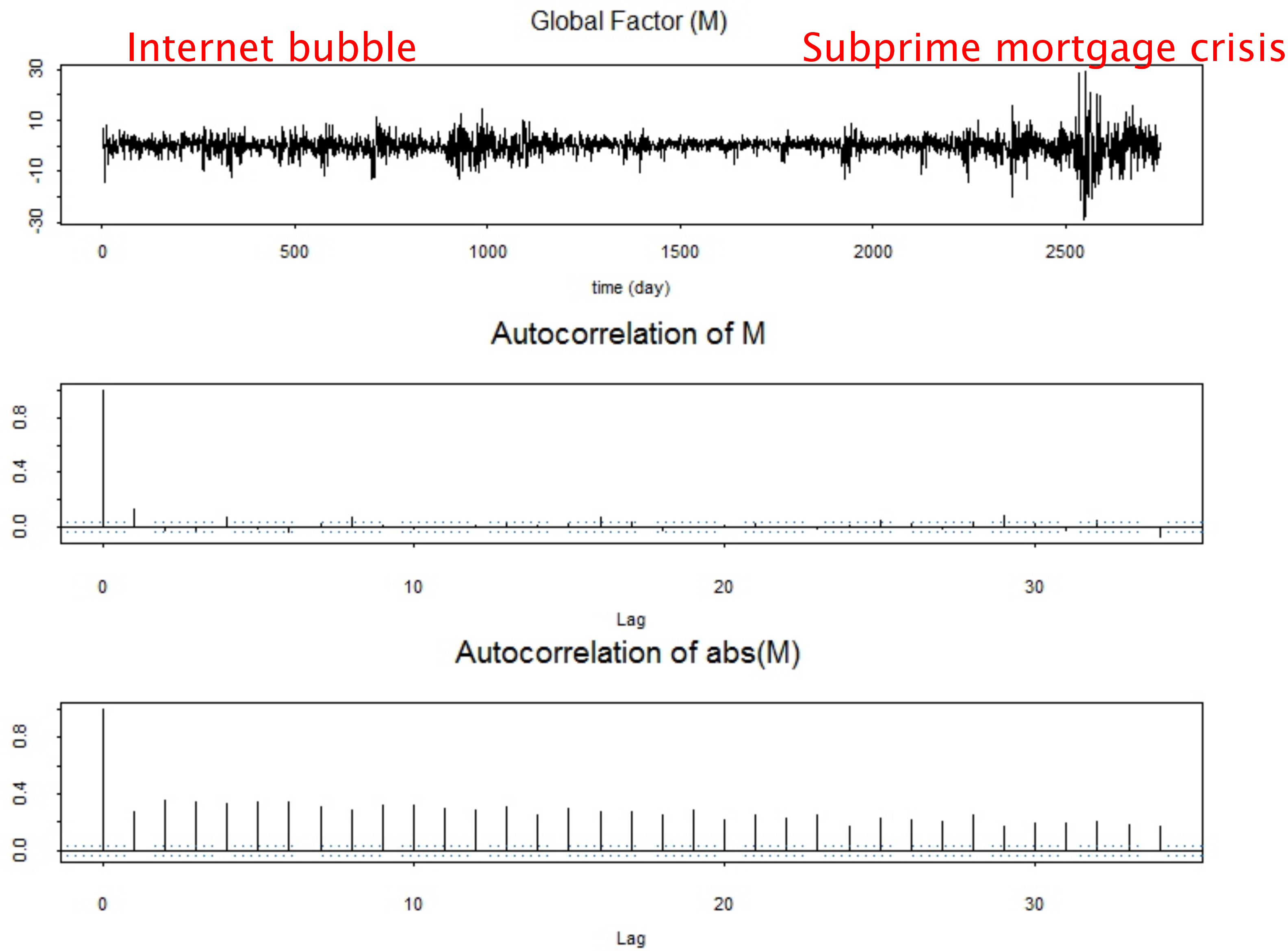
Eigenvalue matrix

- Procedure:

  (1) Find M_t using eigenvalue decomposition.

  (2) Linear regression, find μ, b_i, epsilon.

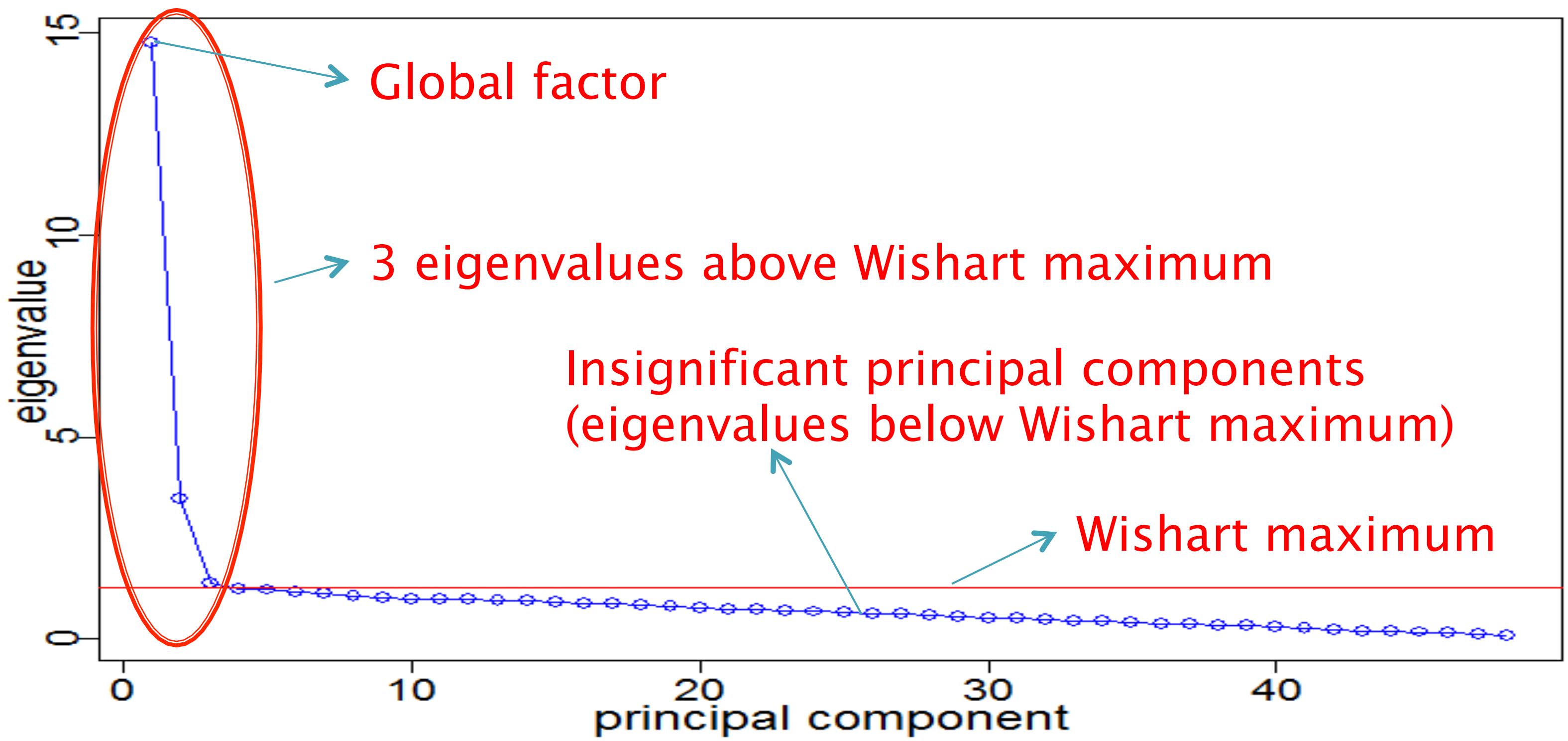
Estimate of the global factor: Result



Significant test of Global Factor

- Only 3 eigenvalues above Wishart maximum (significant).
- The largest eigenvalue (global factor) constitute 31% of total variance, and 75% of total variance of 3 significant principal components.
- Conclusion: the single global factor is sufficient in explaining correlations.

Eigenvalues above and below Wishart maximum.



Application: risk forecasting

- *Define risk: volatility (time dependent standard deviation)*

  volatility=expected standard deviation of a time series at time t given information till time t-1.

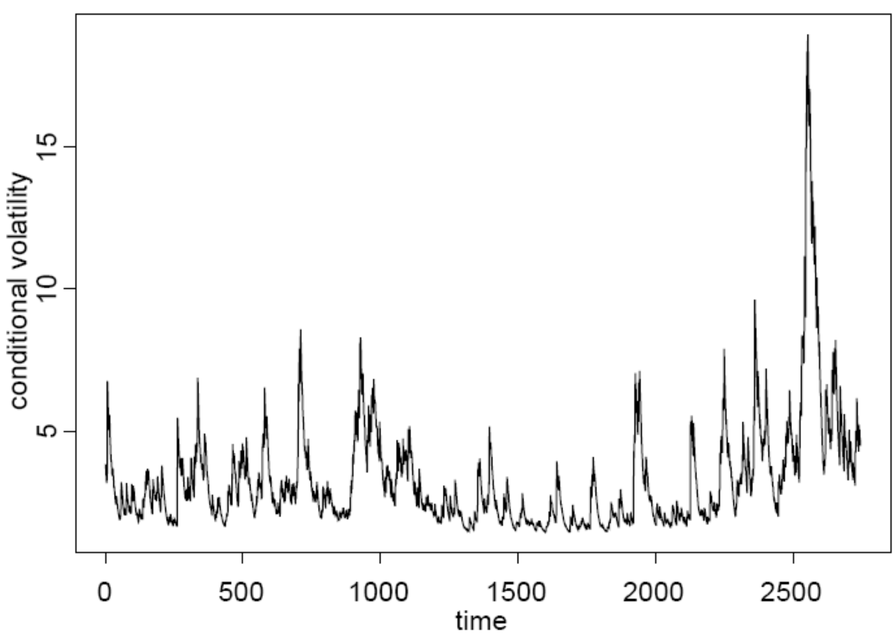
- *How to calculate risk: Apply GARCH to global factor*

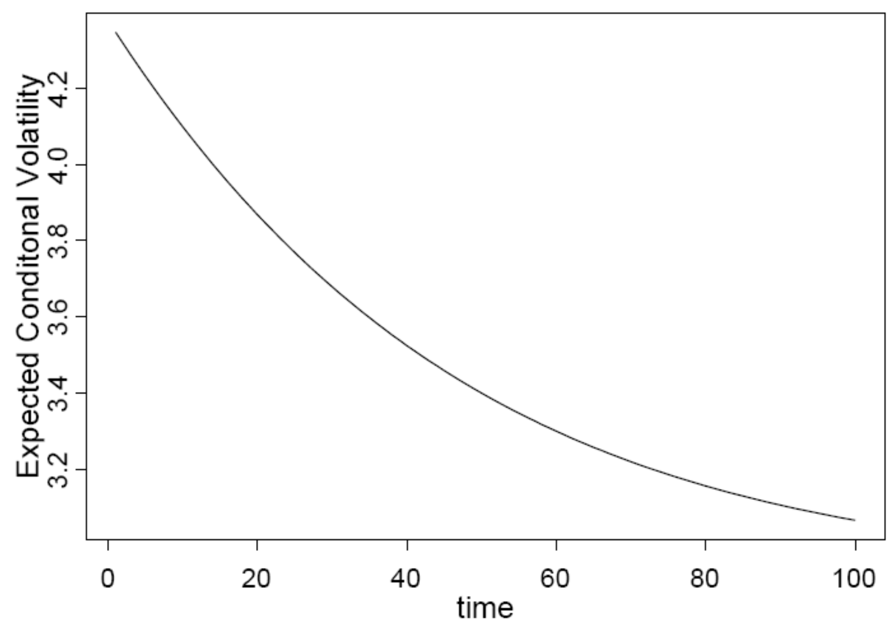
Coefficients estimated using maximum likelihood (MLE)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Forecasted using recursion with estimated coefficients

- *Historical risk and expectation of future risk*





Application: Multiple Global Factors

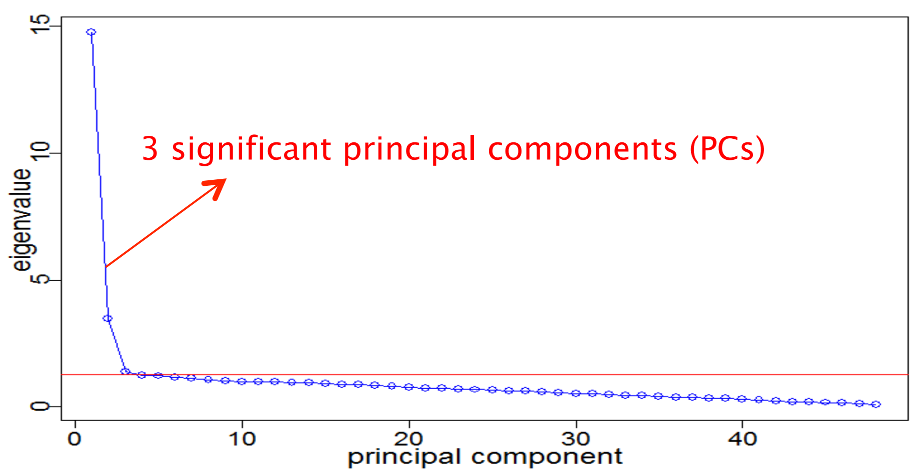
Wang *et., al.,* working paper.

- PC1: American and EU countries
- PC2: Asian and Pacific countries
- PC3: Middle east countries

Statistically, the world economy can be grouped as 3 sectors.

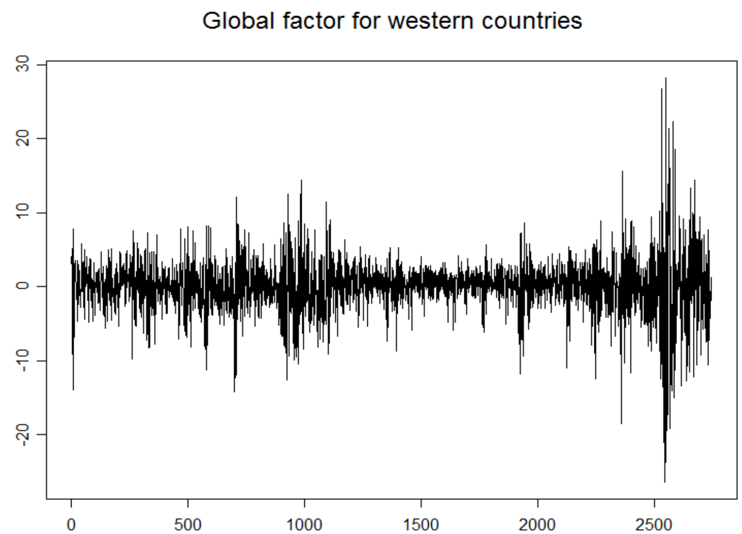
$$r_j = a_j + b_{j1}F_1 + b_{j2}F_2 + \cdots + b_{jn}F_n + \epsilon_j$$

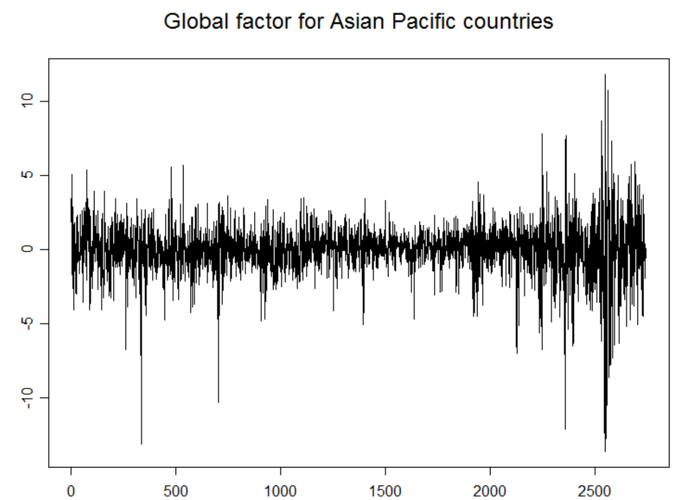
Global factor   Sectors

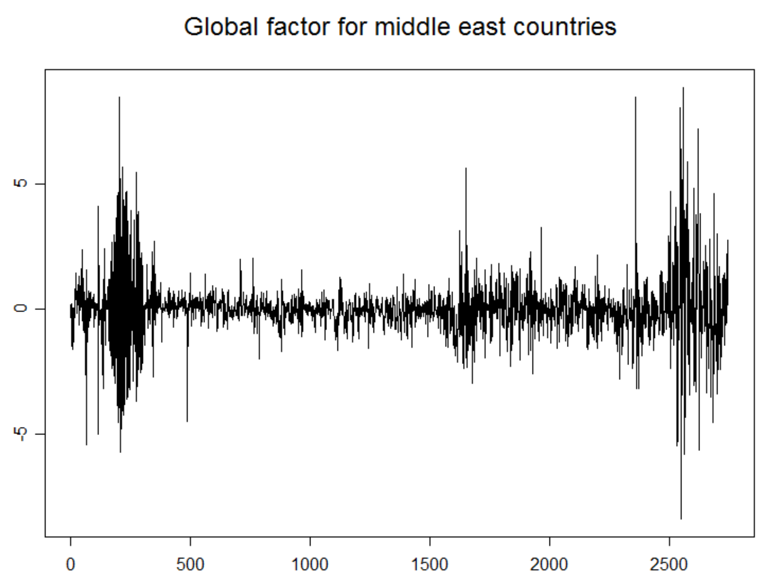


		1st PC	2nd PC	3rd PC	
USA	1				S&P 500 INDEX
Mexico	2				MEXICO BOLSA
UK	3				FTSE 100 INDE
Germany	4				DAX INDEX
France	5				CAC 40 INDEX
Spain	6				IBEX 35 INDEX
Sw itzerland	7				SWISS MARKE
Italy	8				FTSE MIB Index
Portugal	9				PSI 20 INDEX
Ireland	10				IRISH OVERAL
Netherlands	11				AEX-Index
Belgium	12				BEL 20 INDEX
Luxembourg	13				LUXEMBOURG
Denmark	14				OMX COPENHA
Finland	15				OMX HELSINKI
Norw ay	16				OBX STOCK IN
Sw eden	17				OMX STOCKHO
Austria	18				AUSTRIAN TRA
Greece	19				Athex Compos
Poland	20				WSE WIG INDE
Czech Republi	21				PRAGUE STOC
Hungary	22				BUDAPEST ST
Romania	23				BUCHAREST E
Slovenia	24				SBI20 Slovenia
Estonia	25				OMX TALLINN
Turkey	26				ISE NATIONAL
Malta	27				MALTA STOCK
RSA	28				FTSE/JSE AFR
Morocco	29				CFG 25
Nigeria	30				Nigeria Index
Kenya	31				Kenya SE
Israel	32				TEL AVIV 25 IN
Oman	33				MSM30 Index
Qatar	34				DSM 20 Index
Mauritius	35				MAURITIUS ST
Japan	36				NIKKEI 225
Hong Kong	37				HANG SENG IN
China	38				SHANGHAI SE
Australia	39				ALL ORDINARI
New Zealand	40				NZX ALL INDE
Pakistan	41				KARACHI 100
Sri Lanka	42				SRI LANKA CO
Thailand	43				STOCK EXCH O
Indonesia	44				JAKARTA COM
Malaysia	45				FTSE Bursa Ma
Philippines	46				PSEi - PHILIPPI
Mongolia	47				MSE Top 20 Ind

Multiple Global Factors: Western, Asian and Pacific. and Middle East







(1) Large correlation between Western and Asian economies
(2) Small correlation between Middle East Economy and the other 2 groups
(3) Each group has its own financial crises.
(4) The 2008 market crash influenced all 3 groups, indicating globalization (large volatility correlation).

Thanks for listening!

- Duan Wang
- wangduan@bu.edu

Eigenvectors: Implied Participation Number

Definition

$$N_p \overset{\text{def}}{=} 1/\sum_{i=1}^{N} v_i^4$$

Interpretation

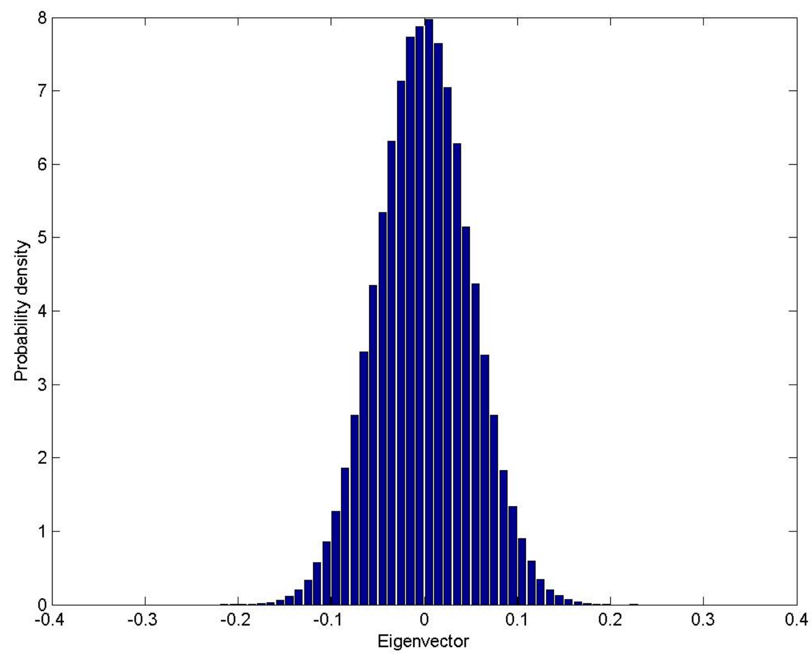
A rough estimation of how many individuals contribute to the PC.

Two extreme cases

(1) $v_i = 1/\sqrt{N}$, for all i's. $N_p = N$.

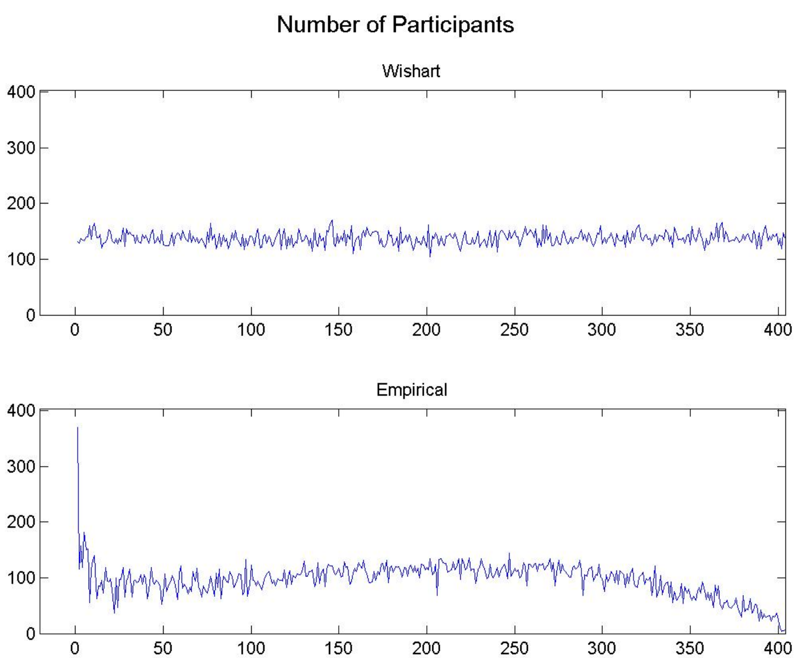
(2) $v_1 = 1$, $v_{i \neq 1} = 0$. $N_p = 1$.

For Wishart matrix



Empirical Implied Participation Number

422 stocks from S&P 500, 1737 daily returns



Wishart Matrix

Similar participation number for all PCs

Empirical Correlation Matrix

- First PC has 370 participants (87.7% of all stocks)

  Global market factor

- First few PCs has large $N \downarrow p$

  Sectors or industry groups

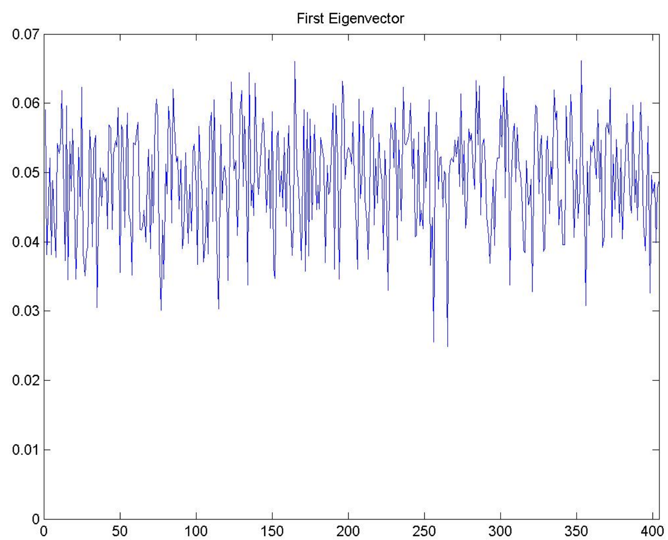
- Last few PCs has small $N \downarrow p$

  Small subgroups

Empirical Implied Participation Number

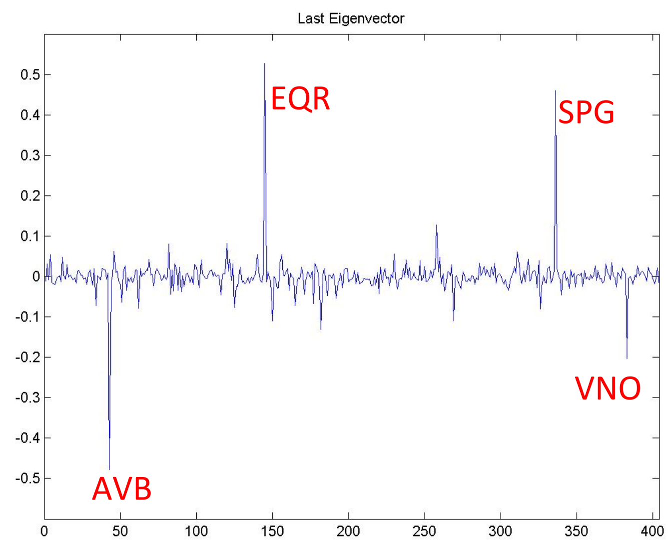
First Eigenvector

- All positive, similar weights
- Indicating global factor



Last Eigenvector

- Small number of participants
- Indicating small subgroup



Efficient Frontier

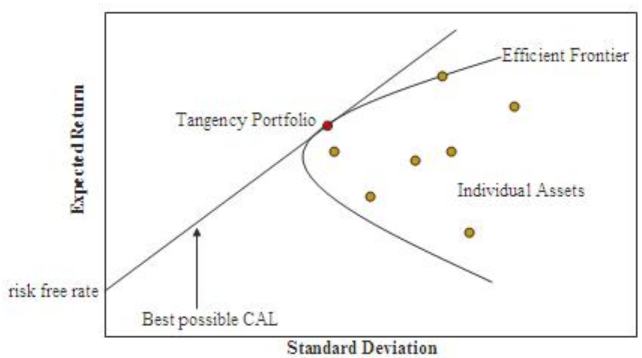
H. Markowitz (1952)

Given:

Individual return $\boldsymbol{\mu}$, Covariance matrix Σ, Portfolio return μ_*

Calculate:

Weigh $\boldsymbol{w}_{eff}$ that minimize the portfolio risk $\boldsymbol{w}^T \Sigma \boldsymbol{w}$



Calculate efficient frontier:

$$\boldsymbol{w}_{\text{eff}} = \operatorname{argmin}_{w} \boldsymbol{w}^T \Sigma \boldsymbol{w}$$

Subject to

$$\boldsymbol{w}^T \boldsymbol{\mu} = \mu_*, \quad \boldsymbol{w}^T \mathbf{1} = 1$$

Solution:

$$A = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

$$B = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

$$C = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

$$D = BC - A$$

$$\boldsymbol{w}_{\text{eff}} = \{B\boldsymbol{\Sigma}^{-1}\mathbf{1} - A\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mu_* (C\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - A\boldsymbol{\Sigma}^{-1}\mathbf{1})\}/D$$

Example: 404 stocks from S&P 500

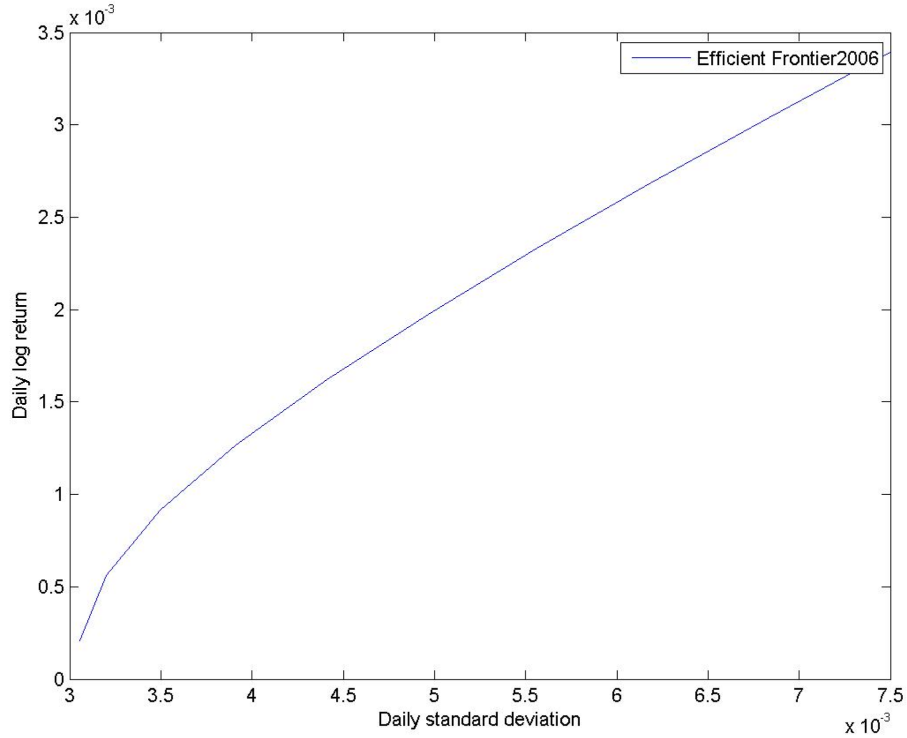
Rebalance every end of year

Use data from past 5 years to calculate w_{eff}

At the end of 2006, use 2002-2006

Rebalance portfolio using w_{eff}

What is realized frontier in 2007?



Example: 404 stocks from S&P 500

Rebalance every end of year

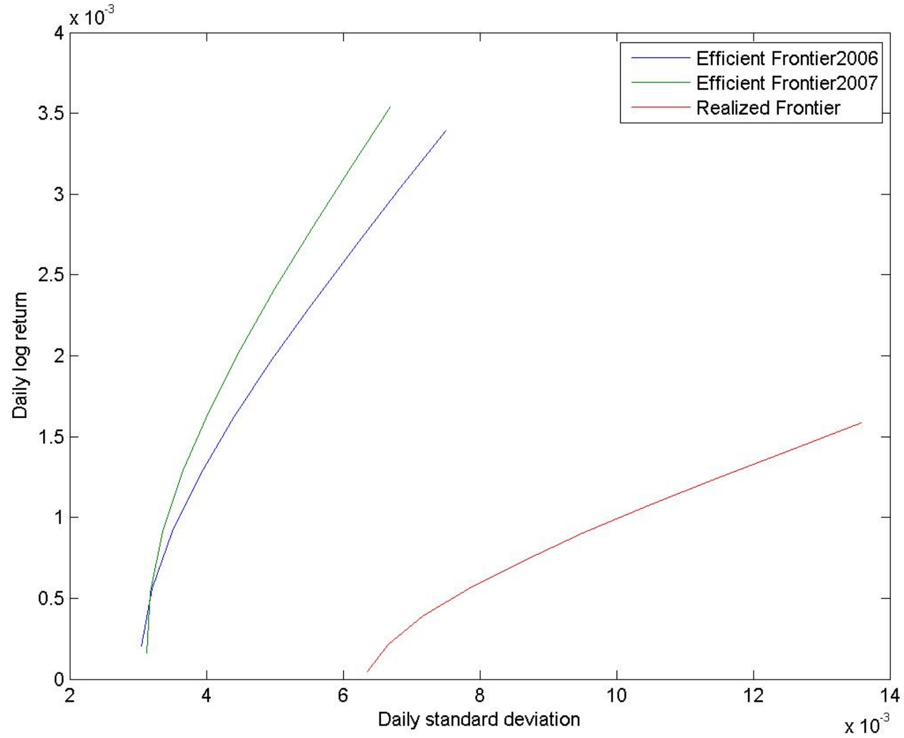
Use data from past 5 years to calculate w_{eff}

At the end of 2006, use 2002-2006

Rebalance portfolio using w_{eff}

What is realized frontier in 2007?

Problem: in sample estimate of the weights does not minimize the out of sample error



Reason for Instability

$\boldsymbol{w}_{\mathrm{minvar}} = \boldsymbol{\Sigma}^{-1}\mathbf{1} / \mathbf{1}^{T}\boldsymbol{\Sigma}^{-1}\mathbf{1}$

Eigenvalue decomposition

- $\boldsymbol{\Sigma} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$
- $\boldsymbol{\Sigma}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^{-1}$
- $\boldsymbol{D} =$ diag($\lambda_1, \lambda_2, \ldots, \lambda_N$) . Eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_N$.
- $\boldsymbol{D}^{-1} =$ diag($\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_N^{-1}$).

Smallest eigenvalues plays larger roles in calculating $\boldsymbol{w}_{\mathrm{minvar}}$

Smaller PCs are dominated by noise

How to solve the problem?

Use RMT to determine the number of eigenvalues we should keep

RMT in Portfolio Optimization: Procedure

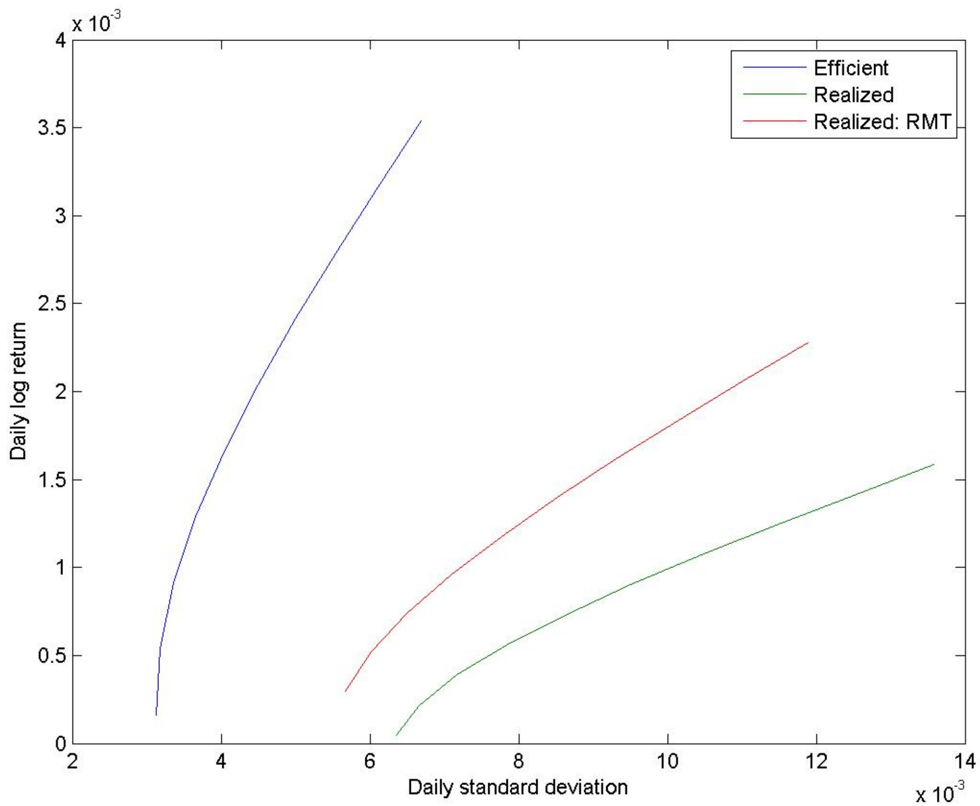
Use RMT to decide number of significant factors M

Dimension reduction

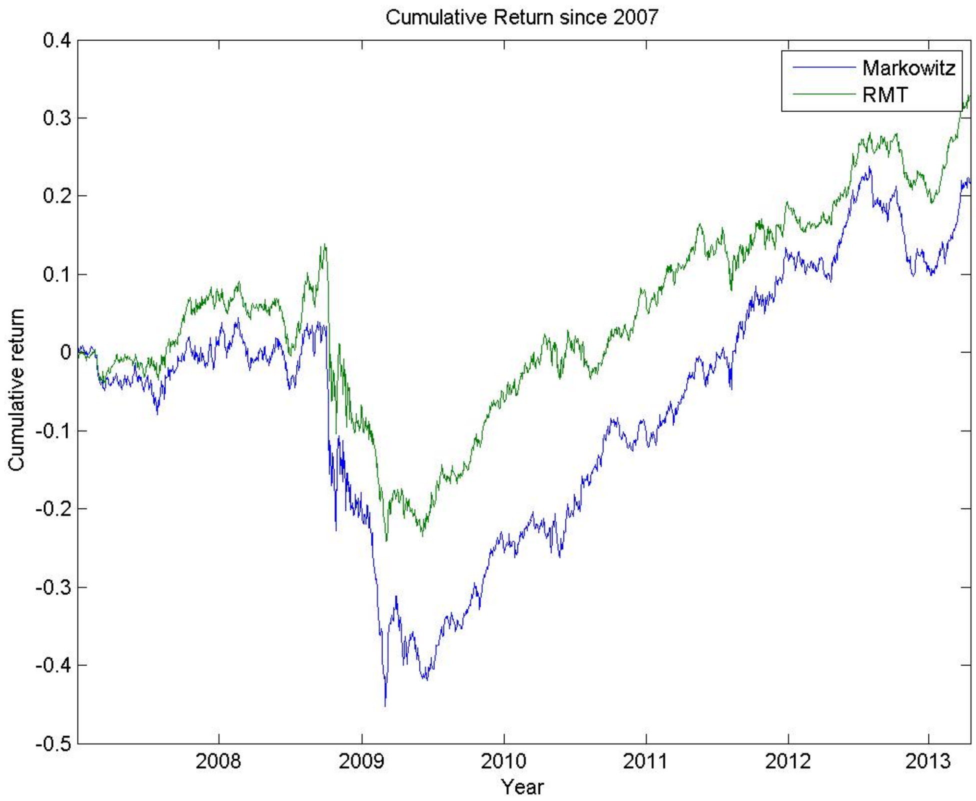
- $\boldsymbol{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_N$.
- $\boldsymbol{D}_{adj} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M, \lambda_r, \lambda_r, \ldots, \lambda_r)$. $\lambda_r \overset{\mathrm{def}}{=} \frac{1}{N-M} \sum_{i=M+1}^{N} \lambda_i$
- $\boldsymbol{\Sigma}_{adj} = \boldsymbol{V} \boldsymbol{D}_{adj} \boldsymbol{V}^{-1}$

Use adjusted covariance matrix to calculate frontier and weights

Realized Frontier: RMT Applied



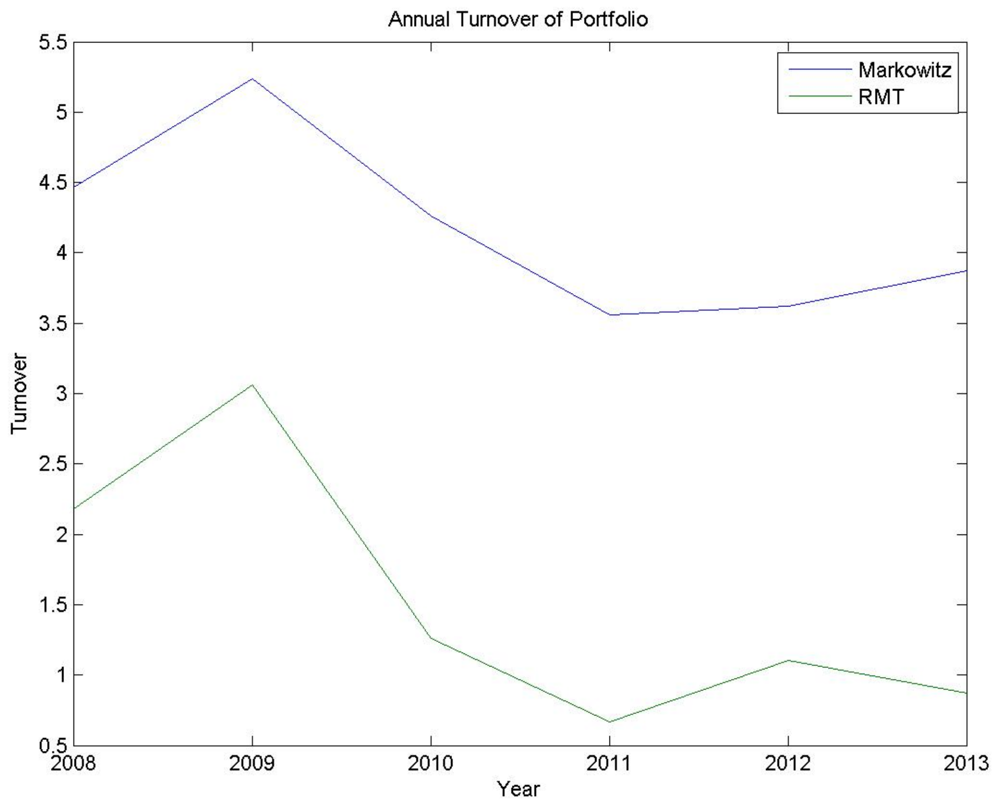
Minimum Variance Portfolio: Cumulative Return



	μ	σ
Markowitz	0.0001443	0.0086652
RMT	0.0002205	0.0080505

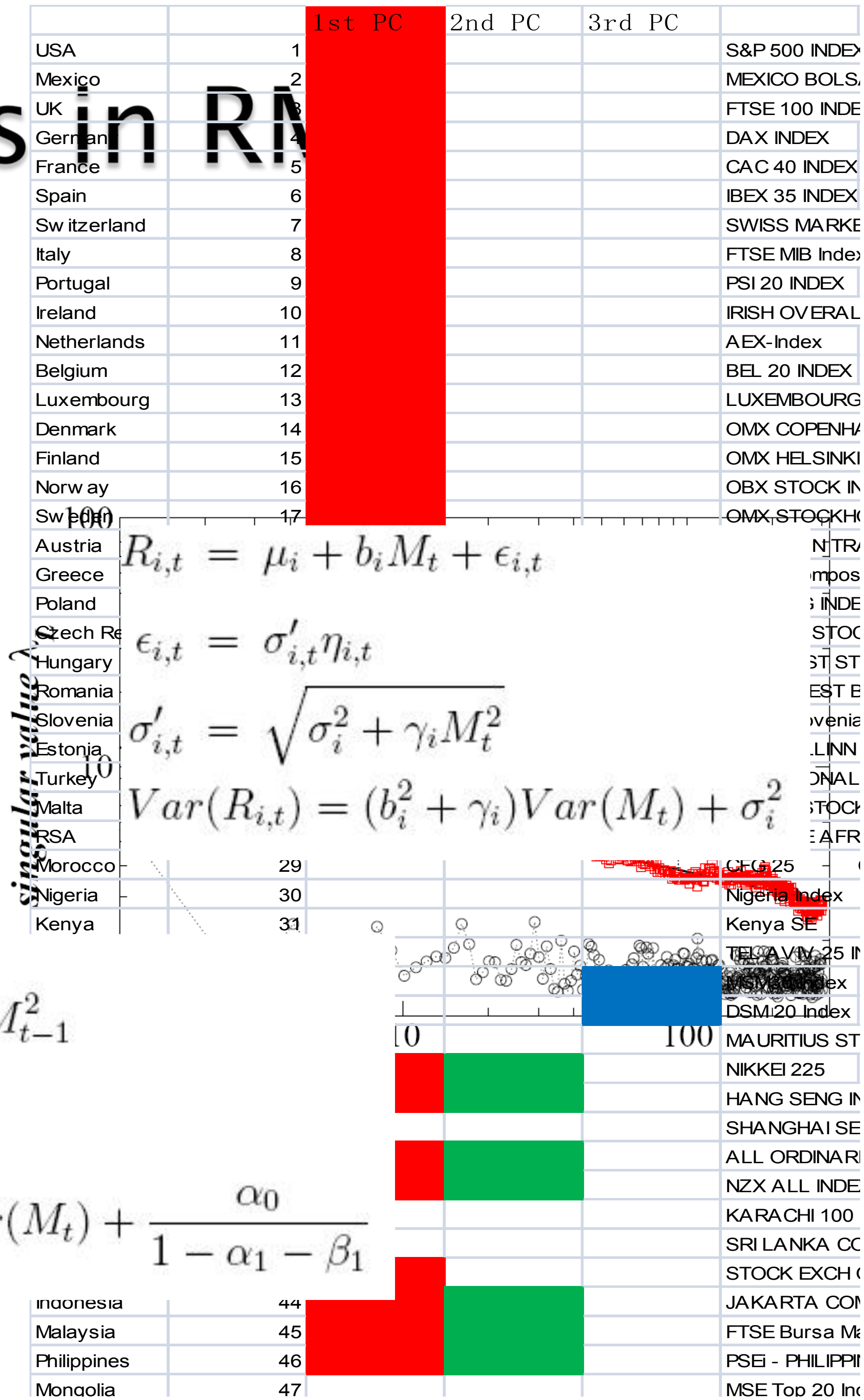
After Applying RMT in constructing the minimum variance portfolio, we increased returns and reduced risk.

Minimum Variance Portfolio: Turnover Rate



Other New Features in RM

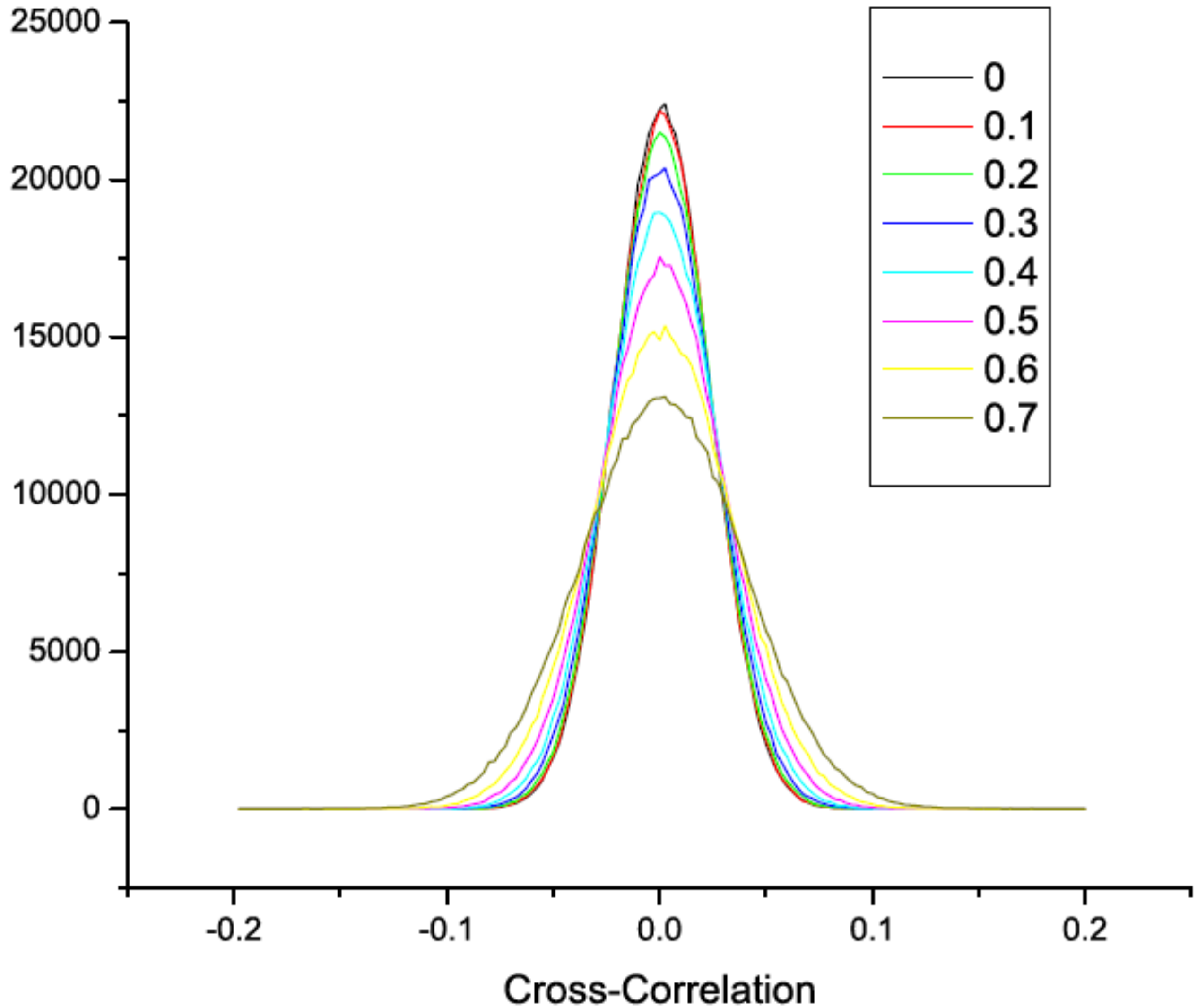
- Time-lag RMT (TLRMT)
- Global Factor Model (GFM)
- Variance Crosscorrelation
  - Conditional Variance Adjusted Regression Model (CVARM)
  - Conditional Heteroskedasticity Adjusted Regression Model (CHARM)



$$
\begin{aligned}
R_{i,t} &= \mu_i + b_i M_t + \epsilon_{i,t} \\
\epsilon_{i,t} &= \sigma'_{i,t}\eta_{i,t} \\
\sigma'_{i,t} &= \sqrt{\sigma_i^2 + \gamma_i M_t^2} \\
Var(R_{i,t}) &= (b_i^2 + \gamma_i)Var(M_t) + \sigma_i^2
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{i,t} &= \sigma_{i,t}\eta_t \\
\sigma_{i,t}^2 &= \alpha_0 + \alpha_1\epsilon_{i,t-1}^2 + \beta_1\sigma_{i,t-1}^2 + \gamma_1 M_{t-1}^2 \\
M_t &= \tilde{\sigma}_t\tilde{\eta}_t \\
\tilde{\sigma_t}^2 &= \tilde{\alpha_0}^2 + \tilde{\alpha}_1 M_{t-1}^2 + \tilde{\beta}_1\tilde{\sigma_{t-1}}^2 \\
Var(R_{i,t}) &= (b_i^2 + \frac{\gamma_1}{1-\alpha_1-\beta_1})Var(M_t) + \frac{\alpha_0}{1-\alpha_1-\beta_1}
\end{aligned}
$$

Crosscorrelation Distribution vs AR(1) Coefficients

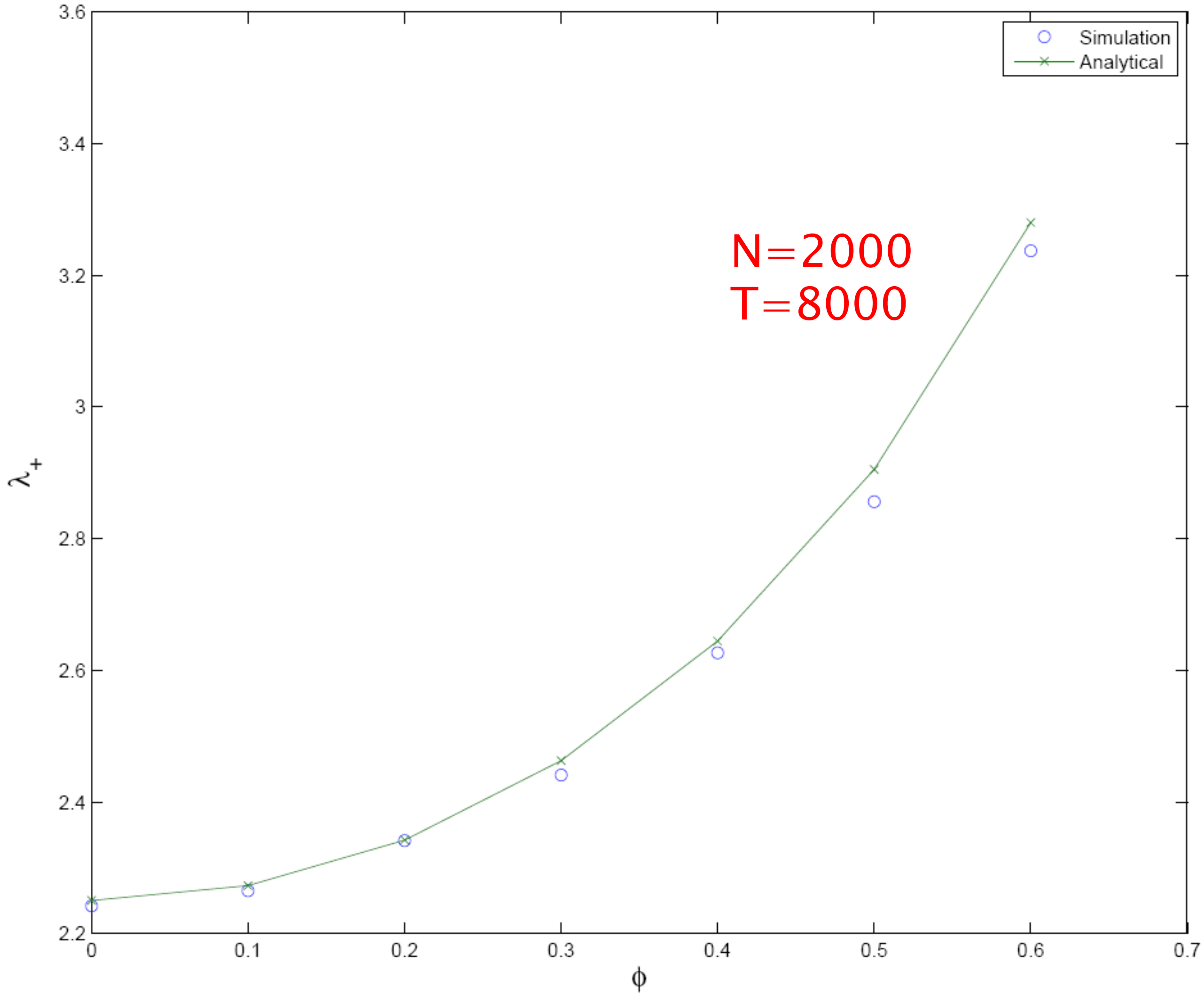


Variance

$$\frac{1}{T}[1 + 2\sum_{\Delta t=1}^{\Delta t_+} A(\Delta t)A'(\Delta t)].$$

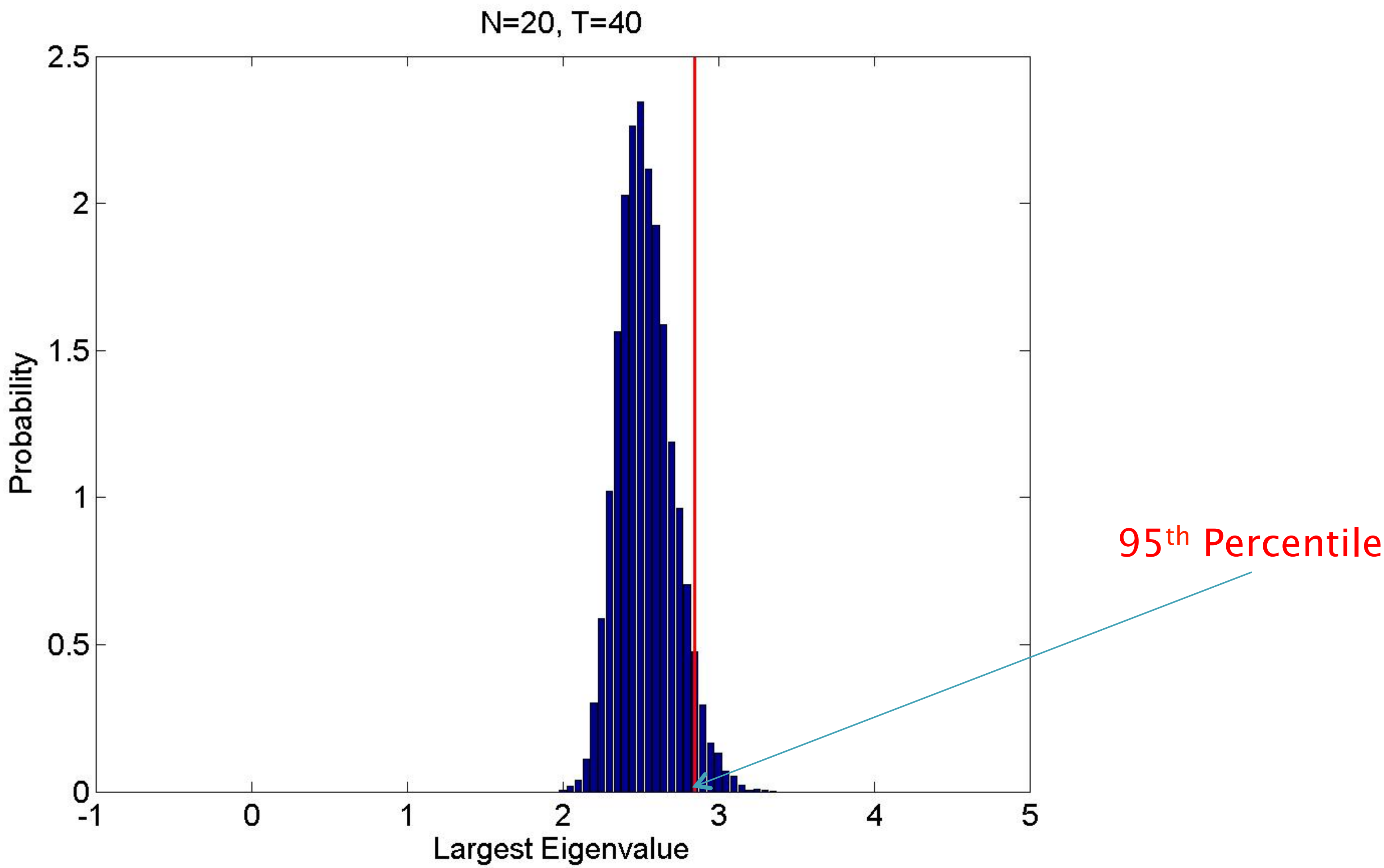
Largest Eigenvalue vs Autocorrelation

$$X_t = \phi X_{t-1} + \epsilon_t$$



$$\lambda_+^* = \sqrt{\frac{1+\phi^2}{1-\phi^2}}\lambda_+^*$$

RMT for Small-Sized Data



Other Possible Extensions:

- RMT with existence of multicolinarity
- RMT with N>T

- (Show Figures Here!)