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PAPER

# Effects of time-delays in the dynamics of social contagions

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## Abstract

Time-delays are pervasive in such real-world complex networks as social contagions and biological systems, and they radically alter the evolution of the dynamic processes in networks. We use a non-Markovian spreading threshold model to study the effects of time-delays on social contagions. Using extensive numerical simulations and theoretical analyses we find that relatively long time-delays induce a microtransition in the evolution of a fraction of recovered individuals, i.e., the fraction of recovered individuals versus time exhibits multiple phase transitions. The microtransition is sharper and more obvious when high-degree individuals have a higher probability of experiencing time-delays, and the microtransition is obscure when the time-delay distribution reaches heterogeneity. We use an edge-based compartmental theory to analyze our research and find that the theoretical results agree well with our numerical simulation results.

## 1. Introduction

Face-to-face contact, Twitter, and Facebook are important channels for such social contagions as the diffusion of public opinion, the adoption of innovations or new behavior, new product market share, and brand awareness [1–5]. All of these systems can be modeled as complex networks [6–8] in which nodes are individuals and links are contacts between them. To investigate the diffusion mechanisms and to predict and control the dynamics of social contagions, many successful models, both Markovian [9, 10] and non-Markovian [11–13], have been proposed. Unlike such biological contagions as epidemic spreading, the social reinforcement effect is ubiquitous in social contagions [9, 13–22]. For example, if an individual has nine friends adopt a new behavior and then a tenth friend adopts it, the individual will take the actions of all ten friends into account when deciding whether to adopt the behavior, since multiple confirmations of the credibility and legitimacy of the behavior are needed. Researchers have found that the social reinforcement effect markedly alters both the type of phase transition that occurs and the resulting final state [9, 13]. The primary factor in these models is the number of individuals who adopt a behavior, which is an order parameter of a phase transition. Watts found that when social reinforcement is introduced into a steady state system the change in average degree causes both continuous and discontinuous phase transitions [9]. Wang *et al* found that the initial seed size (degree exponent) has a critical point below (above) which the order parameter increases discontinuously with the transmission probability, otherwise it increases continuously [13].

Time-delays are pervasive in such real-world complex networks as social contagions and biological systems [23–25]. When individuals seek to accomplish tasks they are limited by the available time and energy, and thus time-delays become an issue. Although previous studies have found that time-delays affect both the evolution and the steady state of biological contagions [26, 27] and synchronizations [28–30], we still lack systematic theoretical and numerical simulations that address the effect of time-delays on the dynamics of social contagions, and especially for the temporal transition of the contagion process. One theoretical challenge is dealing with the *non-Markovian characteristic* of the contagion process and its two factors: memory and time-delays. On the one hand, the social reinforcement induces the memory effect, since for a susceptible to adopt they must remember their accumulated information. On the other, time-delays also cause a non-Markovian effect.

To study the effects of time-delays we use a non-Markovian spreading threshold model in which there are time-delays in the adoption process. The probability that an individual has a time-delay correlates with the number of its contacts (its degree). The time-delay quantifies the exposure time needed after the individual receives enough information to adopt the behavior. We develop a generalized edge-based compartmental theory to study the contagion process. Using extensive numerical simulations and theoretical analyses, we find that a relatively long time-delay causes a microtransition in the temporal evolution of the fraction of individuals in the recovered state in which individuals no longer share the behavioral information with neighbors. This magnitude exhibits multiple phase transitions in the social contagion dynamics, but does not affect the final adoption size. The transition is sharper when high degree nodes (hubs) have a higher probability of experiencing time-delays. We also find that heterogeneity in the time-delay distribution causes the microtransition to become obscure. Our theory accurately predicts these phenomena.

The organization of the paper as follows. We introduce the model in section 2 and develop the theory in section 3. We present extensive numerical simulations in section 4 and then present our conclusions.

## 2. Model description

To realistically describe the dynamics of social contagion, we propose a non-Markovian susceptible-exposed-adopted-recovered spreading threshold model inspired by [31–33]. At each time step individuals are in either a susceptible, exposed, adopted, or recovered state. A susceptible individual has not adopted the new behavior because the amount of behavioral information from neighbors is below its adoption threshold. An individual becomes exposed when the received information exceeds its adoption threshold, but because of time-delays it postpones the adoption of the new behavior. In the adopted state an individual has adopted the behavior and shares the behavioral information with neighbors. A recovered individual is no longer interested in the behavior and no longer shares the behavioral information with neighbors.

In our proposed model each individual $x$ has an adoption threshold $T_x$ and a time-delay value $\tau_x$. The adoption threshold reflects the willingness of the individual to adopt the new behavior, i.e., the lower the adoption threshold the greater its willingness. For simplicity, we set all individuals at the same adoption threshold $T$. The time-delays are caused by the distractions experienced by the individual after they have decided to adopt the behavior but prior to their actually adopting it. In real-world systems, the probability that individuals will experience time-delays is correlated with such inherent characteristics as degree. When an individual is assigned a time-delay value $\tau$ it follows a given distribution $G(\tau)$ within the system, and an individual with a degree $k$ has a time-delay with a probability $\pi_k$. To quantify this probability, we use a well-known family of functions [34, 35]

$$\pi_k = \frac{k^\alpha}{\sum_{i=1}^{N} k^\alpha}, \quad -\infty < \alpha < +\infty, \tag{1}$$

where $N$ is the number of individuals in the network. The parameter $\alpha$ is adjustable and measures the correlations between time-delay and degree. When $\alpha = 0$ there are no correlations. When $\alpha > 0$ ($\alpha < 0$) a large (small) degree individual has a higher (lower) probability of having time-delays. Using equation (1) we assign time-delays to a fraction of $f$ individuals, and no time-delays to the remaining individuals (see figure 1(a)).

At the initial stage we assign a random fraction $\rho_0$ of seed individuals to the adopted state and a fraction $1 - \rho_0$ individuals to the susceptible state. At each time step, each adopted individual $v$ transmits the behavioral information to each susceptible neighbor $u$ with a probability $\lambda$. If individual $u$ receives the information, their information level $m$ increases by one, and we disallow any further transmission of information between these two individuals. When $m \geqslant T$, individual $u$ becomes either exposed, if it has being assigned with a time-delay, or adopted, if it has experienced no time-delays. If individual $u$ does have time-delays, adoption occurs after $\tau$ waiting time steps (see figure 1(b)). At each time step an individual in the adopted state can enter the recovered state with a probability $\gamma$. The spreading dynamics reaches a steady state when there are no longer any exposed or adopted individuals. Because memory and time-delays are present in our model, a non-Markovian characteristic appears.

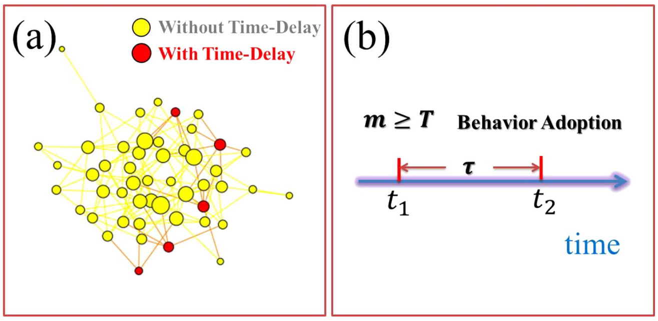


**Figure 1.** An illustration of the spreading threshold model on a social network with $N = 50$ individuals. (a) Assigning 5 individuals with time-delays $\tau$, and the remaining 45 individuals without time-delays. (b) At time step $t_1$, the accumulated pieces of behavior information $m$ of a susceptible individual $u$ is greater than its adoption threshold $T$. If individual $u$ has time-delays, it becomes exposed at time $t_1$, and become adopted at time $t_2$ after waiting $\tau$ time steps. If $u$ has no time-delays, it becomes adopted at time step $t_1$.

## 3. Theoretical analysis

We develop an edge-based compartmental theory [36–40] to analyze the proposed model. In this model, susceptible and exposed individuals are classified non-adopted, i.e., they cannot transmit behavioral information to neighbors and they have not yet adopted the behavior. We denote $S(t)$, $A(t)$, and $R(t)$ the densities of individuals in the non-adopted, adopted, and recovered states, respectively, and $R(t)$ the temporal order parameter of the phase transition [41]. At each time step each individual can be either non-adopted, adopted, or recovered, and thus $S(t) + A(t) + R(t) = 1$.

Because there are time-delays in behavior adoption, the amount of information possessed by individuals adopting the behavior is the sum of their current information and the information accumulated in $\tau$ prior time steps. If individual $u$ has no time-delays, the probability that it will accumulate the required $m$ pieces of information and adopt the behavior by time $t$ is

$$\phi(k, m, t) = (1 - \rho_0)\binom{k}{m}[\theta(t)]^{k-m}[1 - \theta(t)]^m, \tag{2}$$

where $k$ is the degree of $u$, $1 - \rho_0$ the probability that an individual is initially susceptible, and $\theta(t)$ the probability that a randomly chosen edge of individual $v$ has not transmitted the behavioral information to a susceptible neighbor $u$ by time $t$. Here $u$ is in the cavity state, i.e., within time $t$ it can receive information from neighbors but cannot transmit information to neighbors [42]. If an individual $u$ has a time-delay $\tau$, the probability that it accumulates $m$ pieces of information by time $t$ is

$$\varphi(k, m, \tau, t) = (1 - \rho_0)\binom{k}{m}[\theta(t - \tau)]^{k-m}[1 - \theta(t - \tau)]^m. \tag{3}$$

At time $t$ individual $u$ acquires $m$ pieces of information with a probability

$$\chi(k, m, t) = (1 - \ell_k)\, \phi(k, m, t) + \ell_k \sum_{\tau} G(\tau)\varphi(k, m, \tau, t), \tag{4}$$

where $G(\tau)$ is the distribution of time-delays, and $\ell_k$ and $1 - \ell_k$ are the probabilities of $u$ with and without network time-delays, respectively. For the derivation of $\ell_k$ see reference appendix. Individual $u$ remains in the non-adopted state at time $t$ with a probability $s(k, t) = \sum_{m=0}^{T-1} \chi(k, m, t)$. Then the total fraction of individuals in the non-adopted state is

$$S(t) = \sum_k P(k)s(k, t). \tag{5}$$

The expression for $\theta(t)$ can be written $\theta(t) = \xi_S(t) + \xi_A(t) + \xi_R(t)$, where $\xi_S(t)$, $\xi_A(t)$, and $\xi_R(t)$ are the probabilities that a neighbor of $u$ is in the non-adopted, adopted, or recovered states, respectively, and has not transmitted the information to its neighbors by time $t$. If neighbor $v$ of individual $u$ is non-adopted, it cannot transmit the information to $u$, and individual $u$ also cannot transmit the information to individual $v$ because $u$ is in the cavity state. Thus individual $v$ can acquire the information only from neighbors other than $u$. If the degree of individual $v$ is $k'$ and $v$ has no time-delay, the probability that $v$ will at time $t$ acquire $m$ pieces of information is $\phi(k' - 1, m, t)$. If individual $v$ has time-delays, the probability that $v$ will at time $t$ acquire $m$ pieces of information is $\sum_\tau G(\tau)\varphi(k' - 1, m, t)$. Thus individual $v$ obtains $m$ pieces of information with a probability

$$\varepsilon(k', m, t) = (1 - \ell_{k'})\phi(k' - 1, m, t) + \ell_{k'} \sum_{\tau} G(\tau)\varphi(k' - 1, m, t). \tag{6}$$

As a result, individual $v$ remains in a non-adopted state at time $t$ with a probability $\Theta(k', t) = \sum_{m=0}^{T-1} \varepsilon(k', m, t)$. In uncorrelated networks, an edge connects a node with degree $k'$ with a probability $k'P(k')/\langle k\rangle$. Thus

$$\xi_S(t) = \frac{1}{\langle k\rangle} \sum_{k'} k'P(k')\Theta(k', t). \tag{7}$$

The evolution of $\xi_R(t)$ is given by

$$\frac{d\xi_R(t)}{dt} = (1 - \lambda)\gamma\xi_A(t), \tag{8}$$

where $1 - \lambda$ is the probability that the information has not been transmitted through an edge, and $\gamma\xi_A(t)$ indicates that the connected adopted individual has recovered. If an adopted individual transmits the information to susceptible neighbors along an edge, the edge does not meet the definition of $\theta(t)$. Thus the time evolution of $\theta(t)$ is

$$\frac{d\theta(t)}{dt} = -\lambda\xi_A(t). \tag{9}$$

Combining equations (8) and (9) with the initial condition $\theta(0) = 1$ and $\xi_R(0) = 0$, we have $\xi_R(t) = \gamma(1 - \lambda)[1 - \theta(t)]/\lambda$.

Note that non-adopted individuals adopt the behavior and move into the adopted states, and adopted individuals abandon the behavior and become recovered. The time evolution of $A(t)$ is thus

$$\frac{dA(t)}{dt} = -\frac{dS(t)}{dt} - \gamma A(t). \tag{10}$$

Combining equation (10) with equation (5), we have

$$\frac{dS(t)}{dt} = \sum_k P(k) \sum_{m=0}^{T-1} \frac{d\chi(k, m, t)}{dt}. \tag{11}$$

Finally using equation (10), we get the evolution for $R(t)$,

$$\frac{dR(t)}{dt} = \gamma A(t). \tag{12}$$

By numerically integrating equations (5), (10), and (12), we get the order parameter $R(t)$ versus $t$.

## 4. Numerical simulations

In this section we study social contagions on artificial networks that follow the uncorrelated configuration model [43]. We use a degree distribution that follows a power-law, i.e., $P(k) \sim k^{-\gamma_D}$ where $\gamma_D = 3$ is the degree exponent. In our simulations the maximum degree and average degree are set at $k_{\max} \sim \sqrt{N}$ and $\langle k\rangle = 10$, respectively. We use two time-delay distributions, the Dirac delta function $G(\tau) = \delta_{\tau,\langle\tau\rangle}$ and the Gaussian distribution $G(\tau) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\tau - \langle\tau\rangle)^2}{2\sigma^2}\right)$, where $\langle\tau\rangle$ is the average time-delay of individuals and $\sigma$ is the variance.

### 4.1. Time-delays with Dirac delta distribution

Here we use the $G(\tau)$ given by a Dirac delta function with $\langle\tau\rangle$, and study the case of $\alpha = 0$ shown in figure 2. Figure 2(a) shows that time-delays produce temporal microtransitions, i.e., $R(t)$ versus $t$ exhibits multiple transitions, which also exist in percolation [44–46]. When no individuals have time-delays i.e., when $f = 0.0$, the behavior adoption $R(t)$ versus time $t$ grows continuously. Once a finite fraction of individuals have time-delays, $R(t)$ exhibits microtransitions versus $t$. If the behavior can spread, the emergence of microtransitions should fulfill two conditions. (i) A finite fraction of individuals must receive an amount of information that exceeds the adoption threshold at the same time step. (ii) Individuals must experience relatively long time-delays relative to the convergence time of the system without delays. Initially the seeds (adopted individuals) transmit the information to susceptible neighbors, and this causes the amount of information to exceed their adoption threshold and changes their status to 'exposed,' i.e., condition (i) is fulfilled. If these exposed individuals have time-delays, however, they cannot immediately adopt the new behavior and must wait $\tau$ steps. Thus the first microtransition occurs at approximately time $\langle\tau\rangle$. In the successive microtransitions that follow, as $f$ increases more individuals experience time-delays and more microtransitions occur. For a given time $t$, $R(t)$ thus decreases with $f$. Note that, as expected, the final behavior adoption size $R(\infty)$ is not affected by the time-delays because individuals with behavior information levels that exceed their adoption thresholds can adopt the

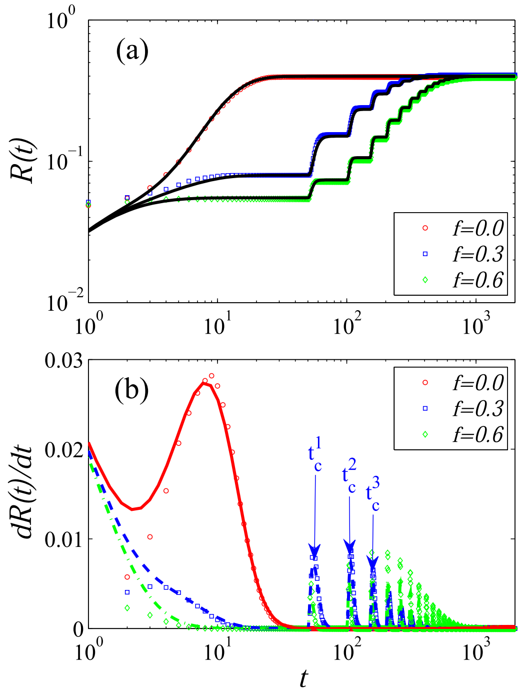


**Figure 2.** Dynamic of social contagions on artificial networks. (a) Fraction of individuals in the recovered state $R(t)$ and (b) the derivative of $R(t)$, i.e., $\mathrm{d}R(t)/\mathrm{d}t$, versus time $t$. Symbols are the simulation results and lines are the corresponding theoretical predictions. In (b), the blue arrows represent the critical points, e.g., $t_c^1$ denotes the critical point of the first microtransition. Other parameters are set to be $N = 10^5$, $\rho_0 = 0.05$, $\gamma_D = 3.0$, $\lambda = 0.5$, $\gamma = 1.0$, $\langle\tau\rangle = 50$, $\alpha = 0$ and $T = 3$, respectively.

behavior when $t \to \infty$. Our edge-based compartmental theory accurately predicts this microtransition phenomenon and time evolution process.

To locate the critical points we use the time evolution of $\mathrm{d}R(t)/\mathrm{d}t$ as a function of $t$ (see figure 2(b)). Previous studies found that $\mathrm{d}R(t)/\mathrm{d}t$ exhibits peaks at critical points [47]. When $f = 0.0$, $\mathrm{d}R(t)/\mathrm{d}t$ has only one peak. The number of peaks increases as $f$ increases because when there are more individuals with time-delays the probability that a microtransition of $R(t)$ occurs increases. Thus we can predict the microtransition by using $\mathrm{d}R(t)/\mathrm{d}t$ to locate the critical points.

Figure 3(a) shows that peak values of $\mathrm{d}R(t_c^i)/\mathrm{d}t$ at peak $i$ versus $f$ exhibit different patterns. In the first peak $\mathrm{d}R(t_c^1)/\mathrm{d}t$ decreases with $f$. At the first critical point, individuals with time-delays adopt the behavior and transmit the information to susceptible neighbors, which causes the susceptible neighbors to become exposed. When $f$ is small, these exposed individuals immediately adopt the behavior, which increases the $\mathrm{d}R(t_c^1)/\mathrm{d}t$ value. For other peaks, i.e., when $i \geqslant 2$, $\mathrm{d}R(t_c^i)/\mathrm{d}t$ first increases with $f$ and then decreases. Figure 3(b) shows $\mathrm{d}R(t_c^i)/\mathrm{d}t$ as a function of $i$ under different $f$ values and shows that $\mathrm{d}R(t_c^i)/\mathrm{d}t$ fits a Gaussian function, i.e., $\mathrm{d}R(t_c^i)/\mathrm{d}t \sim a_1 \mathrm{e}^{-[(i-b_1)/c_1]^2} + a_2 \mathrm{e}^{-[(i-b_2)/c_2]^2}$. Figure 3(c) shows that the number of microtransitions $n$ increases with $f$, and that $n$ as a function of $f$ follows a linear function, i.e., $n \sim f$. Figure 3(d) shows the critical points $t_c^i$ versus $i$ under different values of $f$. We find that critical point $i$ is the same for different $f$ values, and that the values of $t_c^i$ follow a linear function, i.e., $t_c^i \sim i$.

Figure 4 shows how $\langle\tau\rangle$ affects the emergence of microtransitions. Figure 4(a) shows that when the value of $\langle\tau\rangle = 2$ is small there are no microtransitions in the system and that, because in figure 4(b) the value of $\mathrm{d}R(t)/\mathrm{d}t$ exhibits multiple peaks, increasing $\langle\tau\rangle$ causes microtransitions to emerge. This is the case because when $\langle\tau\rangle$ is small individuals who have received a level of information that exceeds the adoption threshold become adopted after experiencing short time-delays $\langle\tau\rangle$, and the microtransitions disappear. Our theory once again agrees with numerical simulations.

Figure 5 shows the effect of relatively long $\langle\tau\rangle$. Figure 5(a) shows that at critical point $i$ there is a linear increase of $t_c^i$ with $\tau$ and that $t_c^i \sim \mathrm{i}\tau$ follows, but the inset of figure 5(a) shows that the number of microtransitions $n$ does not change with $\tau$. Figure 5(b) shows that $\tau$ does not affect $\mathrm{d}R(t_c^i)/\mathrm{d}t$, the peak value of $i$.

Figure 6 shows that when $\alpha < 0$ it is less probable that high-degree individuals will experience time-delays and, in contrast to when $\alpha = 0$, the number of phase transitions decreases and the growth height decreases. When $\alpha > 0$ it is more probable that high-degree individuals will have time-delays and, in contrast to when $\alpha = 0$, the number of phase transitions increases and the growth is sharp. We know that social contagions on

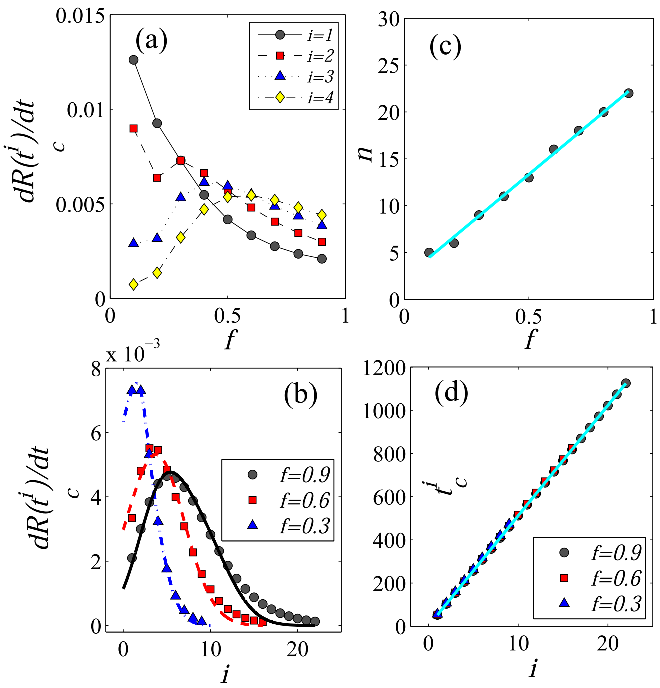


**Figure 3.** Dynamic of social contagions on artificial networks. (a) $\mathrm{d}R(t)/\mathrm{d}t$ versus $f$. (b) $\mathrm{d}R(t_c^i)/\mathrm{d}t$ versus $i$. (c) Number of microtransitions $n$ as a function of $f$. (d) $t_c^i$ versus $i$. Lines in (b)–(d)represent the fitting values. Other parameters are set to be $N = 10^5$, $\gamma_D = 3.0$, $\rho_0 = 0.05$, $\lambda = 0.5$, $\gamma = 1.0$, $\langle\tau\rangle = 50$, $\alpha = 0$ and $T = 3$, respectively.

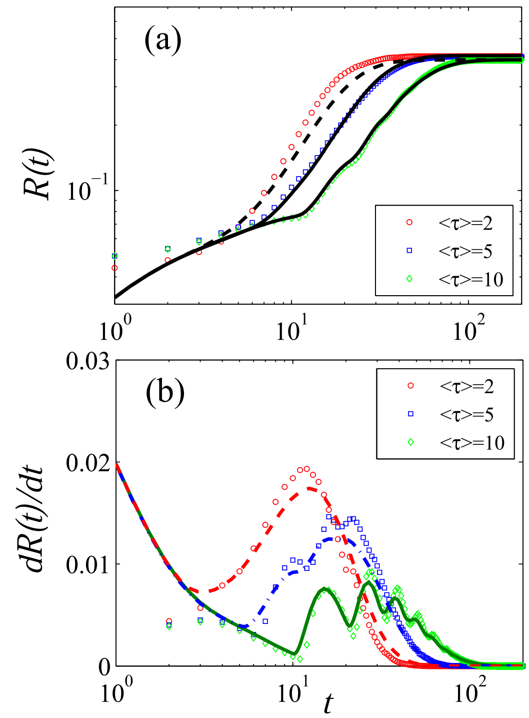


**Figure 4.** Effects of the average time-delays on social contagions. (a) Fraction of individuals in the recovered state $R(t)$ and (b) the derivative of $R(t)$, i.e., $\mathrm{d}R(t)/\mathrm{d}t$, versus time $t$. Symbols are the simulation results and lines are the corresponding theoretical predictions. Other parameters are set to be $N = 10^5$, $\rho_0 = 0.05$, $\gamma_D = 3.0$, $\lambda = 0.5$, $\gamma = 1.0$, $\alpha = 0$, $f = 0.3$ and $T = 3$, respectively.

complex networks behave hierarchically, and that high-degree individuals are more likely to become adopted early in the behavior contagion. Thus when $\alpha = -10$ the value of $R(t)$ is higher than when $\alpha = 0$ or when $\alpha = 10$ at the early stage. When $\alpha = -10$ high-degree individuals adopt the behavior, quickly transmit the information to neighbors, and induce other high-degree individuals and some low-degree individuals to adopt

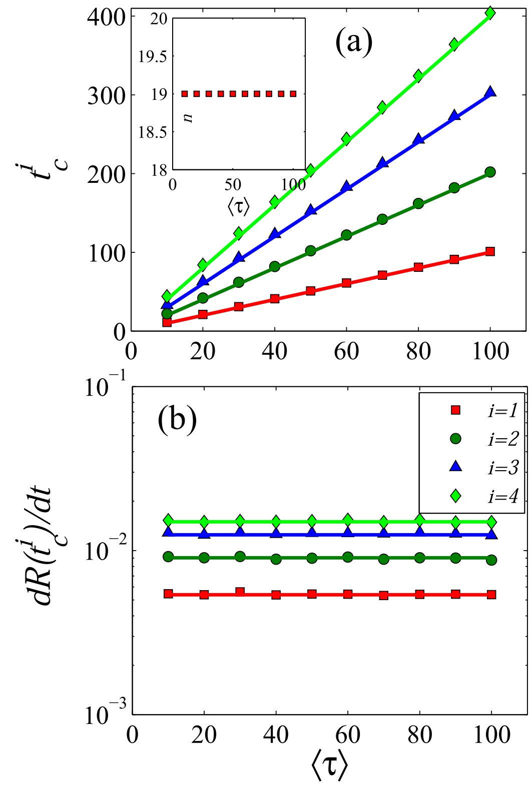


**Figure 5.** Effects of the average time-delays on social contagions. (a) The $i$th critical point $t_c^i$ and (b) the $i$th peak values $\mathrm{d}R(t_c^i)/\mathrm{d}t$ versus time-delays $\langle\tau\rangle$. Other parameters are set to be $N = 10^5$, $\gamma_D = 3.0$, $\rho_0 = 0.05$, $\lambda = 0.5$, $\gamma = 1.0$, $\alpha = 0, f = 0.3$ and $T = 3$, respectively.

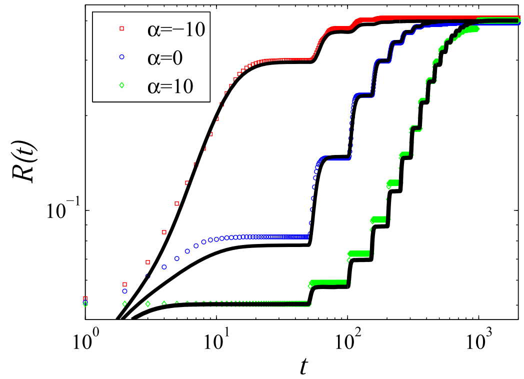


**Figure 6.** Dynamic of social contagions on artificial networks. The fraction of individuals in the recovered state $R(t)$ versus time $t$. Symbols are the simulation results and lines are the corresponding theoretical predictions. Other parameters are set to be $N = 10^5$, $\gamma_D = 3.0$, $\rho_0 = 0.05$, $\lambda = 0.5$, $\gamma = 1.0$, $\langle\tau\rangle = 50, f = 0.3$, and $T = 3$, respectively.

the behavior. After several rounds fewer individuals adopt the behavior, and thus there are fewer phase transitions when $\alpha = -10$. When $\alpha = 10$ high-degree individuals adopt the new behavior after a time delay period and immediately transmit the behavioral information to a large number of neighbors. These neighbors are likely to quickly adopt the behavior, and there is a sharp increase in $R(t)$. Our theory thus accurately predicts the numerical simulation results.

### 4.2. Time-delays with Gaussian distribution

In real-world systems, time-delays for individuals follow are not fixed but follow a distribution we assume to be Gaussian, where $G(\tau) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\tau - \langle\tau\rangle)^2}{2\sigma^2}\right)$, where $\langle\tau\rangle$ is the average number of time-delays with $\langle\tau\rangle = 50$ (see figure 7). We find that the microtransition becomes obscure as $\sigma$ increases. When $\sigma = 0.0$ and all individuals have the same number of time-delays, $R(t)$ exhibits a microtransition versus $t$. When $\sigma = 8$ (a large

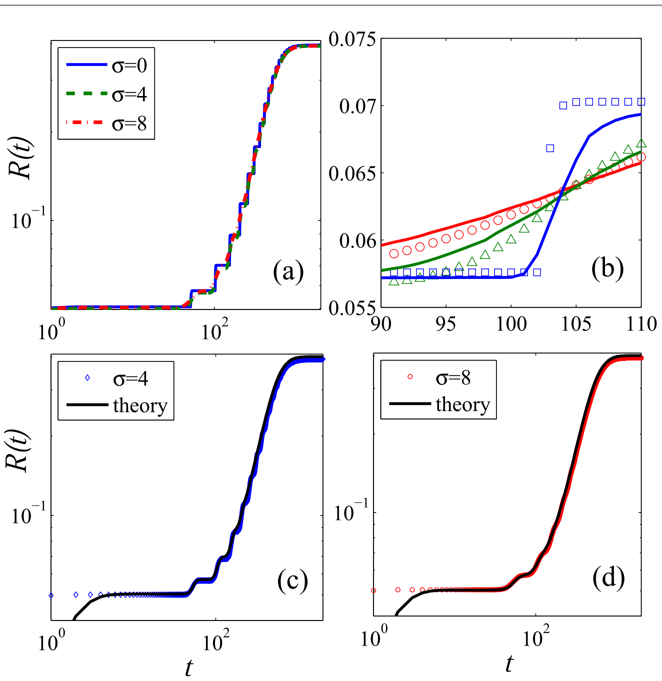


**Figure 7.** Effects of the distribution of time-delays on social contagions. (a), (b) The fraction of individuals in the recovered state $R(t)$ versus $t$ with different variances. $R(t)$ versus $t$ with (c) $\sigma = 4$ and (d) $\sigma = 8$. In (a), the lines are the simulation results. In (b)–(d), symbols are simulation results and lines are the corresponding theoretical prediction. Other parameters are set to be $N = 10^5$, $\gamma_D = 3.0$, $\rho_0 = 0.05$, $\lambda = 0.5$, $\gamma = 1.0$, $\langle\tau\rangle = 50$, $f = 0.3$, $\alpha = 10$ and $T = 3$, respectively.

value) the microtransition becomes obscure because individuals with differing time-delays adopt the behavior only when their received information exceeds their adoption threshold. Once again our theory accurately describes the phenomena.

## 5. Conclusions

We have systematically investigated the effect of time-delays on the dynamics of social contagions and have focused on temporal phase transitions. We propose a non-Markovian spreading threshold model in which each individual is assigned an adoption threshold and a time-delay probability. The adoption threshold takes into account the social reinforcement effect, and time-delays indicate how individual waits before adopting the behavior after the adoption threshold has been reached. We then develop a generalized edge-based compartmental theory with time-delays to describe the non-Markovian model. Using numerical simulations and theoretical analyses we find that relatively long time-delays cause microtransitions in the dynamics of social contagions to emerge but do not affect the final state of the social contagion. When the probability that hubs have time-delays is higher, the microtransition is sharper. Otherwise the microtransition disappears. Finally we find that the heterogeneity of the time-delay distribution causes less obvious microtransitions. Our results provide a deeper understanding of the role of time-delays in the spreading dynamics of complex networks and especially in non-Markovian social contagions. Our results also expand our understanding of phenomena in phase transitions and may provide new insights into spreading dynamics and the connections between human dynamics and social contagions. In our current era of big data there are many challenging issues associated with social contagions that need addressing. For example, verifying the effectiveness of our proposed social contagion model using real data.

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## Appendix. Probability of individuals with time-delays in the network

We denote $A_q(k)$ the number of individuals that have degree $k$ and no time-delays and $g_q(k)$ the degree distribution in the residual network in which all individuals are without time-delays. Here $q$ is the current fraction of individuals without a time-delay. As in [48, 49], the residual degree distribution can be expressed

$$g_q(k) = \frac{A_q(k)}{q\ N}. \tag{A.1}$$

When an additional individual is assigned a time-delay $\tau$ on the basis of $G(\tau)$, using equation (1) $g_q(k)$ becomes

$$A_{q-1/N}(k) = A_q(k) - \frac{g_q(k)k^\alpha}{\langle k^\alpha(q)\rangle}, \tag{A.2}$$

where $\langle k^\alpha(q)\rangle = \sum_k g_q(k)k^\alpha$. In the thermodynamic limit $N \to \infty$, equation (A.2) can be rewritten

$$\frac{\mathrm{d}A_q(k)}{\mathrm{d}q} = N\frac{g_q(k)k^\alpha}{\langle k^\alpha(q)\rangle}. \tag{A.3}$$

Differentiating equation (A.1) with respect to $q$ and substituting it into equation (A.3), we obtain

$$-q\frac{\mathrm{d}g_q(k)}{\mathrm{d}q} = g_q(k) - \frac{g_q(k)k^\alpha}{\langle k^\alpha(q)\rangle}. \tag{A.4}$$

We find by direct integration that [49]

$$g_q(k) = \frac{1}{q}P(k)\beta^{k^\alpha}, \tag{A.5}$$

where $\beta = H_\alpha^{-1}(q)$ and

$$H_\alpha(\beta) = \sum_k P(k)\beta^{k^\alpha},$$

and $\langle k^\alpha(q)\rangle$ can be written

$$\langle k^\alpha(q)\rangle = \frac{\beta H'_\alpha(\beta)}{H_\alpha(\beta)}.$$

We iterate these equations until $q = 1 - f$. Using equation (A.5), the probability $\ell_k$ that an individual with degree $k$ has a time-delay is $\ell_k = 1 - g_f(k)$.

## References

[1] Bakshy E, Hofman J M, Mason W A and Watts D J 2011 *Proc. 4th ACM Int. Conf. on Web Search and Data Mining* vol 65
[2] Watts D J and Dodds P S 2007 *J. Consum. Res.* **34** 441
[3] Iyengar R, Van den Bulte C and Lee J Y 2015 *Mark. Sci.* **34** 408
[4] Zhang Z-K, Liu C, Zhan X-X, Lü X, Zhang C-X and Zhang Y-C 2016 *Phys. Rep.* **651** 1
[5] Liu C, Zhan X-X, Zhang Z-K, Sun G-Q and Hui P-M 2016 *New J. Phys.* **17** 113045
[6] Albert R and Barabási A-L 2002 *Rev. Mod. Phys.* **74** 47
[7] Cohen R and Havlin S 2010 *Complex Networks: Structure, Robustness and Function* (Cambridge: Cambridge University Press)
[8] Centola D 2011 *Science* **334** 1269
[9] Watts D J 2002 *Proc. Natl Acad. Sci.* **99** 5766
[10] Granovetter M S 1973 *Am. J. Sociol.* **78** 1360
[11] Dodds P S and Watts D J 2004 *Phys. Rev. Lett.* **92** 218701
[12] Dodds P S and Watts D J 2005 *J. Theor. Biol.* **232** 587
[13] Wang W, Tang M, Zhang H-F and Lai Y-C 2015 *Phys. Rev.* E **92** 012820
[14] Pastor-Satorras R and Vespignani A 2001 *Phys. Rev. Lett.* **86** 3200
[15] Gallos L K, Liljeros F, Argyrakis P, Bunde A and Havlin S 2007 *Phys. Rev.* E **75** 045104
[16] Moreno Y, Pastor-Satorras R and Vespignani A 2002 *Eur. Phys. J.* B **26** 521
[17] Boguñá M, Castellano C and Pastor-Satorras R 2013 *Phys. Rev. Lett.* **111** 068701
[18] Porter M A and Gleeson J P 2016 *Dynamical Systems on Networks* (Berlin: Springer)
[19] Lü L, Chen D-B and Zhou T 2011 *New J. Phys.* **13** 123005
[20] Pastor-Satorras R, Castellano C, Van Mieghem P and Vespignani A 2015 *Rev. Mod. Phys.* **87** 925
[21] Wang W, Liu Q-H, Zhong L-F, Tang M, Gao H and Stanley H E 2016 *Sci. Rep.* **6** 24676
[22] Liu M, Li D, Qin P, Liu C, Wang H and Wang F 2015 *PLoS One* **10** e0120701
[23] Olfati-Saber R and Murray R M 2004 *IEEE Trans. Autom. Control* **49** 1520
[24] Haerter J O, Jamtveit B and Mathiesen J 2012 *Phys. Rev. Lett.* **109** 168701
[25] Holme P and Saramäki J 2012 *Phys. Rep.* **519** 97
[26] Ortega-Cejas V, Fort J, Méndez V and Campos D 2004 *Phys. Rev.* E **69** 031909
[27] Fort J and Méndez V 2002 *Phy. Rev. Lett.* **89** 178101
[28] Goedgebuer J-P, Larger L and Porte H 1998 *Phy. Rev. Lett.* **80** 2249

[29] Dhamala M, Jirsa V K and Ding M 2004 *Phys. Rev. Lett.* **92** 074104
[30] Earl M G and Strogatz S H 2003 *Phys. Rev.* E **67** 036204
[31] Wang W, Tang M, Shu P and Wang Z 2016 *New J. Phys.* **18** 013029
[32] Wang W, Shu P, Zhu Y-X, Tang M and Zhang Y-C 2015 *Chaos* **25** 103102
[33] Zhu Y-X, Wang W, Tang M and Ahn Y-Y 2017 *Phys. Rev.* E **96** 012306
[34] Gallos L K, Cohen R, Argyrakis P, Bunde A and Havlin S *Phys. Rev. Lett.* **94** 188701
[35] Yuan X, Dai Y, Stanley H E and Havlin S 2016 *Phys. Rev.* E **93** 062302
[36] Volz E 2008 *J. Math. Biol.* **56** 293
[37] Volz E M, Miller J C, Galvani A and Meyers L A 2011 *PLoS Comput. Biol.* **7** e1002042
[38] Miller J C 2011 *J. Math. Biol.* **62** 349
[39] Valdez L D, Macri P A and Braunstein L A 2012 *PLoS One* **7** e44188
[40] Wang W, Tang M, Stanley H E and Braunstein L A 2017 *Rep. Prog. Phys.* **80** 036603
[41] Dorogovtsev S N, Goltsev A V and Mendes J F 2008 *Rev. Mod. Phys.* **80** 1275
[42] Karrer B and Newman M E 2010 *Phys. Rev.* E **82** 016101
[43] Newman M E J 2010 *Networks: An Introduction* (Oxford: Oxford University Press)
[44] Chen W, Schröder M, D'Souza R M, Sornette D and Nagler J 2014 *Phys. Rev. Lett.* **112** 155701
[45] Colomer-de Simón P and Boguñá M 2014 *Phys. Rev.* X **4** 041020
[46] Hackett A, Cellai D, Gómez S, Arenas A and Gleeson J P 2016 *Phys. Rev.* X **6** 021002
[47] Radicchi F 2015 *Nat. Phys.* **11** 597
[48] Huang X, Gao J, Buldyrev S V, Havlin S and Stanley H E 2011 *Phys. Rev.* E **83** 065101
[49] Shao J, Buldyrev S V, Braunstein L A, Havlin S and Stanley H E 2009 *Phys. Rev.* E **80** 036105