A MODEL OF INCOME DISTRIBUTION

Summary

In the models discussed in this paper the distribution of incomes between an enumerable infinity of income ranges is assumed to develop by means of a stochastic process. In most models the stochastic matrix is assumed to remain constant through time. Under these circumstances, and provided certain other conditions are satisfied, the distribution will tend towards a unique equilibrium distribution dependent upon the stochastic matrix but not on the initial distribution. It is found that under fairly general conditions, provided the prospects of change of income as described by the matrix are in a certain sense independent of income for incomes above some limit then the Pareto curve of the equilibrium distribution will be asymptotic to a straight line. This result is preserved even when some of the effects of age on income are allowed for, and also when allowance is made for the effect of an occupational stratification of the population. Some consideration is also given to the fact that changes in the income distribution may cause the stochastic matrix itself to change. Some discussion is also given of cases where the Pareto curve of the equilibrium distribution is not asymptotic to a straight line.

§ 1. Introduction

In a recent article [1] instructions were given for graduating the distribution of personal incomes before tax by means of the distribution function

$$F(t) = \frac{N}{\theta} \tan^{-1} \frac{\sin \theta}{\cos \theta + (t/t_0)^{\alpha}} \tag{1.1}$$

where $F(t)$ is the number of incomes exceeding t, and N, α, t_0 and θ are fitted parameters. For high incomes this formula closely approximates the form

$$F(t) = Ct^{-\alpha} \tag{1.2}$$

with

$$C = \frac{N\pi}{180} t_0^{\alpha} \sin \theta$$

which is the form predicted by Pareto's law.

It has been frequently claimed that actual distributions do

[1] Champernowne [3]. See references at the end of this article.

approximate closely to this form for high income levels, and it is the purpose of this note to seek theoretical reasons for this. I am indebted to Mr. M. Crum of New College, Oxford, for critical advice and enabling me to correct several inaccuracies. Needless to say, he is in no degree responsible for any mis-statements which may remain.

§ 2. The Development of Income Distribution Regarded as a Stochastic Process

The forces determining the distribution of incomes in any community are so varied and complex, and interact and fluctuate so continuously, that any theoretical model must either be unrealistically simplified or hopelessly complicated. We shall choose the former alternative but then give indications that the introduction of some of the more obvious complications of the real world does not seem to disturb the general trend of our conclusions.

The ideas underlying our theoretical model have been briefly indicated in an earlier publication,[1] but a more complete statement may be conveniently put forward at the present time, since recent developments in the theory of stochastic processes involving infinite matrices have enabled more rigorous and neater formulation to be made than was previously found possible.

We shall suppose that the income scale is divided into an enumerable infinity of income ranges, which, for reasons to be later explained, we shall assume to have uniform proportionate extent. For example, we might consider the ranges of income per annum to be £50–£100, £100–£200, £200–£400, £400–£800, . . . although a finer graduation would be more interesting. We shall regard the development through time of the distribution of incomes between these ranges as being a stochastic process, so that the income of any individual in one year may depend on what it was in the previous year and on a chance process. In reality new income-receivers appear every year and old ones pass away, but an obvious and fruitful simplifying assumption to make is that to every "dying" income-receiver there corresponds an heir to his income in the following year, and vice versa. This assumption will imply that the number of incomes is constant through time and that the incomes live on individually, although their recipients are transitory. Not very much difficulty would be involved in allowing more or less than one heir to each dying

[1] Champernowne [1], [2].

person, but on the whole the loss of simplicity would be likely to outweigh the advantages due to the gain in verisimilitude.

Under such assumptions any historical development of the distribution of incomes could be summarily described in terms of the following vectors and matrices, $X_r(0)$, telling us the number $X_r(0)$ of the income-receivers in each range R_r, $r = 1, 2 \ldots$ in the initial year Y_0 and a series of matrices $p'_{rs}(t)$ telling us in each year Y_t, the proportions of the occupants of R_r who are shifted to range R_s in the following year Y_{t+1}. With these definitions the income distributions $x_r(t)$ in the successive years will be generated according to

$$(2.1) \qquad X_s(t+1) = \sum_{r=0}^{\infty} X_r(t) p'_{rs}(t)$$

If we suppose, as is convenient, that the income ranges are paraded in order of size (there being a lowest income range R_0), then there will be some advantage in defining a new set of matrices

$$(2.2) \qquad p_{ru}(t) = p'_{r,\,r+u}(t)$$

and rewriting (2.1) in the form

$$(2.3) \qquad X_s(t+1) = \sum_{u=-\infty}^{s} X_{s-u}(t) p_{s-u,\,u}(t)$$

$p_{ru}(t)$ then tells us the proportion in Y_t of the occupants in R_r who shift up by various numbers u of ranges.

The advantage arises from the fact that in the real world the sizes of such shifts from year to year are mostly fairly limited, so that each $p_{ru}(t)$, regarded as a frequency distribution in u, is likely to be centred round $u = 0$.

In order to make simple models, we should like to be able to assume that the $p_{ru}(t)$ regarded as a frequency distribution in u differed very little in form for variations over a wide range of values of r and t.

When we consider the practical counterpart to this suggestion we see that it means that the prospects of shifts upwards and downwards along the ladder of income ranges differ little as between the occupants of different income ranges, and differ little from year to year.

This obviously cannot apply to all income ranges. For example, a rich man's income must be allowed some risk through death or misadventure of being degraded to a lower range in the following year; but the incomes in the lowest range cannot by definition be allowed this possibility. Again the *absolute* changes in income are liable to be much higher for incomes of £1,000,000 than for incomes of £100, so that the ranges must have a greater absolute width for high than for low incomes if our simplification is to have any plausibility. The obvious choice of ranges is that indicated above whereby each range has equal proportionate extent, for then any universal effects, such as price and interest movements, which are likely to alter income prospects for widely different ranges R_r and R_q in approximately the same manner proportionately, will affect the various functions $p_{ru}(t)$ and $p_{qu}(t)$ in roughly the same fashion.

Our other assumption that the functions $p'_{r,\,r+u}(t) = p_{ru}(t)$ remain constant as t changes through time, takes us far from reality: but an essential preliminary to the study (not here attempted) of the dynamic equilibrium with moving $p'_{rs}(t)$ is to examine the static equilibrium generated by a fixed set of functions $p'_{rs}(t)$.

For it is known that under very general conditions the repeated application of the same set of income-changes represented by an irreducible matrix $p'_{rs}(t)$ will make *any* initial income distribution eventually approach a unique equilibrium distribution which is determined by the matrix $p'_{rs}(t)$ alone. Considerable interest may therefore be found in the question of the type of income distribution which will correspond to the repeated operation of the changes represented by any realistic form of the matrix $p'_{rs}(t)$.

It would be a great advantage in constructing models of income distribution if we had empirical evidence about the matrices $p'_{rs}(t)$ describing actual movements of income in modern

TABLE I

Gross income 1951, £:	0–199	200–399	400–599	600–999	1000 and over
1952 income as percentage of 1951 income.	Proportion of cases per hundred.				
150 and over	7·53	2·62	0·99	0·70	0·36
125 to 149	4·65	5·47	1·90	3·87	0·89
115 to 124	4·68	6·50	6·68	5·64	6·57
105 to 114	24·53	20·69	23·36	23·87	20·35
101 to 104	7·67	16·16	18·13	14·40	7·00
100 exactly	46·51	36·53	30·33	29·57	43·55
96 to 99	0·27	0·96	2·36	2·61	1·10
86 to 95	1·47	4·93	6·69	4·12	5·61
76 to 85	0·74	2·23	5·69	6·93	6·52
57 to 75	1·95	2·41	2·81	7·06	7·32
50 and under	—	1·35	1·06	1·21	0·72
Not available	—	0·14	—	—	—
All ratios	100·00	100·00	100·00	100·00	100·00

communities. Some such evidence could presumably be compiled from the records of the income-tax authorities, but this information has not been tapped for such a purpose. The only figures available to the writer have been kindly supplied by the Institute of Statistics at Oxford and are a by-product of their survey of savings. Unfortunately these figures are regarded as unreliable by the authors of the survey, and they can therefore merely be given in illustration of the discussion which is to follow.

Table I gives a summary of the estimates provided by the Institute of Statistics. With comparatively little manipulation, these figures can be used to provide an estimate of the elements in the matrix $p'_{rs}(t)$ for low values of s and t. Taking for R_0 the range £89–£111, and in general for R_r the range £$10^{1 \cdot 95+r/10}$ to £$10^{2 \cdot 05+r/10}$, the resulting estimates for $p'_{rs}(t)$ for $r = 0$–11, $s = 0$–14 are shown in Table II.

This table shows some degree of regularity in the figures in each diagonal, with a tendency for the lowest incomes to shift upwards by rather more ranges on the average than the high incomes. The reader may find it useful to refer back to it later when considering some of the simplifying assumptions which we will use in constructing our models.

It is unfortunate, however, that the figures tell us virtually nothing about the changes among the incomes of the rich : it is with these that our basic postulate will be mainly concerned.

§ 3. A Simple Model Generating an Exact Pareto Distribution

As an expository device it will be convenient at this stage to consider what will result from very simple assumptions indeed about the matrix $p'_{rs}(t)$ and the corresponding distributions $p_{ru}(t) = p'_{r,r+u}(t)$. Although the assumptions of this section do not approach reality at all, the results they lead to will resemble reality in one respect, and this will assist an understanding of one possible explanation of this aspect of actual distributions.

Let us assume, then, that for every value of t and r, and for some fixed integer n

$$(3.1) \qquad p'_{r,r+u}(t) = p_{r,u}(t) = 0 \text{ if } u > 1 \text{ or } u < -n$$

This means that no income moves up by more than one income range in a year, or down by more than n income ranges in a year,

$$(3.2) \qquad p'_{r,r+ru}(t) = p_{r,u}(t) = p_u > 0$$

if $\qquad -n \leqslant u \leqslant 1 \text{ and } u > -r.$

TABLE II

Estimates of Some Elements in the Matrix p'_{rs} *for England and Wales* 1951–52

Income range.	$s =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	$r =$	p'_{rs}.														
£89–£111	0	n.a.	n.a.	n.a.	n.a.	n.a.	—	—	—	—	—	—	—	—	—	—
£112–£141	1	0·020	0·672	0·202	0·083	0·023	—	—	—	—	—	—	—	—	—	—
£142–£177	2	0·015	0·020	0·674	0·204	0·068	0·019	—	—	—	—	—	—	—	—	—
£177–£221	3	0·007	0·014	0·030	0·676	0·205	0·053	0·015	—	—	—	—	—	—	—	—
£222–£281	5	—	0·012	0·015	0·040	0·672	0·207	0·043	0·011	—	—	—	—	—	—	—
£282–£354	5	—	—	0·016	0·017	0·052	0·666	0·209	0·033	0·007	—	—	—	—	—	—
£355–£445	6	—	—	—	0·015	0·018	0·075	0·666	0·200	0·023	0·004	—	—	—	—	—
£446–£562	7	—	—	—	—	0·013	0·020	0·096	0·658	0·198	0·013	0·002	—	—	—	—
£563–£707	8	—	—	—	—	—	0·016	0·035	0·100	0·632	0·200	0·015	0·002	—	—	—
£708–£892	9	—	—	—	—	—	—	0·018	0·049	0·104	0·607	0·205	0·015	0·002	—	—
£893–£1119	10	—	—	—	—	—	—	—	0·017	0·050	0·106	0·625	0·190	0·010	0·002	—
£1120–£1409	11	—	—	—	—	—	—	—	—	0·015	0·051	0·108	0·646	0·178	0·005	0·001

We may refer to this equation (3.2), and to later modifications of it, as our *basic postulate.* It here means that the prospects of shifts upwards and downwards along the ladder of income ranges are distributed in a manner independent of present income, apart from the limitations imposed by the impossibility of descending below the bottom rung of the ladder. This is the postulate which we shall retain in some modified form in nearly all our models, and which always leads to an income distribution which obeys Pareto's law at least asymptotically for high incomes.

We also need to assume that for each value of r and t

$$(3.3) \qquad \sum_{s=0}^{\infty} p'_{rs}(t) = \sum_{u=-r}^{\infty} p_{ru}(t) = 1$$

which by (3.2) also implies

$$(3.3a) \qquad \sum_{u=-n}^{1} p_u = 1.$$

This assumption (3.3) expresses the fiction that all incomes preserve their identity throughout time in the manner described in Section 2 above.

One other assumption must be introduced in order to ensure that the process is not dissipative, *i.e.*, that the incomes do not go on increasing indefinitely without settling down to an equilibrium distribution. Let us denote

$$(3.4) \qquad g(z) \equiv \sum_{u=-n}^{1} p_u z^{1-u} - z$$

then our stability assumption is that

$$(3.5) \qquad g'(1) \equiv -\sum_{u=-n}^{1} u p_u \text{ is positive.}$$

This means that for all incomes, initially in any one of the ranges R_n, R_{n+1}, R_{n+2} . . ., the average number of ranges shifted during the next year is negative.

This completes the list of assumptions for our first model and when $n = 5$ they give rise to a matrix of Diagram 1.

Now we may determine the equilibrium distribution corresponding to any matrix $p'_{r,r+u}(t) = p_{r,u}(t)$ conforming to our assumed rules. Owing to the uniqueness theorem mentioned above in Section 2, it will be sufficient to find any distribution which remains exactly unchanged under the action of the matrix $p'_{rs}(t)$ for one year. For this distribution when found must (apart from an arbitrary multiplying constant) be the unique distribution which will be approached by all distributions under

$s =$	0	1	2	3	4	5	6	7	8 . . . .
$r =$	$p'_{rs}(t)$								
0	$1 - p_1$	p_1	0	0	0	0	0	0	0 . . . .
1	$1 - p_0 - p_1$	p_0	p_1	0	0	0	0	0	0 . . . .
2	$p_{-5} + p_{-4} + p_{-3} + p_{-2}$	p_{-1}	p_0	p_1	0	0	0	0	0 . . . .
3	$p_{-5} + p_{-4} + p_{-3}$	p_{-2}	p_{-1}	p_0	p_1	0	0	0	0 . . . .
4	$p_{-5} + p_{-4}$	p_{-3}	p_{-2}	p_{-1}	p_0	p_1	0	0	0 . . . .
5	p_{-5}	p_{-4}	p_{-3}	p_{-2}	p_{-1}	p_0	p_1	0	0 . . . .
6	0	p_{-5}	p_{-4}	p_{-3}	p_{-2}	p_{-1}	p_0	p_1	0 . . . .
7	0	0	p_{-5}	p_{-4}	p_{-3}	p_{-2}	p_{-1}	p_0	p_1 . . . .
8	0	0	0	p_{-5}	p_{-4}	p_{-3}	p_{-2}	p_{-1}	p_0 . . . .
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

DIAGRAM 1

the repeated action of the matrix multiplier $p'_{rs}(t)$ year after year.

Our assumptions (3.1) to (3.5) have deliberately been chosen so as to make the solution obvious. Indeed, if X_s is the desired equilibrium distribution, we need by (2.3), (3.1) and (3.2)

$$X_s = \sum_{-n}^{1} p_u X_{s-u} \quad \text{for all } s > 0 \tag{3.6}$$

and

$$X_0 = \sum_{-n}^{0} q_u X_{-u} \quad \text{where } q_u = \sum_{v=-n}^{u} p_r. \tag{3.7}$$

We need only satisfy (3.6), since (3.6), (3.1), (3.2) and (3.3) ensure the satisfaction of (3.7) as well.

Now an obvious solution of (3.6) is

$$X_s = b^s \tag{3.8}$$

where b is the real positive root other than unity of the equation

$$g(z) = \sum_{u=-n}^{1} p_u z^{1-u} - z = 0 \tag{3.9}$$

where $g(z)$ was already defined in (3.4) above.

Descartes' rule of signs establishes the fact that (3.9) has no more than two real positive roots : since unity is one root, and $g(0) = p_0 > 0$, and $g'(1) > 0$ by (3.5), the other real positive root must satisfy

$$0 < b < 1. \tag{3.10}$$

Hence the solution (3.8) implies a total number of incomes given by

$$N' = \frac{1}{1-b} \tag{3.11}$$

and, to arrange for any other total number N, we need merely modify (3.8) to the form

$$X_s = N(1 - b)b^s. \tag{3.8a}$$

Now suppose that the proportionate extent of each income range is 10^h, and that the lowest income is $y_{\text{min.}}$: then X_s is the number of incomes in the range R_s whose lower bound is given by

$$y_s = 10^{sh}y_{\text{min.}} \text{ whence } \log_{10}y_s = sh + \log_{10}y_{\text{min.}} \tag{3.12}$$

By summing a geometrical progression, using (3.8a), we now find that in the equilibrium distribution the number of incomes exceeding y_s is given by

$$F(y_s) = Nb^s \text{ whence } \log_{10}F(y_s) = \log_{10}N + s\log_{10}b \tag{3.13}$$

Now put

$$\alpha = \log_{10}b^{-1/h} \text{ and } \gamma = \log_{10}N + \alpha\log_{10}y_{\text{min.}} \tag{3.14}$$

Then it follows from (3.12) and (3.13) that

$$\log_{10}F(y_s) = \gamma - \alpha\log_{10}y_s \tag{3.15}$$

This means that for $y = y_0, y_1, y_2 \ldots$, the logarithm of the number of incomes exceeding y is a linear function of y. This states Pareto's law in its exact form.

Thus if all ranges are of equal proportionate extent, our simplifying assumptions ensure that any initial distribution of income will in the course of time approach the exact Pareto distribution given by (3.14), (3.15).

The very simple model discussed in this section brings out clearly the tendency for Pareto's law to be obeyed in a community where, above a certain minimum income, the prospects of various amounts of percentage change of income are independent of the initial income.

Most of the remainder of the article will be spent in generalising this very simple model so that it is less unrealistic.

In actual income distributions, Pareto's law is not even approximately obeyed for low incomes: if logarithm of income is measured along the horizontal axis, the frequency distributions found in practice are not J-shaped like that obtained in our model, but single humped and moderately symmetrical. The first modification which we make to our model is to remove the assumption that there is a lowest income range R_0 and to set up conditions which lead to a two-tailed distribution, one for the poor and one for the rich.

In these simple models, Pareto's law is obeyed exactly, not merely asymptotically. We next introduce two generalisations which limit observance of the law to the occupants of high income groups and render it no longer exact but asymptotic. These generalisations consist in :

(i) allowing incomes to shift upwards by more than one range in a year;

(ii) limiting our basic assumption (3.2) that the prospects of various amounts of percentage change of income are independent of initial income to apply to higher incomes only.

These two generalisations bring our model much closer to the conditions indicated by Table II above.

In real life a man's age has a great influence on his prospects of increasing his income. Our next generalisation takes this into account. We now allow a man's prospects of change of income to depend on his age. Finally, we use the same technical device to allow a man's occupation to influence his prospects of change of income.

Despite these generalisations of the model, it is still found that the Pareto curve must be asymptotic to a straight line. Is it then possible that the approximate linearity over high income ranges of the Pareto curves found for many modern communities is due to the approximate fulfilment in the real world of our basic assumption? This question is briefly discussed in the final sections of the paper.

§ 4. A Model Generating a Two-tailed Income Distribution Obeying Pareto's Law

The simple model described in the last section generated a distribution with only one Pareto tail. The essential modifications required to introduce a two-tailed distribution are the following :

(i) We drop the assumption that there is a lowest income-range R_0, and adopt an infinite sequence of income ranges R_r, of equal proportionate extent, allowing r to run from minus infinity to plus infinity.

(ii) We adopt assumptions about that part of the matrix $p'_{rs}(t)$ for which r is negative analogous to those adopted about that part for which r is positive.

(iii) We allow for some movement of incomes to and fro between ranges R_r for which r is positive and those for which r is negative.

In particular, we assume as in (3.1)

(4.1) $p'_{r,r+u}(t) = p_{ru}(t) = 0$ when $r \geqslant 0$ and $u \geqslant 1$ or $u < -n$

and we retain our basic postulate (3.2)

(4.2) $p'_{r,r+u}(t) = p_{ru}(t) = p_u > 0$ if $-n \leqslant u \leqslant 1$ and $u > -r$

We further assume that

(4.3) when $r \geqslant 0$ and $u < -r$, and when $r < 0$ and $u > r$, $u < -1$ or $u > v$,
$$p'_{r,r+u}(t) = p_{ru}(t) = 0$$

We now introduce a positive integer r and non-negative constants $\pi_{-1}\,\pi_0\,\pi_1\,\ldots\,\pi_v$ and satisfying

(4.4) $$\pi_{-1} > 0 \quad \pi_0 > 0 \quad \pi_1 > 0 \quad \pi_v > 0 \quad \lambda > 0 \quad \sum_{s=1}^{v} \pi_s = 1 \quad \sum_{u=1}^{v} u\pi_u > 1$$
$$1 - \lambda - \pi_{-1} > 0 \quad 1 - \lambda - p_1 > 0$$

and put

(4.5) $$p_{ru}(t) = \pi_u \text{ when } r < 0 \text{ and } u < -r - 1 \text{ and } -1 \leqslant u \leqslant v$$
$$p_{0,-1}(t) = p_{-1,2}(t) = \lambda \quad p_{00}(t) = 1 - \lambda - p_1 > 0$$
$$p_{-10}(t) = 1 - \lambda - \pi_{-1} > 0$$

and assume as before that for all r

(4.6) $$\sum_{u=-\infty}^{\infty} p_{ru}(t) = 1 \text{ and } p'_{r,r+u}(t) = p_{ru}(t).$$

These assumptions can best be understood by considering their effects when n and v take particular values. Thus when $n = 3$ and $v = 2$, they give rise to a matrix for $p'_{rs}(t)$ whose centre is of the following form :

$s =$	-4	-3	-2	-1	0	1	2	3	4	5
$r =$										
-5	π_1	π_2	0	0	0	0	0	0	0	0
-4	π_0	π_1	π_2	0	0	0	0	0	0	0
-3	π_{-1}	π_0	π_1	π_2	0	0	0	0	0	0
-2	0	π_{-1}	π_0	$1 - \pi_0 - \pi_{-1}$	0	0	0	0	0	0
-1	0	0	π_{-1}	$1 - \lambda - \pi_{-1}$	λ	0	0	0	0	0
0	0	0	0	λ	$1 - \lambda - p_1$	p_1	0	0	0	0
1	0	0	0	0	$1 - p_0 - p_1$	p_0	p_1	0	0	0
2	0	0	0	0	$p_{-3} + p_{-2}$	p_{-1}	p_0	p_1	0	0
3	0	0	0	0	p_{-3}	p_{-2}	p_{-1}	p_0	p_1	0
4	0	0	0	0	0	p_{-3}	p_{-2}	p_{-1}	p_0	p_1
5	0	0	0	0	0	0	0	p_{-3}	p_{-1}	p_0
6	0	0	0	0	0	0	0	p_{-3}	p_{-2}	p_{-1}

DIAGRAM 2

We retain our assumption (3.5) that

(4.7) $$g'(1) = -\sum_{u=-n}^{1} up_u \text{ is positive}$$

and introduce the analogous assumption that

$$\gamma'(1) = -\sum_{v=-1}^{m} v\pi_v \text{ is negative} \tag{4.8}$$

where

$$\gamma(z) \equiv \sum_{v=-1}^{1} \pi_v z^{1-v} - z \tag{4.9}$$

Then, by an argument analogous to that of Section 3, the equation

$$\gamma(z) = 0 \tag{4.10}$$

must have a single real root β satisfying

$$\beta > 1 \tag{4.11}$$

It may be easily verified that the distribution

$$\begin{aligned} X_s &= Ab^s \text{ when } s \geqslant 0 \\ X_s &= A\beta^{s+1} \text{ when } s < 0 \end{aligned} \tag{4.12}$$

satisfies the equilibrium condition

$$X_s = \sum_{r=-\infty}^{\infty} p'_{rs}(t)X_r \tag{4.13}$$

Hence for some value of A, this must be the equilibrium distribution towards which any actual distribution must tend under the repeated action of the multiplying matrix $p'_{rs}(t)$ determined by the various assumptions of this model.

To secure any total number of incomes, N say, we need only put

$$A = \frac{(1-b)(\beta-1)N}{2\beta - b\beta - 1} \tag{4.14}$$

In this solution there are two Pareto tails; one relating to high incomes and one to low incomes. The distribution is kept stable by the two conditions (4.7), (4.8), which ensure that for large incomes the expected change u is negative and for small incomes it is positive. This pair of conditions is needed to offset the continual dispersal of incomes due to the variance of the frequency distributions in u, p_u and π_u.

This example has been hand-picked so as to yield a crystal-clear solution, but one essential feature of this solution, namely the conformity to Pareto's law of the distribution, will be found to be approximately preserved through a series of modifications and relaxations of our simplifying assumptions. The basic postulate which leads to the approximate obedience of this law was retained in assumption (4.2), which determines that the functions of type $p_{ru}(t)$ should be the same for all values of r relating to high income ranges.

One minor generalisation which can be made to the above example without essentially altering the form of the solution is to enlarge the channels of communication between income ranges with $r \geqslant 0$ and those with $r < 0$, hitherto limited to the flow of a proportion λ from each of R_{-1} and R_0. We may adjust the values of $p'_{r,-1}(t)$ and $p'_{r0}(t)$ in such a manner as to allow the flow of incomes from each of $R_0R_1 \ldots R_{n-1}$ to R_{-1} and from each of $R_{-1}R_{-2} \ldots R_{-r}$ to R_0 without altering the solution further than to the form

$$(4.15) \qquad \begin{aligned} x_s &= Bb^s \text{ when } s \geqslant 0 \\ x_{-s} &= B\beta^{s-1} \text{ when } s < 0 \end{aligned}$$

where the ratio between β and B will now depend on the value of $p'_{r,-1}(t)$ and $p'_{r0}(t)$, and need no longer be unity.

§ 5. A More General Model Generating a Distribution Asymptotic to a Pareto Distribution

One of the most restrictive assumptions in the example we have discussed was that

$$p_{ru}(t) = 0 \text{ when } u > 1 \text{ and } r \geqslant 0$$

The abandonment of this assumption destroys the complete simplicity of the solution.

In order to concentrate attention on the new generalisation, let us first restore the assumption that R_0 is the minimum income range so that we shall only have to consider one tail of the distribution. Then let us replace the assumption (3.1) by

(5.1) $p_{ru}(t) = 0$ if $u > m$ a given positive integer or $u < -n$ and modify (3.2) to the form

(5.2) $p_{ru}(t) = p_u$ (defined for $u = -n, 1-n, \ldots, m$) if $u + r \geqslant m$ where $\sum_{u=-n}^{m} p_u = 1$ and no p_u is negative.

We may retain assumptions (3.3), (3.4) and (3.5) (extending the summation in (3.4) and (3.5) from $-n$ to m) and add the additional assumption, $p_m > 0$.

The assumptions made so far have defined $p'_{rs}(t)$ for $s \geqslant m$ and determined $\sum_{s=0}^{m-1} p'_{rs}(t)$ for each $r = 0, 1, \ldots, n$ and $\sum_{s=r-n}^{m-1} p'_{rs}(t)$ for each $r = n+1, \ldots, n+m-1$. But the individual values $p'_{rs}(t)$ for $r = 0, 1, \ldots, n+m-2$ and $r - n \leqslant s < m$ are, subject to these linear restraints, still at our disposal. We shall make no further assumption about these individual values, except that none are negative and that when $1r - s1 \leqslant 1$, $p'_{rs}(t)$ is positive.

The effect of these assumptions in the case $n = 2$ $m = 3$ is to give the matrix $p'_{rs}(t)$ the following form:

$p'_{00}(t)$	$p'_{01}(t)$	$p'_{02}(t)$	p_1	0	0	0	0 . . . .
$p'_{10}(t)$	$p'_{11}(t)$	$p'_{12}(t)$	p_2	p_3	0	0	0 . . . .
$p'_{20}(t)$	$p'_{21}(t)$	$p'_{22}(t)$	p_1	p_2	p_3	0	0 . . . .
0	$p'_{31}(t)$	$p'_{32}(t)$	p_0	p_1	p_2	p_3	0 . . . .
0	0	p_{-2}	p_{-1}	p_0	p_1	p_2	p_3 . . . .
0	0	0	p_{-2}	p_{-1}	p_0	p_1	p_2 . . . .
0	0	0	0	p_{-2}	p_{-2}	p_0	p_1 . . . .
0	0	0	0	0	p_{-2}	p_{-1}	p_0 . . . .
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

DIAGRAM 3

subject to the conditions:

(i) that the sum of the elements in each row are unity;

(ii) that the elements in the three central diagonals are all positive, and no elements are negative.

In the general case where m is some positive integer, the equation

$$g(z) = \sum_{\theta=0}^{m+n} p_{m-\theta} z^{\theta} - z^m = 0 \tag{5.3}$$

will still have just one positive root b_1 less than unity, provided we retain our assumption (3.5) in the form

$$g'(1) = - \sum_{u=-n}^{m} u p_u > 0. \tag{5.4}$$

This assumptions ensures, by a theorem due to Mr. F. G. Foster,[1] that the matrix $p'_{rs}(t)$ is "non-dissipative," and hence that a unique finite non-zero equilibrium distribution will be approached in the limit under the repeated application of the changes embodied in this matrix.

Let the $m + n$ roots of (5.3) be $b_1 b_2 \ldots b_{m+n}$; and let x_s $(s = 0, 1, 2 \ldots)$ denote the equilibrium distribution, which must satisfy the equilibrium equations

$$\left\{ \begin{aligned} x_s &= \sum_{r=0}^{\infty} p'_{rs}(t) x_r \qquad s = 0, 1, 2 \ldots \\ \sum_{s=0}^{\infty} x_s &= N \end{aligned} \right. \tag{5.5}$$

[1] Foster [4].

This set of equations may be subdivided into

$$(5.6) \qquad x_s = \sum_{r=0}^{\infty} p'_{rs}(t)x_r \qquad s = 0, 1, 2 \ldots m-1$$

$$(5.7) \qquad x_s = \sum_{u=-n}^{m} p_u x_{s-u} \qquad s = m, m+1, m+2 \ldots$$

$$(5.8) \qquad \sum_{s=0}^{\infty} x_s = N$$

The solution is known, because of Foster's theorem, to exist: hence there exist coefficients $B_1 \ldots B_{m+n}$ such that

$$(5.9) \qquad x_s = \sum_{k=1}^{m+n} B_k b_k^s \qquad s = 0, 1, 2 \ldots m+n-1$$

It will then follow from (5.3) and (5.7) that

$$(5.10) \qquad x_s = \sum_{k=1}^{m+n} B_k b_k^s \qquad s = m+n, m+n+1, \ldots$$

Amongst those roots whose coefficients in (5.10) are not zero there will be one (at least) whose modulus is not exceeded by that of any other. Let its modulus be b: then for large s, (5.10) will reduce to the form

$$(5.11) \qquad x_s = b^{-s}\{B'_1 + B'_2(-1)^s + \sum_{k=3}^{k} B'_k \cos(s\theta_k + \phi_k) + o(1)\}$$

Since no x_s can be negative

$$(5.12) \qquad B'_1 > 0$$

and b itself must be one root of (5.3). But since all the coefficients of non-zero powers of z in the expansion of $z^{-m}g(z)$ are positive, there can then be no other root of modulus b than b itself. Hence (5.11) may be reduced to

$$(5.13) \qquad x_s = b^{-s}\{B'_1 + o(1)\}$$

It follows from (5.8), (5.12) and (5.13) that $|b| < 1$ and hence

$$(5.14) \qquad x_s = b_1^{-s}\{B_1 + o(1)\}$$

where b_1 is the real root lying between 0 and 1. (5.14) expresses the fact that the Pareto curve for the equilibrium distribution is asymptotic to a straight line.

It is always possible to find the coefficients B_k in the exact solution (5.9), (5.10) by finding b_1 and the roots $b_2 \ldots b_m$ of modulus less than b_1 and fitting B_1 to B_k so as to satisfy (5.6).

It is perhaps of some interest to state that by a suitable choice of the elements $p'_{rs}(t)$ in the top left-hand corner of the matrix illustrated in Diagram 3, which were left with arbitrary values, it is always possible to arrange that all the terms except the first vanish in the expansion for x_s so that in this case Pareto's law is exactly obeyed throughout the whole income scale as in the example of Section 3. A variety of such suitable choices is available, but the result has so little practical relevance that it may be left to the curious reader to verify it if he so wishes.

It is now convenient to relax our assumption (5.2) so that the distribution $p_{ru}(t)$ need only conform to the standard form p_u for high incomes. We replace (5.2) by

$$(5.15) \qquad p_{ru}(t) = p_u \text{ defined for } u = -n, 1-n, \ldots, m \quad \text{if } u + r \geqslant m + w$$

where w is a non-negative integer

$$\sum_{u=-n}^{m} p_u = 1 \text{ and no } p_u \text{ is negative.}$$

The $p_{ru}(t)$ thus freed from the restriction (5.3) are those for which $m - r \leqslant u \leqslant m + w - r$ and $-n \leqslant u \leqslant m$, and these may be left free, apart from the usual requirements that no $p_{ru}(t)$ is negative, all $p_{r1}(t)$, $p_{r0}(t)$ and $p_{r-1}(t)$ are positive and the survival assumption (5.2).

In the case $n = 1$ $m = 2$ $w = 2$, the effect of these assumptions on the appearance of the matrix $p'_{rs}(t)$ is shown below.

$r =$	0	1	2	3	4	5	6
$s =$							
0	$p'_{00}(t)$	$p'_{01}(t)$	$p'_{02}(t)$	0	0	0	0
1	$p'_{10}(t)$	$p'_{11}(t)$	$p'_{12}(t)$	$p'_{13}(t)$	0	0	0
2	0	$p'_{21}(t)$	$p'_{22}(t)$	$p'_{23}(t)$	p_2	0	0
3	0	0	$p'_{32}(t)$	$p'_{33}(t)$	p_1	p_2	0
4	0	0	0	$p'_{43}(t)$	p_0	p_1	p_2
5	0	0	0	0	p_{-1}	p_0	p_1
6	0	0	0	0	0	p_{-1}	p_0

DIAGRAM 4

The reader may compare this diagram with the figures of Table II.

The effect of the change in our model on the solution is in principle not very great. As before, we find $b_1 \ldots b_m$ those m roots of (5.3) which have modulus less than unity and we try solutions of the form (5.9) for $x_w x_{w+1} \ldots$

But we can no longer expect $x_0 x_1 \ldots x_{w-1}$ to conform to the rule (5.9), and we need w more equations to determine these w further unknowns. These equations are provided by extending the equations (5.6) to cover $s = 0, 1, \ldots, m + w - 1$.

Subject to these modifications, the solution is exactly the same as before, and again the Pareto curve for high incomes must be asymptotic to a straight line.

The extension of these results to the case where there are two tails in a generalised form of the example discussed in Section 4 above involves no difficulty in principle.

§ 6. A Model Making Allowance for Some Effects of Age- and Occupation-Structure

An obvious objection to a theory based on the constancy over time of the movement matrices $p_{rs}(t)$ is the fact that age and death play such an important part in determining the changes in an income. In this section we shall modify our assumptions so as to go some way towards meeting this difficulty.

Our method will be to suppose that our population is divided between C "colonies," and that income-receivers can migrate from one colony to another, the prospects of change of income varying from colony to colony. When we wish to discuss the effect of age on income distribution, the "colonies" will represent age-groups: if the width of the age-groups exceeds one "year," then an income attached to one individual may either remain in that age-group or pass on to the next age-group above, or if the individual dies, pass with an appropriate reduction in size to the age-group containing the heir.

But the method could be used also to study the effects on income distribution of the tendency for families to remain in the same occupation: for this purpose we would make the "colonies" represent occupations. As in occupations, the income prospects in some colonies would be better than in others: most incomes would remain in one colony, but there would again be some movement between colonies.

We shall find that provided within each colony the $p_{ru}(t)$ functions have a form independent of the income range, for all large incomes the asymptotic approach of the Pareto curve to a straight line will be preserved under these far more general assumptions.

We now set down formally the notation for our model modified to include colonies.

Let $$ {}_{cd}p'_{rs}(t) = {}_{cd}p_{rs-r}(t) $$

denote the proportion of the incomes in range R_r in colony c in year t which move into range R_s in colony d in year $t + 1$.

If ${}_{c}x_r(t)$ denotes the number of incomes in range R_r in colony c in year t, then by definition

$$ (6.1) \qquad {}_{d}x_s(t+1) = \sum_{c-1}^{C} \sum_{r-0}^{\infty} {}_{cd}p'_{rs}(t)\,{}_{c}x_s(t) $$

and if an equilibrium distribution ${}_dx_s$ exists it must satisfy the condition

(6.2) $$ {}_dx_s = \sum_{c=1}^{C} \sum_{r=0}^{\infty} {}_{cd}p'_{rs}(t) {}_cx_r $$

for every $d = 1, 2, \ldots C$ and $s = 0, 1, 2, \ldots$

The assumptions for our model can now be set down in a form closely analogous to those of the simpler model of Section 5, We assume

(6.3) $$ {}_{cd}p_{rs}(t) = {}_{cd}p'_{r,\, r+s}(t) = 0 $$

when $s > m_d$ or $s < -n_d$

where m_d and n_d are positive integers $d = 1, 2, \ldots C$

(6.4) $$ {}_{cd}p_{ru}(t) = {}_{cd}p'_{r,\, r+u}(t) = {}_{cd}p_u \geqslant 0 $$

if $s = r + u \geqslant m_d + w_d$

and $m_d \geqslant u \geqslant -n_d$

where the w_d are non-negative integers for $d = 1, 2, \ldots C$, and the ${}_{cd}p_u$ are constants satisfying for each $c = 1, 2, \ldots C$, the survival condition

(6.5) $$ \sum_{d=1}^{c} \sum_{u=-n_d}^{m_d} {}_{cd}p_u = 1 $$

It is convenient at this stage to introduce the notion of the accessibility of one income range R_s in one colony C_d from the income range R_r in colony C_c.

Range R_s in C_d will be called accessible in one step from range R_r in C_c if ${}_{cd}p'_{rs}(t)$ is positive : it will be called accessible in two steps from R_r in C_c if it is accessible in one step from any range in any colony which itself is accessible from R_r in C_c in one step. In general, the definition may be extended one by one to any larger number of steps, by always defining R_s in C_d to be accessible in n steps from R_r in C_c, if it is accessible in one step from any range in any colony, which itself is accessible from R_r in C_c in $(n - 1)$ steps.

Finally, R_s in C_d will be termed accessible from R_r in C_c if for any n it is thus accessible in n steps.

We now make the further assumption

(6.6) Each range in any colony is accessible from each range in every colony.

The purpose of this assumption is to ensure that the equilibrium income distribution is unique.

The survival postulate now takes the form

(6.7) $$ \sum_{d=1}^{c} \sum_{u=-n_d}^{m_d} {}_{cd}p_{ru}(t) = 1 $$

and we require one further postulate in order to rule out solutions involving periodic fluctuations from one distribution to another. This postulate may take the form that

(6.8) There is some pair of ranges R_s in C_d and R_r in C_c and some integer n such that R_s in C_d is accessible from R_r in C_c both in n steps and in $(n + i)$ steps.

The effect of these assumptions on the matrices ${}_{cd}p'_{rs}(t)$ may be illustrated by a numerical example with $C = 3$. In this example, the three colonies represent the young, middle-age and old-age groups 20–35, 35–50 and 50–65 years, and the unit of time during which the matrix ${}_{cd}p'_{rs}(t)$ operates once is taken as fifteen calendar years. It is supposed that all the young survive to middle-age, but half the middle-aged die and their incomes pass with suitable reduction to young heirs in the next period, while the other half reappear as the old in the next period, then to die and transmit their incomes to the young in the following period.

We arrange $m_1 = 0 \quad m_2 = 1 \quad m_3 = 1 \quad C = 3$

$w_1 = 1 \quad w_2 = 0 \quad w_3 = 0$

and choose the following nine matrices ${}_{cd}p'_{rs}(t)$ for $c = 1, 2, 3$ and $d = 1, 2, 3$.

We put identically equal to zero those five of the matrices for which either $c = d$ or $c = 1$ and $d = 3$, or $c = 3$ and $d = 2$. The other four matrices we choose as follows :

$$
{}_{12}p'_{rs}(t) = \begin{array}{lllll}
0\cdot1 & 0\cdot9 & 0 & 0 & \text{etc.} \\
0 & 0\cdot1 & 0\cdot9 & 0 & \text{etc.} \\
0 & 0 & 0\cdot1 & 0\cdot9 & \text{etc.} \\
0 & 0 & 0 & 0\cdot1 & \text{etc.} \\
 & \text{etc.} & & &
\end{array}
\qquad
{}_{23}p'_{rs}(t) = \begin{array}{lllll}
0\cdot1 & 0\cdot4 & 0 & 0 & \text{etc.} \\
0 & 0\cdot1 & 0\cdot4 & 0 & \text{etc.} \\
0 & 0 & 0\cdot1 & 0\cdot4 & \text{etc.} \\
0 & 0 & 0 & 0\cdot1 & \text{etc.} \\
 & \text{etc.} & & &
\end{array}
$$

$$
{}_{21}p'_{rs} = \begin{array}{lllll}
0\cdot5 & 0 & 0 & 0 & \text{etc.} \\
0\cdot4 & 0\cdot1 & 0 & 0 & \text{etc.} \\
0\cdot2 & 0\cdot2 & 0\cdot1 & 0 & \text{etc.} \\
0 & 0\cdot2 & 0\cdot2 & 0\cdot1 & \text{etc.} \\
0 & 0 & 0\cdot2 & 0\cdot2 & \text{etc.} \\
 & \text{etc.} & & &
\end{array}
\qquad
{}_{31}p'_{rs} = \begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & \text{etc.} \\
0\cdot9 & 0\cdot1 & 0 & 0 & 0 & \text{etc.} \\
0\cdot8 & 0\cdot1 & 0\cdot1 & 0 & 0 & \text{etc.} \\
0\cdot5 & 0\cdot3 & 0\cdot1 & 0\cdot1 & 0 & \text{etc.} \\
0 & 0\cdot5 & 0\cdot3 & 0\cdot1 & 0\cdot1 & \text{etc.} \\
0 & 0 & 0\cdot5 & 0\cdot3 & 0\cdot1 & \text{etc.} \\
 & & \text{etc.} & & &
\end{array}
$$

DIAGRAM 5

It may be noted that R_0 in C_1 is accessible from R_0 in C_2 both in one step and in two steps. Thus the matrices satisfy the postulate (6.8).

In any generalised model of this type we shall have C sets of equilibrium equations to be satisfied by the equilibrium distributions ${}_dx_s$, namely

$$
\text{(6.9)} \qquad {}_dx_s = \sum_{c=1}^{C} \sum_{r=0}^{\infty} {}_{cd}p'_{rs}(t)\,{}_cx_r \qquad d = 1, 2, \ldots C
$$

For $s > m_d + w_d$, these conditions become

$$(6.10) \qquad {}_d x_s = \sum_{c=1}^{c} \sum_{u=-n_d}^{m_d} {}_{cd}p_u \, {}_d x_{s-u} \qquad d = 1, 2, \ldots C$$

We are thus led to investigate the C simultaneous equations

$$(6.11) \qquad A_d = \sum_{c=1}^{c} \sum_{u=-n_d}^{m_d} {}_{cd}p_u A_c z^{-u} \qquad d = 1, 2, \ldots C$$

which we may write again as

$$(6.12) \qquad \sum_{c=1}^{c} P_{cd}(z) A_c = 0 \qquad d = 1, 2, \ldots C$$

where

$$(6.13) \qquad P_{cd}(z) = \left\{ \sum_{u=-n_d}^{m_d} {}_{cd}p_u z^{-u} - 1 \right\} \text{ if } c = d$$

$$= \sum_{u=-n_d}^{m_d} {}_{cd}p_u z^{-u} \qquad \text{if } C \neq d$$

Elimination of the coefficients A_c leads to

$$(6.14) \qquad \text{Det.}|P_{cd}(z)| = G(z), \text{ say} = 0$$

This function $G(z)$ can be expressed in the form

$$(6.15) \qquad G(z) = \sum_{u=-n}^{m} p_u z^{-u} \qquad m = \sum_{c=1}^{C} m_c \;\; n = \sum_{c=1}^{c} n_c$$

and plays a similar role in the theory to the function $g(z)$ in earlier sections. In particular, if we postulate

$$(6.16) \qquad G'(1) = -\sum_{u=-n}^{m} u p_u > 0$$

it can be proved by an application of Foster's theorem that the process is non-dissipative and that a unique equilibrium distribution exists.

We can again prove by the methods of Section 5 that for large s, where x_s is the equilibrium distribution

$$(6.17) \qquad x_s = b_1{}^s\{B_1 + o(1)\}$$

where b_1 is a real positive root of (6.14).

Thus again the Pareto curve is asymptotic to a straight line in the region of high incomes.

The procedure for finding an exact solution is the following. First find b_1 the largest real positive root and $b_2 \ldots b_m$ the roots of lesser modulus. Let B_{ce} denote the co-factor of $P_{c1}(b_e)$ in the determinant $G(z)$. Then

$$(6.18) \qquad \sum_{c=1}^{C} B_{ce} P_{c1}(b_e) = G(b_e) = 0 \qquad e = 1, 2 \ldots m$$

and (6.12) will clearly be satisfied, provided

$$(6.19) \qquad A_c = \sum_{e=1}^{m} B_{ce}\lambda_e, \text{ where } \lambda_e \text{ are any numbers.}$$

We have still to determine not only the m values λ_e but also those values ${}_dx_s$ for which $s < w_d$: the number of these is given by

$$(6.20) \qquad w = \sum_{d=1}^{c} w_d$$

We have thus $(w + m)$ unknowns to find. To discover them we have the $w + m$ equations determining the equilibrium of those ${}_cx_s$ for which $s < m_c + w_c$: namely (6.2) for these values of c and s. These equations are not linearly independent, and only determine the ratios between the $w + m$ unknowns. Apart from a scale factor, this is sufficient to determine all the ${}_cx_s$, and the scale factor can then be found if we know the original total population.

It can be proved exactly as before that the distribution

$$(6.21) \qquad x_s = \sum_{c=1}^{C} {}_cx_s$$

so determined is the unique equilibrium distribution and that the term involving $b_1{}^s$ will dominate the whole value of x_s for sufficiently large s. Thus the Pareto curve for sufficiently large incomes will preserve its property of being asymptotic to a straight line, despite the greater generalisation introduced in this model.

As in simpler models, we could remove the restriction that there is a minimum income range R_0 and elaborate the model so as to secure an equilibrium distribution with two Pareto tails, one for the poor and one for the rich. The exposition is tedious, and since our conclusions would not be substantially affected, this refinement is eschewed.

§ 7. Numerical Example Involving the Effect of Age-structure on Income Distribution

The general method of solution indicated in the last section can be made much clearer by applying it to the numerical example described above in Diagram 4.

In this example, $C = 3$, $m = 2$, and the determinant of $P_{cd}(z)$ is

$$(7.1) \quad G(z) = \begin{vmatrix} -1 & 0{\cdot}9z^{-1} + 0{\cdot}1 & 0 \\ 0{\cdot}1 + 0{\cdot}2z + 0{\cdot}2z^2 & -1 & 0{\cdot}4z^{-1} + 0{\cdot}1 \\ 0{\cdot}1 + 0{\cdot}1z + 0{\cdot}3z^2 + 0{\cdot}5z^3 & 0 & -1 \end{vmatrix}$$

$$= 0{\cdot}36z^{-2} + 0{\cdot}139z^{-1} - 0{\cdot}688 + 0{\cdot}420z + 0{\cdot}088z^2 + 0{\cdot}005z^3$$

By differentiation

$$(7.2) \qquad G'(1) = 0{\cdot}4 > 0$$

so that, since $m = 2$, the equation $G(z) = 0$ must have just two roots of modulus less than unity. These may be found by Horner's method

$$(7.3) \qquad \text{as } b_1 = 0{\cdot}4563136 \qquad b_2 = -\,0{\cdot}1453788$$

The six co-factors B_{ce} of $p_{c1}(b_e)$ in the determinants $D(b_e)$ are

$$(7.4) \quad B_{1e} = 1 \quad B_{2e} = 0{\cdot}9b_e^{-1} + 0{\cdot}1$$
$$B_{3e} = (0{\cdot}9b_e^{-1} + 0{\cdot}1)(0{\cdot}4b_e^{-1} + 0{\cdot}1) \quad e = 1, 2$$

and their numerical values are

$$(7.5) \quad \begin{array}{lll} B_{11} = 1 & B_{21} = 2{\cdot}072\ 327 & B_{31} = 2{\cdot}023\ 814 \\ B_{12} = 1 & B_{22} = -\,6{\cdot}090\ 724 & B_{32} = -\,6{\cdot}310\ 8 \end{array}$$

We now put

$$(7.6) \quad \begin{cases} {}_1x_s = \lambda_1 b_1{}^s + \lambda_2 b_2{}^s & \text{for } s = 1, 2, 3 \ldots \\ {}_2x_s = \lambda_1 B_{21} b_1{}^s + \lambda_2 B_{22} b_2{}^s & \text{for } s = 0, 1, 2 \ldots \\ {}_3x_s = \lambda_1 B_{31} b_1{}^s + \lambda_2 B_{32} b_2{}^s & \text{for } s = 0, 1, 2 \ldots \end{cases}$$

and we still have to determine λ_1, λ_2 and ${}_1x_0$.

We have available for this purpose the three equations

$$(7.7) \quad \begin{cases} {}_1x_0 = 0{\cdot}5\,{}_2x_0 + 0{\cdot}4\,{}_2x_1 + 0{\cdot}2\,{}_2x_2 + {}_3x_0 + 0{\cdot}9\,{}_3x_1 \\ \qquad\qquad + 0{\cdot}8\,{}_3x_2 + 0{\cdot}5\,{}_3x_3 \\ {}_2x_0 = 0{\cdot}1\,{}_1x_0 \\ {}_3x_0 = 0{\cdot}1\,{}_2x_0 \end{cases}$$

Fortunately, any two equations contain all the fresh information provided by the three, and we accordingly take the two simple ones and rewrite them as

$$(7.8) \quad \begin{cases} \lambda_1 B_{21} + \lambda_2 B_{22} = 0{\cdot}1\,{}_1x_0 \\ \lambda_1 B_{31} + \lambda_2 B_{32} = 0{\cdot}1(\lambda_1 B_{21} + \lambda_{22} B_{22}) \end{cases}$$

If we leave aside the scale factor we may arbitrarily put $\lambda_1 =$ 10,000 and, substituting our numerical values for $B_{21}B_{22}B_{31}$ and B_{32}, we then find from the second equation that

$$(7.9) \qquad \lambda_2 = 3185{\cdot}939$$

and then from the first equation that

$$(7.10) \qquad {}_1x_0 = 13185{\cdot}939$$

Using our equations (7·6) for the other ${}_cx_s$ we may now obtain the numerical values of as many ${}_cx_s$ as we please. Here are the first few values, with λ_1 put equal to 10000 :

TABLE III

	Young.	Middle-aged.	Old.	Total.	Income range.
S	${}_1X_s$	${}_2X_s$	${}_3X_s$		
0	13,186	1,319	132	14,637	£125– £200
1	4,100	12,277	1,755	18,132	£200– £316
2	2,150	3,995	5,301	11,356	£316– £500
3	940	2,028	1·765	4,734	£500– £800
4	435	890	901	2,225	£800–£1250
5	198	411	397	1,006	£1250–£2000
6	90	187	183	460	£2000–£3160
7 and over	75	157	153	385	£3160–
TOTAL :	21,174	21,174	10,687	52,935	

The income scale put in on the extreme right assumes that the minimum income is £125 and that the upper limit of each income range is nearly 60% greater than the lower limit.

It will be noted that although the equilibrium distributions of incomes for young, middle-aged and old are very different for small incomes, yet already at income levels of £1,250 and over each is rapidly approaching a Pareto distribution with

$$(7.11) \qquad \alpha = -\frac{\log_{10} b_1}{\log_{10} 1{\cdot}58} = -5 \log_{10} b_1 = 1{\cdot}7041$$

This is well brought out by Chart 1 which shows for each age group and for all ages the following cumulative totals plotted on the double logarithmic paper.

TABLE IV

Income level.	Number of incomes exceeding this level.			
	Young.	Middle-aged.	Old.	Total.
£125 . . .	21,174	21,174	10,587	52,935
£200 . . .	7,988	19,855	10,455	38,298
£316 . . .	3,888	7,578	8,700	20,166
£500 . . .	1,738	3,674	3,399	8,810
£800 . . .	798	1,645	1,634	4,076
£1250 . . .	363	755	733	1,851
£2000 . . .	165	344	336	845
£3160 . . .	75	157	153	385

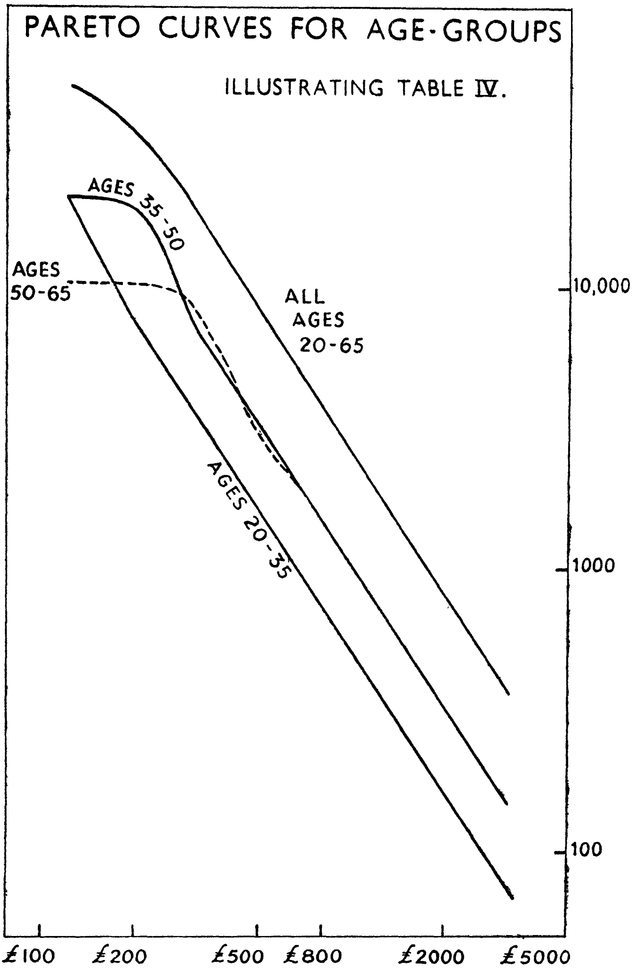
We may read off from the chart that the median incomes in the three age-groups differ considerably: they are £175, £280 and £410 approximately. Yet for incomes over £500 it is clear from Chart 1 that the *proportionate* distributions are almost identical for the three age-groups.[1]

[1] For similar charts of actual distributions see Lydall [5].

§ 8. Numerical Example Involving the Effect of Occupational Structure on Income Distribution

In our last model we made the assumption that every income range in every colony was accessible from every other. If we relax this assumption to state that within a certain group G of

Chart I



pairs (rc) every R_r in every colony c is accessible from every other, then our results will still hold, provided that the initial distribution was confined to this group.

If we maintain our other assumptions, then the effective new possibility introduced by this relaxation is the inclusion of colonies where there is an upper limit to the possible income.

This is useful when our colonies represent groups of occupations. Thus, let our society be composed of persons classified according to their main sources of income into :

1. Unskilled labour.
2. Semi-skilled work; skilled work and clerical.
3. Salaries and professional.
4. Profits, land, property.

Then we might assume that there was an effective ceiling on the incomes of classes 1 and 2, and 3. If for simplicity we ignore the complication of age considered in our last example and take broad income groups so that group R_r extends from 2^{r-3} thousand pounds to 2^{r-2} we might set up the following model within our relaxed assumption (see Diagram 6).

It is assumed that no one can obtain incomes higher than £500 in occupation 1, or higher than £1000 in occupation 2. It will be seen that ${}_{33}p'_{rs}$, ${}_{34}p'_{rs}$, ${}_{43}p'_{rs}$ and ${}_{44}p'_{rs}$ assume repetitive forms when r becomes large: we might therefore expect to find the Pareto curves for occupations 3 and 4 to be asymptotic to straight lines.

It will be noted that ${}_{43}p'_{r2} > 0$ for all $r > 0$ so that (6.3) is not satisfied for finite n_3 when $c = 4$ $d = 3$ $s = 2$. Nevertheless, a solution can be found by the following method, which is similar to that of Section 7.

The matrix $P_{rs}(z)$ for this example is only concerned with the values 3 and 4 of r and s. It is

$$(8.1)\quad P_{rs}(z) = \begin{vmatrix} 0{\cdot}020z^{-1} - 0{\cdot}218 + 0{\cdot}180z & 0{\cdot}000139z^{-1} + 0{\cdot}01786 \\ 0 & 0{\cdot}030z^{-1} - 0{\cdot}115 + 0{\cdot}075z \end{vmatrix}$$

The survival condition (6·5) is not, however, satisfied, since 1% of those in each income group $s > 3$, in C_4, escape each year into income group R_2 of C_3. Consequently, writing

$$\begin{aligned}(8.2)\quad G(z) &= \mathrm{Det}|P_{rs}(z)| \\ &= 0{\cdot}00060z^{-2} + 0{\cdot}0088z^{-1} + 0{\cdot}03197 - 0{\cdot}03705z \\ &\quad + 0{\cdot}01350z^2\end{aligned}$$

we find that $G(1) \neq 0$ and that the non-dissipative condition (6.15) is not satisfied.

Nevertheless, two real positive roots of $G(z) = 0$ of modulus less than unity can be found and the usual method of solution proves adequate. The two roots are

$$(8.3)\qquad b_1 = 0{\cdot}333\ 333 \qquad b_2 = 0{\cdot}100\ 000$$

$s =$ / $r =$	${}_{11}p'_{rs}$		${}_{12}p'_{rs}$		
	0	1	0	1	2
0	0·856	0·144	0	0	0
1	0·250	0·593	0·057	0·110	0

$s =$ / $r =$	${}_{21}p'_{rs}$		${}_{22}p'_{rs}$			${}_{23}p'_{rs}$				${}_{24}p'_{rs}$			
	1	2	0	1	2	0	1	2	2	0	1	2	2
0	0·050	0·100	0·750	0·100	0	0	0	0	0	0	0	0	0
1	0·050	0·220	0·075	0·640	0·025	0	0·010	0	0	0	0	0	0
2	0	0·075	0	0·125	0·500	0	0	0·250	0	0	0	0·050	

$s =$ / $r =$	${}_{32}p'_{rs}$				${}_{33}p'_{rs}$						
	0	1	2	2	0	1	2	3	4	5	6
0	0·482	0	0	0	0·132	0·1875	0	0	0	0	0
1	0·02657	0·075	0	0	0·025	0·500	0·250	0	0	0	0
2	0	0	0·05	0	0·00486	0·055	0·79956	0·020	0	0	0
3	—	—	—	—	0	0	0·180	0·782	0·020	0	0
4	—	—	—	—	0	0	0	0·180	0·782	0·020	0
5	—	—	—	—	0	0	0	0	01·80	0·782	0·020
6	—	—	—	—	0	0	0	0	0	0·180	0·782
						etc.			etc.		

$s =$ / $r =$	${}_{34}p'_{rs}$							
	0	1	2	3	4	5	6	
0	0·100	0·120	0	0	0	0	0	
1	0·0025	0·050	0·07075	0	0	0	0	etc.
2	0	0·010	0·060	0·000139	0	0	0	
3	0	0	0	0·01786	0·000139	0	0	
4	0	0	0	0	0·01786	0·000139	0	
5	0	0	0	0	0	0·01786	0·000139	
				etc.				

$s =$ / $r =$	${}_{43}p'_{rs}$			
	0	1	2	>2
0	0·500	0·050	0	0
1	0·050	0·300	0·150	0
2	0	0·0081	0·010	0
3	0	0	0·010	0
4	0	0	0·010	0
	0	0	0·010	0

$s =$ / $r =$	${}_{44}p'_{rs}$							
	0	1	2	3	4	5	6	
0	0·250	0·200	0	0	0	0	0	
1	0·100	0·370	0·030	0	0	0	0	etc.
2	0	0·010	0·8519	0·030	0	0		
3	0	0	0·075	0·885	0·030	0	0	etc.
4	0	0	0	0·075	0·885	0·030	0	
5	0	0	0	0	0·075	0·885	0·030	
			etc.		etc.			etc.

${}_{13}p'_{rs} = {}_{14}p'_{rs} = 0$ for all $r.s.$
${}_{31}p'_{rs} = {}_{41}p'_{rs} = {}_{42}p'_{rs} = 0$ for all $r.s.$

Diagram 6

and we calculate

$$(8.4)\qquad \begin{aligned} B_{3e} &= 0{\cdot}30b_e^{-1} - 0{\cdot}115 + 0{\cdot}075b_e \\ B_{4e} &= -\,0{\cdot}000139b_e^{-1} - 0{\cdot}01786 \qquad e = 1, 2 \end{aligned}$$

to give

$$B_{31} = 0 \quad B_{32} = 0{\cdot}1925 \quad B_{41} = -\,0{\cdot}18278 \quad B_{42} = -\,0{\cdot}01925$$

Hence we are led to try solutions of the form

$$(8.5)\qquad \begin{aligned} {}_3x_s &= 0{\cdot}1925\lambda_2 10^{-s} \qquad s = 2, 3, 4 \ldots \\ {}_4x_s &= -\,0{\cdot}18278\lambda_1 3^{-s} - 0{\cdot}01925\lambda_2 10^{-s} \quad s = 3, 4, 5 \ldots \end{aligned}$$

It will slightly simplify the algebra to fix the scale arbitrarily at this stage by choosing λ_2 so that

$$(8.6)\qquad \begin{aligned} {}_3x_s &= 10^{8-s} \qquad s = 2, 3, 4 \ldots \\ {}_4x_s &= \lambda'_1 3^{-s} - 10^{7-s} \qquad s = 3, 4 \ldots \end{aligned}$$

λ'_1 is still undetermined and we also have still to find ${}_4x_0$, ${}_4x_1$, ${}_3x_0$, ${}_3x_1$, ${}_2x_0$, ${}_2x_1$, ${}_2x_2$, ${}_1x_0$ and ${}_1x_1$.

To find these ten constants we have the eleven equations associated with the equilibrium of ${}_1x_0$, ${}_1x_1$, ${}_2x_0$, ${}_2x_1$, ${}_2x_2$, ${}_3x_0$, ${}_3x_1$, ${}_3x_2$, ${}_4x_0$, ${}_4x_1$ and ${}_4x_2$, namely

$$(8.6)\qquad \left\{\begin{aligned} {}_1x_0 &= 0{\cdot}856\,{}_1x_0 + 0{\cdot}250\,{}_1x_1 + 0{\cdot}050\,{}_2x_0 + 0{\cdot}050\,{}_0x_1 \\ {}_1x_1 &= 0{\cdot}144\,{}_1x_0 + 0{\cdot}593\,{}_1x_1 + 0{\cdot}100\,{}_2x_0 \\ &\qquad + 0{\cdot}200\,{}_2x_1 + 0{\cdot}075\,{}_2x_2 \\ &\qquad \text{(seven equations omitted)} \\ {}_4x_1 &= 0{\cdot}120\,{}_3x_0 + 0{\cdot}050\,{}_3x_1 + 10^4 + 0{\cdot}200\,{}_4x_0 \\ &\qquad + 0{\cdot}370\,{}_4x_1 + 0{\cdot}010\{\lambda'_1 3^{-2} - 10^5\} \\ \lambda' 3^{-2} - 10^5 &= 0{\cdot}7075\,{}_3x_1 + 6000 + 0{\cdot}030\,{}_4x_1 \\ &\qquad + 0{\cdot}0852\{\lambda'_1 3^{-2} - 10^5\} + 0{\cdot}075\{\lambda'_1 3^{-3} - 10^4\} \end{aligned}\right.$$

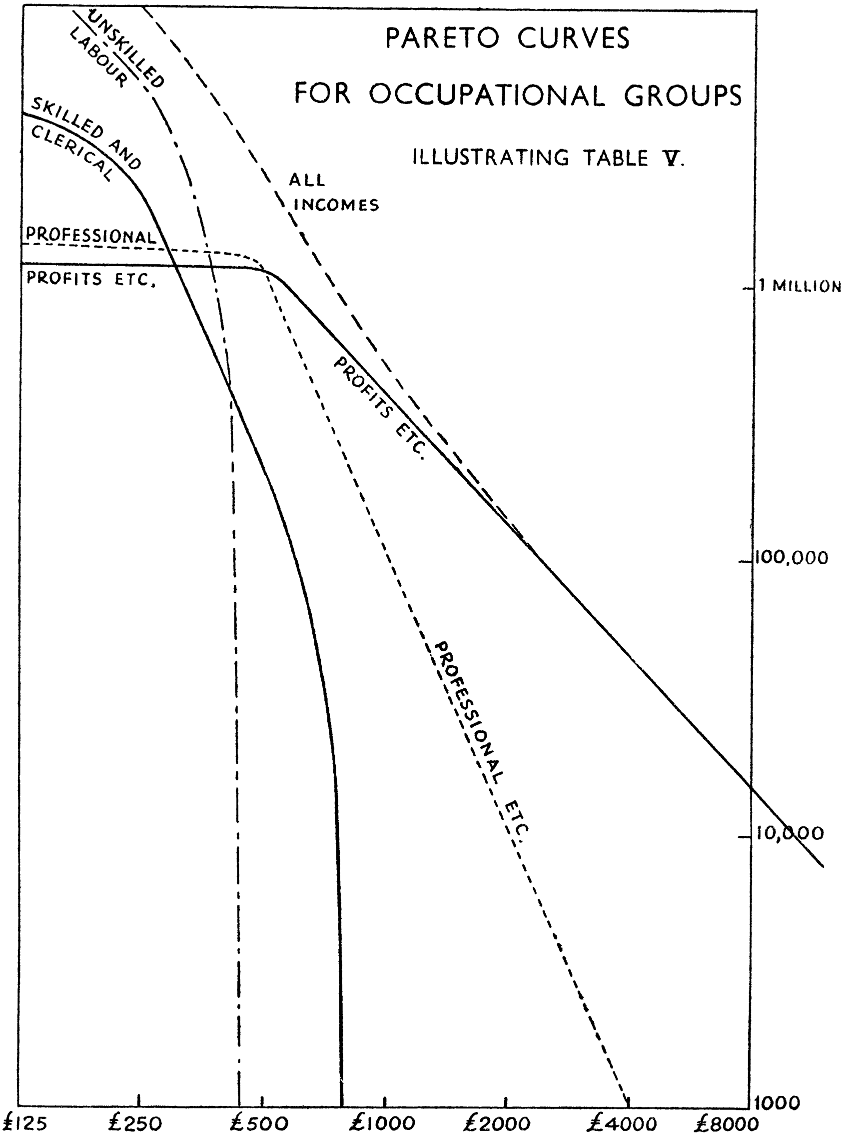
Only ten of these equations provide independent information: any one is implied by the other ten. The solution of ten simultaneous equations is no mean undertaking, but in this example the solution can be found as

$$(8.7)\qquad \left\{\begin{aligned} \lambda'_1 &= 7{\cdot}29\,.\,10^6 &\qquad {}_1x_0 &= 10^7 \\ {}_1x_1 &= 5\,.\,10^6 & {}_2x_0 &= 2{\cdot}10^6 \\ {}_2x_1 &= 1{\cdot}8\,.\,10^6 & {}_2x_2 &= 2.10^5 \\ {}_3x_0 &= 2.10^4 & {}_3x_1 &= 2{\cdot}10^5 \\ {}_4x_0 &= 10^4 & {}_4x_1 &= 5{\cdot}10^4 \end{aligned}\right.$$

The values of ${}_3x_s$ and ${}_4x_s$ for $s > 1$ may now be found from the equations

$$(8.8)\qquad \begin{aligned} {}_3x_s &= 10^{3-s} \\ {}_4x_s &= 3^{6-s}10^4 - 10^{7-s} \end{aligned}$$

Chart II


PARETO CURVES
FOR OCCUPATIONAL GROUPS
ILLUSTRATING TABLE V.
UNSKILLED LABOUR
SKILLED AND CLERICAL
ALL INCOMES
PROFESSIONAL
PROFITS ETC.
PROFITS ETC.
PROFESSIONAL ETC.
1 MILLION
100,000
10,000
1000
£125
£250
£500
£1000
£2000
£4000
£8000

From the complete set of ${}_c x_s$ thus obtained, it is a quick matter to compile the following table showing for each occupation the number with incomes exceeding various levels.

TABLE V

Income level.	Number of incomes exceeding this level.				
	Unskilled labourers.	Skilled workers and clerical.	Salaried and professional.	Profits, land, etc.	Total.
£125 . .	15,000,000	4,000,000	1,331,111	1,163,889	21,495,000
£250 . .	5,000,000	2,000,000	1,311,111	1,153,889	9,465,000
£500 . .	0	200,000	1,111,111	1,103,889	2,415,000
£1,000 . .	0	0	111,111	393,889	505,000
£2,000 . .	0	0	11,111	133,889	145,000
£4,000 . .	0	0	1,111	44,889	46,000
£8,000 . .	0	0	111	14,989	15,100
£16,000 .	0	0	11	4,999	5,010
£32,000 .	0	0	1	1,667	1,668

The corresponding Pareto curves are shown in Chart 2. An interesting feature of this solution is that the slopes of the Pareto lines for occupations 3 and 4 are not the same, that for the professional and salaried classes being much steeper than that for those whose income came from profits, land, etc. It is the latter distribution, of course, which determines the slope of the Pareto line for all incomes.[1]

§ 9. A Model Generating a Distribution in which Pareto's Law is not Obeyed

The above examples are probably sufficient to illustrate the theory that the approximate observance of Pareto's law which has so often been remarked upon is not an illusion or coincidence, but has its explanation in a similarity at different high income-levels of the prospects of given proportionate changes of income.

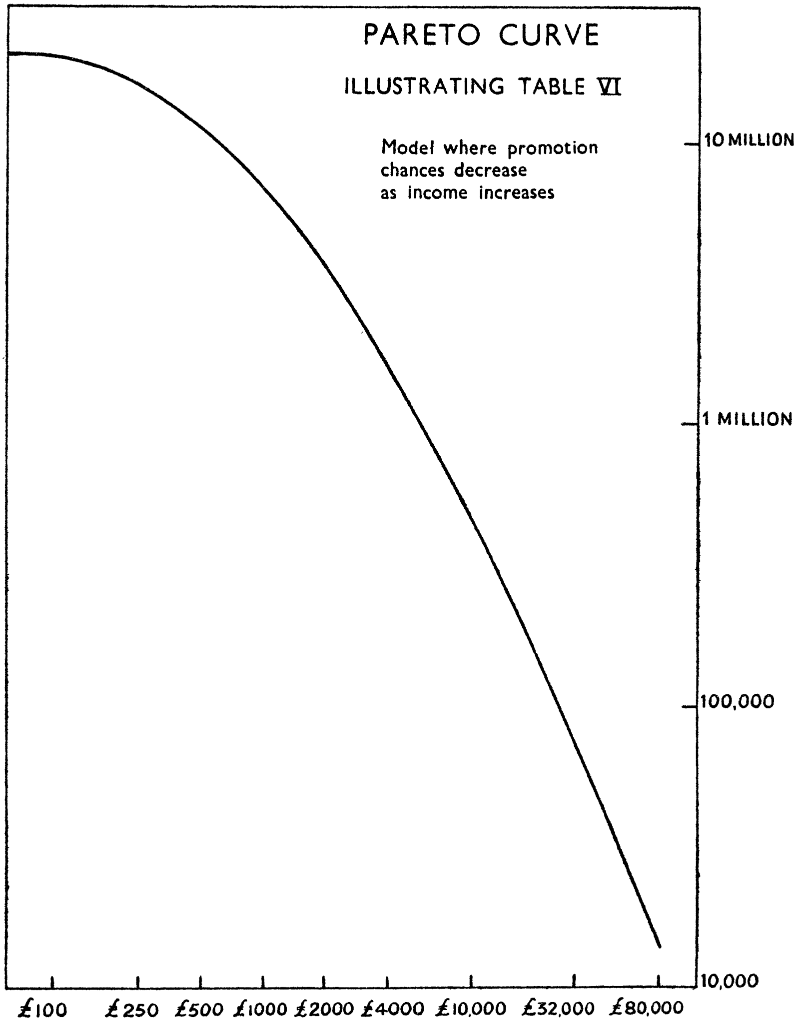
They can do little more than illustrate the theory, since they are built on the artificial simplifying assumption that these prospects of change remain constant through time at each income level. It will be readily appreciated that any model catering for prospects which are not constant through time is much more complicated and the results obtainable are far less clear: the investigation of such models must form the subject of another article than this. The importance of such change in prospects has already been hinted at in the suggestion that changes in the

[1] For similar charts of actual distributions see Lydall [5].

income distribution affect the influence described by the matrices $p'_{rs}(t)$ just as much as the influence affect the incomes.

Another gap in our discussion so far has been any consideration

CHART III



of models which do *not* lead to a Pareto distribution. There is a noticeable tendency recently for the Pareto curves of the United Kingdom and other countries to curve very slightly downwards at the tail, and it would be interesting to have a model illustrating how this could come about. The explanation is probably that the prospects of increasing are proportionately less rosy nowadays for the very large incomes than for the large incomes. This is not necessarily because the owners of vast incomes are any less abstemious and accumulative than their forerunners used to be, but may be because income tax and death duties are now at a level which makes the piling up of huge fortunes a more gradual and less-rewarding undertaking.

A very simple model will suffice to illustrate the effect on the Pareto curve that would result from a progressive worsening of the chance of (say) doubling the income as one considered larger and larger incomes.

We shall suppose that R_0, R_1, R_2 . . . are the income ranges £62 10*s*.–£125, £125–£250, £250–£500, etc., etc. We shall suppose that the chance of going *down* one range is the same in all ranges (except R_0), and is 10%. In R_0 the chance of going *up* one range is 30%, but in R_1 it is only 15%, in R_2 it is 10%, and in general in R_r it is only $\frac{30\%}{r+1}$.

The equilibrium condition for this model is

$$(9.1) \qquad x_r = \frac{0{\cdot}3x_{r-1}}{r} + \left\{0{\cdot}9 - \frac{0{\cdot}3}{r+1}\right\} x_r + 0{\cdot}1x_{r+1}$$

It can easily be checked that the solution is $r = 1, 2 \ldots$

$$(9.2) \qquad x_r = \frac{3^r x_0}{r} \quad r = 1, 2, \ldots$$

The cumulative distribution corresponding to $x_0 = 10^6$ is given in Table VI, and the corresponding Pareto curve is shown in Chart 3. As one would expect, the steadily worsening prospects of income-promotion that are found as consideration passes up the income scale are reflected in a continuous downward curvature of the Pareto curve.

TABLE VI

Income.	Number of incomes exceeding x.
x	(Fx) (thousands)
£62 10*s*.	20,085
£125	19,085
£250	16,085
£500	11,585
£1,000	7,085
£2,000	3,710
£4,000	1,685
£8,000	673
£16,000	239
£32,000	76·4
£64,000	22·1
£135,000	5·9

This example has been chosen so as to provide a very simple solution. In general, it will be difficult to obtain the equilibrium solutions in models where the promotion prospects, as reflected in the matrix $p_{ru}(t)$, vary throughout the income scale.

§ 10. A Model Illustrating the Reaction of Changes in the Income Distribution on the Matrix Depicting the Influences Shaping that Distribution

In conclusion, a warning must be given that although the models discussed above throw some light on the reasons why an approximate obedience of Pareto's law is so often found in actual income distributions, they do not throw much light on the mechanism determining the actual values observed for Pareto's a. It is tempting to draw conclusions from the fact that in equilibrium

$$a = \frac{-\log b_1}{\log (1 + h)} \tag{10.1}$$

where b_1 is the positive root of the equilibrium equation

$$\sum_{u=-n}^{m} p_u z^{-u} = 1 \tag{10.2}$$

and h is the proportionate width of each income range. One might suppose that one had only to estimate the p_u corresponding to various economic situations in order to deduce the slopes of the Pareto lines in the consequent income distributions. But it would be just as sensible to guess at the consequent income distributions and deduce how much the p_u functions would have to be modified before that equilibrium was reached.

The point may be illustrated by a final model. Suppose that initially the income distribution is

$$x_s = 2^{23-s} \tag{10.3}$$

where R_s is the range $2^{s/2}$ £100 to $2^{s+1/2}$ £100; then the total income will be approximately 2^{25} £100.

Now suppose that the real income of the community is held constant and that the $p_{ru}(t)$ are given by

$$\begin{aligned} p_{ru}(t) &= p_u, \quad r = 1, 2, 3 \ldots \\ p_{01}(t) &= p_1 \quad p_{00}(t) = 1 - p_1 \end{aligned} \tag{10.4}$$

where $p_{-1} = 0{\cdot}3$ $p_0 = 0{\cdot}5$ $p_1 = 0{\cdot}2$ other $p_u = 0$.

The corresponding value of b_1 is $2/3$, so that the equilibrium distribution must be

$$x^s = \frac{2^{24}}{3}\left(\frac{2}{3}\right)^s \tag{10.5}$$

which involve a total income of approximately $\frac{2^{24}}{1-\theta}$ £100 where

$$(10.6) \qquad \theta = \frac{\log 2}{\log 9 - \log 4} = 0{\cdot}8547$$

The numerical value of this total income is about $3{\cdot}44 \times 2^{25}$ £100. Thus money income will have to rise in ratio 3.44, so that if total real income is to remain constant, prices must rise in ratio 3.44, so that the real income of those in R_0 will be only about 30% of what it was originally in R_0.

Now suppose that originally an income at the lower end of R_0 represented the subsistence level. Then directly prices tend to rise some policy must be adopted to subsidise those in R_0: let us suppose that prices are subsidised at the expense of prospects of increasing income. More precisely

$$(10.7) \quad p_{-1} = 0{\cdot}3 + T \quad p_0 = 0{\cdot}5 \quad p_1 = 0{\cdot}2 - T \quad p_{01}(t) = p_1 p_{00}(t) = 1 - p_1$$

where T is continually adjusted so as to keep prices and total money income stable.

It is intuitively plausible that this policy will lead eventually to an equilibrium distribution.

The corresponding value of b_1 is given by

$$(10.8) \quad (0{\cdot}3 + T)b_1^2 + 0{\cdot}5b_1 + (0{\cdot}2 - T) = b_1$$

whence since $b_1 \neq 1$

$$(10.9) \qquad b_1 = \frac{0{\cdot}2 - T}{0{\cdot}3 + T}$$

The total income will be $\frac{2^{24}}{1-\theta}$ £100

where

$$\theta = \frac{+\log 2}{-2\log b_1},$$

and in order that this should be equal to the initial total income of 2^{25} £100 so as to obviate the need for higher prices we need $\theta = \frac{1}{2}$ and hence $b_1 = \frac{1}{2}$ and hence $T = 0{\cdot}1$.

When we work out the equilibrium distribution we find, of course, that it is simply the initial distribution unchanged. Hence, it is truer, under our extreme simplifying assumptions, that the initial distribution determined the $p_{ru}(t)$ than that the $p_{ru}(t)$ determined the equilibrium distribution.

Had we allowed some increase in total real income, a lower value of T would, of course, have been necessary, and had we allowed for a continuous expansion of real income an altogether more advanced model with a shifting $p_{ru}(t)$ function for low values of r would have been required.

These illustrations remind one of the impossibility of drawing any simple conclusions about the effect on Pareto's α of various redistributive policies by *merely* considering the effects of these policies on the functions $p_{ru}(t)$ representing the prospects of increase of income.

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