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Domany-Kinzel Model of Directed Percolation: Formulation as a Random-Walk Problem and Some Exact Results

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It is shown that the directed percolation on certain two-dimensional lattices, in which the occupation probability is unity along one spatial direction, is related to a random-walk problem, and is therefore exactly solvable. As an example, the case of the triangular lattice is solved. It is also shown that the square-lattice solution obtained previously by Domany and Kinzel can be derived using Minkowski's "taxicab geometry."

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Directed percolation[1] has aroused considerable recent interest among workers from many fields of physics, because of its applications ranging from Reggeon field theory[2] to Markov processes involving branching, recombination, and absorption that arise in chemistry and biology.[3] The combination of renormalization-group, Monte Carlo computer-simulation, and series-expansion procedures has led to a great deal of progress.[4-10]

Relatively little is known in the way of exact solutions for the directed percolation problem. However, in a recent Letter, Domany and Kinzel[11] have proposed a particularly elegant model of directed percolation for a square lattice which is amenable to exact solution. Consider a bond percolation process for which the horizontal and vertical bonds are intact (occupied) with respective probabilities p_H and p_V. Adopt the "sun-belt" convention of placing westward and southward arrows, respectively, on all horizontal and vertical bonds.[12] Domany and Kinzel considered general p_H, p_V and also obtained for $p_H=1$, $p_V=p$ a closed form expression for the probability, $P(\vec{R},p)$, that a site $\vec{R}$ located to the south and west of the origin could be reached by one or more connected paths. They found that for large R, there exists a $p_c(\vec{R}/|\vec{R}|)$ such that $P(\vec{R},p\geq p_c)=1$ and that $P(\vec{R},p\to p_c^-)\sim\exp(-R/\xi)$ with $\xi=(p_c-p)^{-2}$.

Here we present the following further exact results on the Domany-Kinzel problem:

(i) We show that the Domany-Kinzel model of directed percolation is related to a random-walk problem.

(ii) We show more generally that directed percolation on certain two-dimensional nets in which the occupation probability is unity along one spatial direction can also be formulated as a random-walk problem, leading to a simple derivation and analysis of the solution. As an example, the triangular lattice is treated.

Consider first the Domany-Kinzel problem of an infinite square lattice whose sites are denoted by the coordinates (i,j), and let $\vec{O}=(0,0)$, $\vec{R}=(N-1,L)$, so that point $\vec{R}$ is $N-1$ units to the west of the origin and L units to the south of the origin.[12] A bond configuration of the lattice is percolating if there exists at least one directed path running from $\vec{O}$ to $\vec{R}$. Then the key to the Domany-Kinzel solution lies in the fact that a *unique* path can be singled out for each percolating configuration. This can be accomplished by adopting the convention of following the downward arrow whenever possible. Thus, starting from $\vec{O}$, one traverses horizontally, unless there is a down arrow originating from $\vec{O}$, in which case one follows the down arrow immediately. Generally, one follows the first down arrow en route to the next row, and repeats the process. Clearly, a unique path connecting $\vec{O}$ to $\vec{R}$ will be singled out by this process in each percolating configuration. (The path shown in Fig. 1 of Ref. 11 follows the opposite convention, going from $\vec{R}$ to $\vec{O}$, but the effect is the same.)

Since $p_H = 1$, a given configuration must be percolating as soon as the path reaches row L at any point (n, L) with $0 \leq n \leq N-1$. Hence one can write

$$P(\vec{R}, p) = \sum_{n=0}^{N-1} p W_{n, L-1}, \tag{1}$$

where $W_{n, L-1}$ is the probability that the path shall reach the point $(n, L-1)$ on row $L-1$. In writing (1), we have already summed over all percolating configurations corresponding to the same path. Consider now the paths running from $(0, 0)$ to $(n, L-1)$. There are precisely n horizontal and $L-1$ vertical arrows in such paths, with each vertical arrow carrying a weight (probability) p and each horizontal arrow a weight (probability) $q = 1 - p$. It follows that

$$W_{n, L-1} = p^{L-1} q^n C_{n, L-1}, \tag{2}$$

where $C_{n, L-1}$ is the number of distinct paths connecting $(0, 0)$ and $(n, L-1)$. Since the vertical and horizontal arrows can occur in any order, we have

$$C_{n, L-1} = \binom{n + L - 1}{L - 1}. \tag{3}$$

This is the result of Domany and Kinzel who derived it using a different (and more involved) method of counting and analyzed it using a method whose generalization to other lattices is not apparent.

It is of interest to point out here that the number $C_{n, L-1}$ also arises in taxicab geometry, a metric system first proposed by Minkowski over 70 years ago,[13] as the number of "straight" lines between two fixed points.[14] Specifically, $C_{n, L-1}$ is the total number of distinct "taxicab routes" from point $(0, 0)$ to point $(n, L-1)$ on a directed lattice; that $C_{n, L-1}$ is simply given by Eq. (3) is demonstrated clearly in a recent popular account of Minkowski's taxicab geometry.[15]

The paths connecting $(0, 0)$ and $(n, L-1)$ can also be regarded as those traced by a random walker on a directed lattice. Then $W_{n, L-1}$ is the probability that the walker will eventually reach $(n, L-1)$. The formulation as a random-walk problem offers a natural and clean way to analyze the results (2) and (3); it can also be extended to other two-dimensional lattices when the occupation probability is unity along one spatial direction.

As an example, consider the directed percolation problem on a triangular lattice in which the horizontal bonds are present with probabilities $p_H = 1$, the vertical bonds with probabilities $p_V = p$, and the diagonal bonds with probabilities $p_D = p'$. All bonds are directed in the south, west, and southwest directions as shown in Fig. 1. We again compute the probability $P(\vec{R}, p, p')$ that the sites $\vec{O} = (0, 0)$ and $\vec{R} = (N-1, L)$ are connected by at least one directed path. The Domany-Kinzel case is recovered by taking $p' = 0$.

As in the Domany-Kinzel problem, a key step of the solution is to devise a convention which will generate a unique path connecting $\vec{O}$ and $\vec{R}$ in percolating configurations. For this purpose we adopt the convention of following the arrows in the order of vertical, diagonal, and horizontal at each site. Thus, starting from $\vec{O}$ and following arrows according to the order just described, we shall always reach $\vec{R}$ in configurations which are percolating.[16] This convention also assigns the weights p, qp', and qq', respectively, to

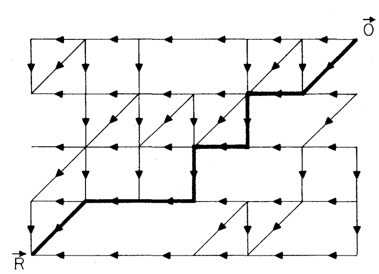

FIG. 1. A typical percolating configuration for the triangular lattice with $N - 1 = 6$ and $L = 4$. The bonds are all oriented, and are intact with respective probabilities $p_H = 1$, $p_V = p$, and $p_D = p'$. The heavy lines denote the *unique* path associated with this configuration.

the vertical, diagonal, and horizontal arrows along the path, where $q=1-p$ and $q'=1-p'$.

In analogy to (1), we now have

$$P(\vec{R},p,p')=(1-qq')\sum_{n=0}^{N-2} W_{n,L-1}+pW_{N-1,L-1}=1-(1-qq')\sum_{n=N-1}^{\infty} W_{n,L-1}+pW_{N-1,L-1}, \tag{4}$$

where we have distinguished the case $n=N-1$ from the cases $0\leq n\leq N-2$. The second equality follows from the elementary fact that the point (∞, L) is connected to the origin with probability 1.

To proceed further, we now regard $W_{n,L-1}$ as the probability that a walker will reach $(n, L-1)$ from $(0,0)$ in a random walk on the triangular lattice with anisotropic probabilities $0,0,0,qq',p,qp'$ for the six directions. Then $W_{n,L-1}$ can be computed by standard means, leading to the expression[17]

$$W_{n,L-1}=\frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} d\varphi_1\,d\varphi_2\,\frac{\exp[-in\varphi_1-i(L-1)\varphi_2]}{1-qq'\exp(i\varphi_1)-p\exp(i\varphi_2)-qp'\exp[i(\varphi_1+\varphi_2)]}. \tag{5}$$

Simply stated, $W_{n,L-1}$ is the coefficient of $x^n y^{L-1}$ in the expansion of $[1-qq'x-py-qp'xy]^{-1}$. If we introduce $z_1=\exp(-i\varphi_1)$ and $z_2=\exp(-i\varphi_2)$, the integrations in (5) become contour integrals around the unit circles in the z_1 and z_2 planes. It is readily established that the simple pole in the z_2 plane is always within the unit circle $|z_2|=1$. Therefore, after carrying out the z_2 integration, we obtain

$$W_{n,L-1}=(\tfrac{1}{2}i)\oint_{|z|=1} F_{n,L}(z)\,dz, \tag{6}$$

where

$$F_{n,L}(z)=z^n(pz+qp')^{L-1}/(z-qq')^L. \tag{7}$$

For the Domany-Kinzel case, $p'=0$, $q'=1$, and (7) reduces to $F_{n,L}(z)=p^{L-1}z^{n+L-1}/(z-q)^L$, so that a straightforward integration of (6) leads to the result (2) and (3). For the general problem, a direct evaluation of (6) and (4) yields a double series which is not easily analyzed. But we can substitute (6) into (4) and deform the contour in the first term to $|z|=r$, $qq'<r<1$, to permit carrying out the summation. This leads to

$$P(\vec{R},p,p')=1-\frac{1-qq'}{2\pi i}\oint_{|z|=r}\frac{dz}{1-z}F_{N-1,L}(z)+\frac{p}{2\pi i}\oint_{|z|=1} dz\,F_{N-1,L}(z). \tag{8}$$

The integrals in (8) can be evaluated by the method of steepest descent. For $N=\alpha L$ large, α finite, we write $F_{N,L}(z)\simeq[f_\alpha(z)]^L$, $f_\alpha(z)=z^\alpha(pz+qp')/(z-qq')$, and find that the integrand is stationary at z_0 determined by $f'(z_0)=0$. It can be verified that $f_\alpha(z_0)$ attains its maximum value of 1 at $z_0=1$ and that $z_0\gtrless 1$ for $\alpha\lessgtr\alpha_c$, where $\alpha_c=q/(1-qq')$. Therefore $F_{N,L}(z_0)=0$ for $z_0\neq 1$ and $F_{N,L}(1)=(1-qq')^{-1}$. In the second integral in (8), we can always deform the contour to pass z_0 so that it gives rise to zero contribution after using the method of steepest descent. The first integral is again zero for $\alpha>\alpha_c$, since in this case $z_0<1$ and the contour can be deformed continuously to the stationary point. But for $\alpha<\alpha_c$ this deformation sweeps past the simple pole at $z=1$, generating a residue $-(1-qq')F_{N-1,L}(1)=-1$. For $\alpha=\alpha_c$ only half of the residue is generated. Thus we find[18]

$$\begin{aligned} P(\vec{R},p,p')&=1, \quad \alpha>\alpha_c,\\ &=0, \quad \alpha<\alpha_c,\\ &=\tfrac{1}{2}, \quad \alpha=\alpha_c. \end{aligned} \tag{9}$$

In fact, for $\alpha<\alpha_c$, we find after Taylor expanding about $z_0=1$, $P(\vec{R},p,p')\simeq[f_\alpha(z_0)]^L\sim\exp(-R/\xi)$, with $\xi\sim(\alpha_c-\alpha)^{-2}$. This leads to the same critical exponent $\nu=2$ as in the Domany-Kinzel solution,[19] reflecting a Gaussian distribution of the profile of the percolation cone.[20] It is also instructive to note that for $p=0$ $(q=1)$ the lattice is again simple quadratic with $\vec{R}$ situated at $(N-L-1, L)$. The resulting critical value then reads $\alpha_c-1=q'/(1-q')$, in agreement with that of Domany and Kinzel.

The connection of directed percolation with a random-walk problem is more general and appears to be applicable whenever a *unique* path can be associated with a percolating configuration. In particular, it is applicable to the percolation problem for which *any* number of bonds are present between the sites (i,j) and $(i-k, j+1)$, $k=-1,0,1,2,3,\ldots$, with respective probabilities p_k, in addition to $p_H=1$. In such cases $P(\vec{R},\{p_k\})$ can always be computed by considering a corresponding random-walk problem.

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[16]Other conventions (such as the order of diagonal, vertical, and horizontal) may lead to a "trap," even though the configuration is percolating.

[17]See, e.g., E. W. Montroll, in *Applied Combinatorial Mathematics*, edited by E. F. Beckenbach (Wiley, New York, 1964), Chap. 4.

[18]Note that our analysis does not involve the cancellation of large factors and the use of the Stirling approximation; it is always tricky to perform such cancellations when the end result is a number of the order of magnitude of unity [cf. the discussion in Ref. 11 following Eq. (8)].

[19]The result $\nu = 2$ was also recently obtained with use of position-space renormalization-group arguments (W. Klein, to be published).

[20]B. C. Harms and J. P. Straley, "Directed percolation: Shape of the percolation cone, conductivity exponents, high dimensionality behavior, and the nature of the phase diagram" (to be published).

Stationary System of Two Masses Kept Apart by Their Gravitational Spin-Spin Interaction

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An exact vacuum solution of Einstein's field equations is presented, describing two isolated bodies balanced by their gravitational spin-spin interaction.

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It is well known that Curzon's static bipolar solution of the Einstein field equations[1] which describes the axisymmetric gravitational field of two separated masses fails to satisfy the condition of elementary flatness on the part of the axis between the two masses. In physical terms this can be interpreted as meaning that the masses are held apart by a strut.[2,3] With the recently developed techniques for generating stationary axisymmetric solutions from static ones, the question arose whether it is possible to stabilize two masses by addition of angular momentum. This is indeed the case.

The metric for space-time has the usual form

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega\, d\varphi)^2,$$

and Curzon's bipolar solution for equal masses is given by

$$f_0 = e^{2\chi} = \exp[-4mx/(x^2 - y^2)], \quad \gamma_0 = -m^2 \frac{1-y^2}{(x^2-y^2)^4}[(x^2-1)(x^4+6x^2y^2+y^4)+(x^2-y^2)^3], \quad \omega_0 = 0,$$

