# An Empirical study on Markowitz Modern Portfolio Theory

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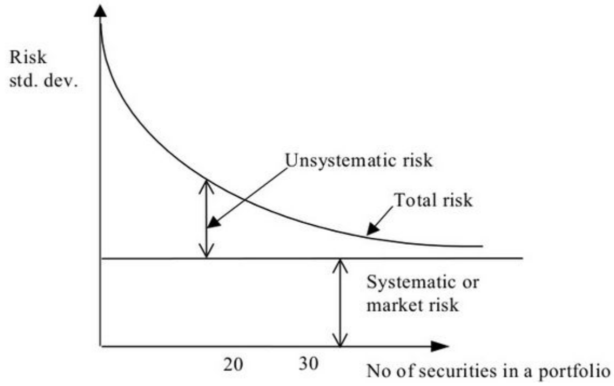
This paper is an empirical study on Markowitz's work of Modern Portfolio Theory. The paper traces out the the Efficient Frontier of a set of portfolio with basic Markowitz model, then introduce the short-selling and free-risk asset into the model to adjust the result. We try to figure out how these two factors affect efficient frontier.
Key words: Mean-Variance analysis, short-selling, free-risk asset,

## I. INTRODUCTION

Since the introduction of Markowitz (1952) Mean-Variance (MV) model, variance has become the most common risk measure in portfolio optimization. However, this model relies strictly on the assumptions that there are no short selling and risk-free assets. Nonetheless, in practice these two assumptions do not hold.

The article is composed of THREE sections. The first part explains the basic Markowiitz Model and data used to empirically achieve the study objectives. Then the results are adjusted by adding short selling and free-risk asset into the model. The final section summarizes the observed results and describes their implications.

**Figure 1. Two types of risks in market.**




## II. DATA AND METHODS

The basic model was developed by Markowitz using mean-variance analysis. He states that the expected return and variance of returns of a portfolio are the whole criteria for portfolio selection. Markowitz model relies on the following assumptions;

- Investors seek to maximize the expected return of total wealth.
- All investors have the same expected single period investment horizon.
- All investors are risk-averse, that is they will only accept a higher risk if they are compensated with a higher expected return.
- Investors base their investment decisions on the expected return and risk.
- All markets are perfectly efficient.

### Risk Measures

There are two types of risks (Figure 1), one is systematic risk which affects the overall market, it cannot be mitigated through diversification, the other one is unsystematic risk, which can be reduced through diversification. So when we mainly talk about unsystematic risk here.

## Variance

Markowitz model uses variance as the measure of risk while mean return as the expected return. The expected risk of a combination of assets is often lower than the individual asset, because the assets always have correlation below than 1, or even better, below than 0, which means that a shock may cause some of the assets' risk increase, while lead the others' returns decrease. So the diversification reduce the variability of returns around the expected return. So in order to have a diversified portfolio it is important that the assets in a portfolio do not have a perfect positive correlation. Markowitz was the first to examine the role and impact of diversification and put forward the modern portfolio theory. The idea of this theory is to find the weight of assets that will minimize the portfolio risk for a given rate of return.

Assume portfolio weights are：

$$\omega_t = (\omega_1, \omega_2, \cdots, \omega_n)^T \qquad (1)$$

s.t. $\omega_1 + \omega_2 + \cdots + \omega_n = 1$ (2)

Expected returns are：

$$r = \omega_1 r_1 + \omega_2 r_2 + \cdots + \omega_n r_n \qquad (3)$$

Variance (Standard Deviation) of returns are:

$$\sigma_p^2 = E[(\sum_{i=1}^{n} \omega_i r_i - \sum_{i=1}^{n} \omega_i E[r_i])^2]$$

$$= \sum_{i,j=1}^{n} \omega_i \omega_j E[(r_i - E[r_i])(r_j - E[r_j])]$$

$$= \sum_{i,j=1}^{n} V_{i,j} \omega_i \omega_j \qquad (4)$$

where $r_i$ is the mean return of asset i; $\omega_i$ is the weight of asset i and $\omega_j$ is the weight of asset j; $V_{i,j}$ is the covariance between assets i and j, that is:

$$V = \begin{vmatrix} Var(r_1) & Cov(r_1, r_2) & \dots & Cov(r_1, r_n) \\ Cov(r_2, r_1) & Var(r_2) & \dots & Cov(r_2, r_n) \\ \dots & \dots & \dots & \dots \\ Cov(r_n, r_1) & Cov(r_n, r_2) & \dots & Var(r_n) \end{vmatrix} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{1n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{vmatrix} \qquad (5)$$

## DATA

The data to investigate consists of 12 stocks which have relatively higher returns. We utilize the monthly close price data chosen for a period from January 2014 to April 2015. As it is hard to predict the returns in the future, so we use the average return in the last 16 months instead. All the data come from Google Finance. The mean returns and covariances are shown below.(Table 1. and Table 2.)

**Table 1. Mean return and variance of each stock**

| Portfolio | Average return% | risk |
|---|---|---|
| IRCP | 4.96 | 0.087721132 |
| APPL | 4.11 | 0.052699075 |
| LEVYU | 4.39 | 0.127440516 |
| LEVY | 3.31 | 0.085976235 |
| NIKE | 2.33 | 0.051217275 |
| PBCP | 2.10 | 0.066363293 |
| QURE | 8.49 | 0.274116132 |

| | | |
|---|---|---|
| SCMP | 6.11 | 0.158430564 |
| MTDR | 2.76 | 0.139361612 |
| CHR | 3.80 | 0.085369942 |
| NHTC | 13.93 | 0.213445932 |
| TZF | 3.57 | 0.193861817 |

### Model

The objective of Markowitz model is to find the weight of assets that will minimize the portfolio variance for a given expected return. This model is a quadratic programming model. The mathematical model is as follows:

$$\begin{cases} \text{Min} \quad \sigma_p^2 = \omega^T V \omega = \sum_{i,j=1}^{n} V_{ij} \omega_i \omega_j & (6) \\ \text{S.t.} \quad \omega_1 + \omega_2 + \cdots + \omega_n = 1 & (7) \\ u_\omega = \omega^T u = \omega_1 u_1 + \omega_2 u_2 + \cdots + \omega_n u_n = \bar{u} & (8) \\ \omega_i \geq 0, j = 1,2,\cdots,n & (9) \end{cases}$$

I trace out the efficient frontier by varying the desired return and produce many efficient portfolios and plotting in the risk/return space. Figure 2 shows the efficient frontier, the set of portfolios with the maximum return for a given risk or the minimum risk given a return, for 12 stocks without short selling and free-risk assets. Point A represents the portfolio with minimum risk for the given return, while point B is the portfolio with maximum returns for the given risk.

In the process, we found some unusual monthly returns, especially in March 2015 (Table 3), the returns are extraordinary big, maybe due to the shock government policy. So we need to remove that outlier month to get a more accurate result . Figure 3 shows the new efficient frontier of portfolios with adjusted data. By comparing the efficient frontiers with and without the outlier month ( Figure 4), we can see that the modified efficient frontier moves downward. And obviously, the modified efficient frontier has a lower maximum expected returns.

# III ADJUSTED MODEL

### Short selling

As we mentioned before, the basic Markowitz Model isn't include short selling and risk-free assets, so in order to enhance the practical usefulness of the Markowitz Optimal Portfoliio. Short selling is a regulated type of market transaction which means selling shares of a stock that are borrowed in expectation of a fall in the stock's price. When the price declines, the investor buys an equivalent number of shares of the same stock at the new lower price and then returns the stock that was borrowed to the lender. By introducing short selling into the basic model, we get the adjusted efficient frontier in Figure 5.

The short -selling does actually expand the set of optimal portfolios, investor could hold portfolios with higher returns, but also with much higher risks. In fact, If the number of short sales is unrestricted, then by a continuous short selling, investor could generate an infinite expected return. As the Markowitz Model assumes that all the investors are risk aversion, the short selling benefit more to the investor with low degree of risk aversion, because they are more likely to hold the portfolios with higher risks. While for the investors with high degree of risk aversion, the introducing of short selling almost change nothing.

## Risk-free assets

So far, we assume that all the portfolios on the efficient set are risky. Alternatively, investors could easily combine a risky portfolio with an investment in a risk-free security. In this way, investors are no longer restricted to their initial wealth when investing in risky portfolios.

**Table 2. Covariance form**

| COV% | IRCP | APPL | LEVYU | LEVY | NIKE | PBCP | QURE | SCMP | MTDR | CHR | NHTC | TZF |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| IRCP | 0.72 | -0.02 | 0.67 | 0.43 | -0.01 | 0.30 | 0.27 | -0.23 | 0.04 | 0.03 | 0.34 | 0.28 |
| APPL | -0.02 | 0.26 | -0.26 | -0.18 | 0.04 | -0.14 | 0.1 | 0.16 | -0.20 | 0.15 | 0.34 | -0.23 |
| LEVYU | 0.67 | -0.26 | 1.52 | 1.00 | -0.01 | 0.68 | 0.17 | -0.17 | 0.15 | 0.07 | 0.72 | -0.49 |
| LEVY | 0.43 | -0.18 | 1.00 | 0.69 | -0.02 | 0.43 | 0.28 | -0.06 | 0.13 | 0.05 | 0.38 | -0.30 |
| NIKE | -0.01 | 0.04 | -0.01 | -0.02 | 0.24 | -0.05 | -0.16 | 0.18 | -0.23 | -0.01 | 0.05 | 0.33 |
| PBCP | 0.30 | -0.14 | 0.68 | 0.43 | -0.05 | 0.41 | 0.27 | -0.13 | 0.16 | 0.03 | 0.25 | -0.27 |
| QURE | 0.27 | 0.12 | 0.17 | 0.28 | -0.16 | 0.27 | 6.98 | 1.70 | 0.47 | 0.66 | -0.42 | -0.30 |
| SCMP | -0.23 | 0.16 | -0.17 | -0.06 | 0.18 | -0.13 | 1.70 | 2.34 | -0.50 | 0.18 | -0.09 | -1.27 |
| MTDR | 0.04 | -0.20 | 0.15 | 0.13 | -0.23 | 0.16 | 0.47 | -0.50 | 1.81 | -0.06 | -0.58 | 0.51 |
| CHR | 0.03 | 0.15 | 0.07 | 0.05 | -0.01 | 0.03 | 0.66 | 0.18 | -0.06 | 0.68 | -0.11 | -0.49 |
| NHTC | 0.34 | 0.34 | 0.72 | 0.38 | 0.05 | 0.25 | -0.42 | -0.09 | -0.58 | -0.11 | 4.25 | -1.10 |
| TZF | 0.28 | -0.23 | -0.49 | -0.30 | 0.33 | -0.27 | -0.30 | -1.27 | 0.51 | -0.49 | -1.10 | 3.51 |

**Figure 2. Estimated Efficient Frontier of Portfolios consisting 12 stocks**

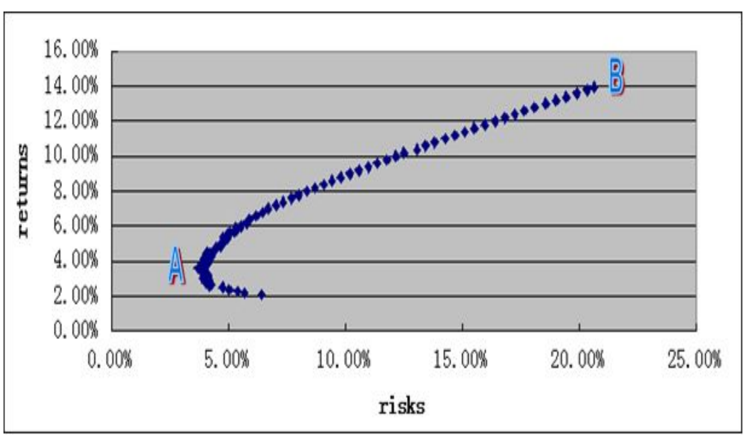



**Figure 3. Modified Efficient Frontier of Portfolio without returns in March 2015**

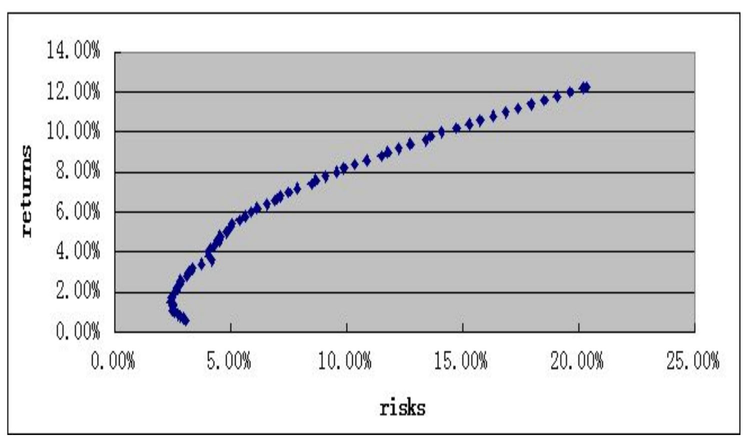


**Table 3. Returns in March 2015**

| return % | IRCP | APPL | LEVYU | LEVY | NIKE | PBCP | QURE | SCMP | MTDR | CHR | NHTC | TZF |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 201503 | 21.43 | -3.14 | 48.32 | 31.50 | 3.31 | 23.38 | 4.65 | 1.17 | 1.20 | 5.84 | 37.38 | -18.8 |

**Figure 4. Comparison between original and modified efficient frontier**

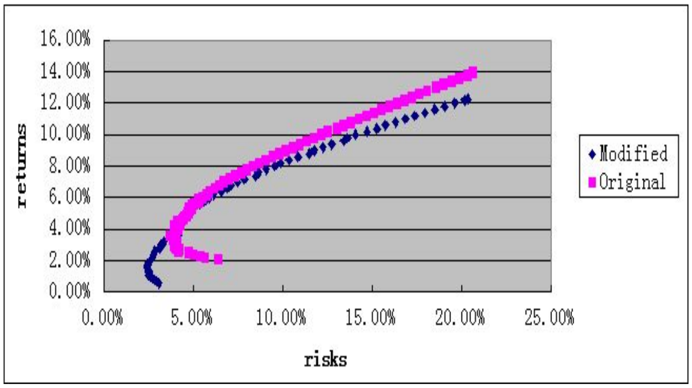
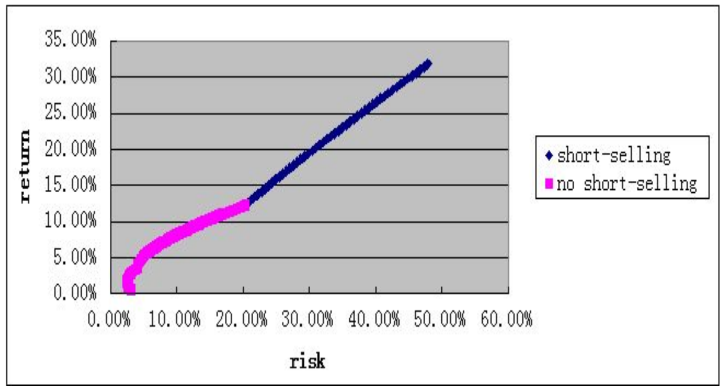



**Figure 5. Efficient Frontier of Portfolio with short selling**




Before we continue, one thing to note about the risk-free asset is that since it is risk free, it has no correlation to other securities. thus it provides no diverisfication.

Assume portfolio P consists of a risk-free asset and (n-1) risky assets. Then:

$$r_p = \omega_F r_F + (1-\omega_F) r_R \tag{10}$$

$$\sigma_p = (1-\omega_F)\sigma_F \tag{11}$$

where $r_F$ and $r_R$ are returns of risk-free asset and risky portfolio respectively; $\omega_F$ and $\omega_R$ are the weights of risk-free asset and risky portfolio respectively; $\sigma_R$ is the variance of the risky portfolio. and $r_P$, $\sigma_P$ are returns and variance of the whole portfolio respectively.

Combining equation (10) and equation (11) gives:

$$r_p = r_F + \frac{r_R - r_F}{\sigma_R}\sigma_P \tag{12}$$

Equation (12) is a straight line going through the point (0, $r_F$ )and the slope is $\frac{r_R - r_F}{\sigma_R}$ .The portfolio giving the $r_R$ and $\sigma_R$ that maximizes $\frac{r_R - r_F}{\sigma_R}$ is called the optimal portfolio of risky assets. So the problem is:

$$\text{Max} \quad \frac{r_R - r_F}{\sigma_R} = \sum_{i=1}^{n} r_i \omega_i \tag{13}$$

S.t. $$\sigma_R^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i,j}\omega_i\omega_j \quad (14)$$

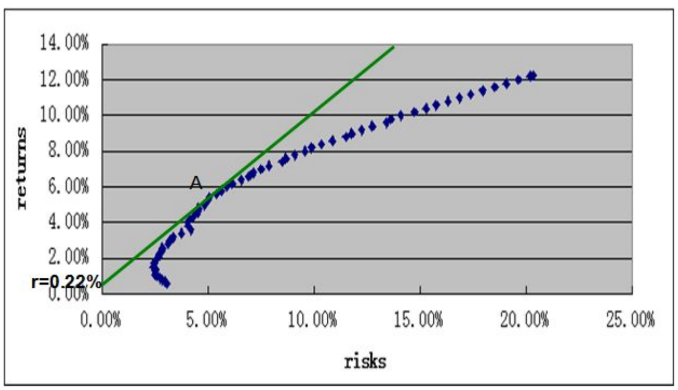
$$\sum_{i=1}^{n}\omega_i = 1 \quad (15)$$

As shown in Figure 6, we could see that it generate a line which is called the capital market line(CML). CML intersects with y-axis at 0.22%, which is the monthly risk free rate. The point of tangency (A) between CML and Efficient Frontier is called super-efficient portfolio. Investors will always invest at risky portfolio at point A, regardless their tolerance of risk. Investors can place themselves anywhere on this line in two ways:

-To leverage their position by borrowing free-risk asset and invest them in risky portfolio A.

-To deleverage their position by selling some portfolio A, then invest more in risk-free asset.

**Figure 6. Optimal Portfolio of Risky Assets**




# IV CONCLUSION

The Markowitz Model relies on the assumption that there is no short selling and risk-free assets, which make it useless in practice. So we introduce these two important factors into Markowitz efficient frontier model to make it much more useful. The first modified model that considering short selling expand the set of efficient portfolios, which enable investors hold a portfolio with expected returns, as well as risks. Then, the final model with risk-free asset determines the optimal risky portfolio for investors in all degree of risk aversion. So if an investor wants to use Markowitz Model to choose a ideal portfolio, it is essential to do some complementary calculations by taking short selling and risk-free asset into account.

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