Methods of Pricing American Options

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1 Introduction

This report accompanies the talk given on 4/18/2016 about the methods of pricing American Options. To begin, we start with a broad introduction of options. First, we must differentiate between two of the most popular types of options; American and European Options. At their core, American and European options are similar in many ways, yet in regard to options pricing, we must focus on their differences. Though several differences exist, the most pertinent is the difference in the right to exercise. With a European option, the holder of the option only has the right to exercise the option at the predetermined maturity date. On the other hand, with an American option, the holder of the option has the right to exercise the option on any day on or before the maturity date. This added degree of complexity makes pricing American options notably more difficult to price. Nest, we must differentiate between what is known as a long position, and a short position. A long position means that you have the right but not the obligation to exercise the option, whereas the short position means that if the person who holds the long position exercises the option, you are obligated to exercise it. Next, we differentiate between the call and put option. A call option allows the holder to buy an asset, and the put option allows the holder to sell an asset. The main purpose of options began as a way to hedge risk; if a farmer wanted to sell wheat at a certain price without worrying about price fluctuations, he'd buy a put option. Nowadays, the option is also used to speculate on prices. Financial firms are extremely good at putting together intricate option portfolios that make them money. The last topic to touch on is that of arbitrage. Arbitrage at its most basic definition, is when two things are priced differently in a market. Anyone lucky enough to spot this disconnect can make a profit out of nothing, buying at the lower price and turning around immediately and selling at a higher price. In many ways, arbitrage can be seen as a form of inefficiency in the market, and is therefore assumed to not exist. This assumption is actually quite a good one, because the computers of today's traders level out arbitrage opportunities by capitalizing on them before they have the chance to get too large. Arbitrage in today's economy may be decimal places of cents, but over millions and millions of assets, it can still turn a profit. With this very basic introduction to options and the assumptions used in modeling, we may now move on to the actual models.

2 The Binomial Options Pricing Model

The Binomial Options Pricing Model, or BOPM, is a simple discrete-time model to value options. It's based mostly on probability and algebra, and is therefore a very easy model to begin with. In the BOPM framework, "the value of the call can be interpreted as the expectation of its discounted future value in a risk-neutral world" (Cox et al 1979). So what does that mean? Economic jargon can at times become confusing, so let's unpack that phrase. We should begin with what a 'discounted future value' is. In economics, and more specifically behavioral economics, we see that humans generally prefer 'payoffs' in a sooner time period. That is to say, they 'discount' the benefit they receive from a future payoff simply because it is not in the present. A classic example of this is when offering test subjects $20 today or $200 in 5 years, most people will choose the $20 today. The dollar amounts and the time to the future payoff are variable, but the pattern of preferences remains. Now that we understand a 'discounted future value', let's move to 'a risk-neutral world'. A risk-neutral world is not one in which there exists no risk, rather it is one in which the consumer attitude towards risk is neutral. That is to say, nobody is very risk-loving or very risk-averse. This assumption is not true, and is known to be false, yet it is required for the BOPM framework. This is a limitation of the Binomial Options Pricing Model. Putting that all together, we can begin to see how to BOPM works.A stock S can either go up or down in the next time period. The probability of it going up is q, and the probability it goes down in 1-q. if it goes up, the new value of S is uS, and if it goes down, the new value is dS, where u¿r¿d. r is the risk free rate, and u and d are just constants. The binomial tree can easily be extrapolated to a long call option, where instead of uS and ds, the payoffs are $max[0, uS - K]$ and $max[0, dS - K]$ for the up and down states, respectively. The payoffs are maximum functions, because as this is a long call, the holder will not exercise the option if the payoff is below 0. The next step is to go from a stock binomial tree to a portfolio, comprised of both stocks and risk-free bonds.This portfolio is denoted as $\Delta S + B$ where Δ is the amount of stock held, and B is the dollar amount held in risk-free bonds. Similarly, the portfolio can either go up or down, and that is shown by the up state payoff being $\Delta uS + rB$, and the down state being $\Delta dS + rB$. Since Δ and B are endogenous, that is they are for us as the holder of the portfolio to choose, we select Δ and B such that $\Delta uS + rB = Cu$ and $\Delta uS + rB = Cd$. Solving for Δ and B, we get $\Delta = (C_u - C_d)/((u - d)S)$ and $B = (uC_d - dC_u)/(u - d)r$. This combination of Δ and B is known as the "hedging portfolio" which means it should be properly covered from risk. Using these values for Δ and B, plugging into $C = \Delta S + B$, we get $C = (C_u - C_d)/((u - d)) + (uC_d - dC_u)/(u - d)r$, factoring and simplifying, we get $C = [((r - d)/(u - d))C_u + ((u - r)/(u - d))C_d] * (1/r)$. To simplify further, we define $p \equiv ((r - d)/(u - d))$ and $(1 - p) \equiv ((u - r)/(u - d))$. Thus, we get the final value of the call as

$$C = [pC_u + (1 - p)C_d] * (1/r)$$

if $C > (S-K)$; otherwise $C = S-K$. It should be noted, that this is only a one time-step case. Obviously, more time steps are possible. They follow the same logic as the one time-step case; however, they get very complicated very quickly. For example, the final value for only a two-step case is given by

$$C = (p^2 max[0, u^2(S-K)] + 2p(1-p)max[0, du(S-K)] + (1-p)^2 max[0, d^2(S-K)]] * (1/r^2)$$

It can therefore be seen then, that these valuations for 60 time-steps for example, would become impossibly complicated. That's where quadratic approximations and simulations come in.

3 Quadratic Approximations/Simulations

Quadratic Approximations are not something we will focus on for too long, but they were the first attempt to tackle American options pricing from a heavily analytical sense. When Black, Scholes, and Merton had their seminal work on pricing European options published, there was a rush to find a way to link that work to American options, as an equation to price American options as the BSM equation prices European options would be met with equal levels of fame. Quadratic approximations were a method of that rush that has so far proven unfruitful, but what they did do is they directed economists to simulation methods, like that of Longstaff and Schwartz, known unsurprisingly as the Longstaff-Schwartz method. The Longstaff-Schwartz method (LSM) works by using least squares regression to estimate the expected payoff of an option. The method is to regress discounted future option cash flows on the current price of the underlier associated with in-the-money sample paths. The idea behind the method of only regressing on in-the-money sample paths (an in the money sample path is one that would pay off a profit if exercised today) is that the holder will not exercise an option unless it will bring them some profit, so only in the money paths matter. In this simulation the risk-neutral market model used is the stochastic differential equation $dS = rSdt + \sigma SdZ$, where r is the riskless rate,σ is the exposure matrix; both are constants, and Z follows a standard Brownian motion. It is important to note that simulation methods such as the method of Longstaff and Schwartz are accurate, but only under assumptions known to be wrong, such as risk neutrality.