Current flow in random resistor networks: The role of percolation in weak and strong disorder

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We study the current flow paths between two edges in a random resistor network on a $L \times L$ square lattice. Each resistor has resistance $x_{ij}e^{s_{ij}}$, where $s_{ij}$ is a uniformly distributed random variable and $x_{ij}$ controls the strength of the distribution. We find that: (a) The scaled variable $u = x_{ij}/a$, where $a$ is the percolation connectedness exponent, fully determines the distribution of the current path length $\ell$ for all values of $u$. For $a \sim 1$, the behavior corresponds to the weak disorder limit and $\ell$ scales as $\sim L$, while for $a \gg 1$, the behavior corresponds to the strong disorder limit with $\ell \sim L^{d_u}$, where $d_u = 1.22 \pm 0.01$ is the optimal path exponent. (b) In the weak disorder regime, there is a length scale $\xi \sim a'$, below which strong disorder and critical percolation characterize the current path.

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Transport in disordered media is a classic problem in statistical physics which attracts much attention due to its broad range of applications. Examples include flow through porous material, oil production, and conductivity of semiconducting materials or metal-insulator mixtures [1–10]. These problems have been studied using a random resistor network model with bonds that have a resistance chosen from a probability distribution mimicking the nature of the physical problem under consideration. Among the different classes of disorder distributions used, the most common is percolation disorder, in which the resistance of a bond is either 1 or 0,1. Gaussian distributions and power-law distributions have also been studied extensively [12,13].

Here, we study a random resistor network with exponential disorder [14]. We consider the two opposite edges of a $L \times L$ lattice as source $A$ and sink $B$. Each bond connecting adjacent nodes $i$ and $j$ corresponds to one resistor, whose resistance $r_{ij}$ is given by [1–4,15]

$$r_{ij} = x_{ij}e^{s_{ij}},$$

where $a$ controls the disorder strength and $x_{ij}$ is a random number taken from a uniform distribution $x_{ij} \in [0, 1]$. Recent experiments show that for quenched condensed granular Ni thin films, the conductivity is well described by exponential disorder with large $a$ [2]. Exponential disorder enables us to understand the magnetoresistance phenomenon that out of $10^9$ grains, only a few govern the electric conductivity [5]. Optimal paths in networks have also been studied with exponential disorder, where the optimal path is the path between two sites that minimizes the total weight $\Sigma_{(ij)}e^{s_{ij}}$ [16–20], where the sum is over the bonds $(ij)$ along the path. The length of the optimal path $\ell_{\text{opt}}$ has been shown to scale with the system size as $L^{d_u}$ for the strong disorder limit ($a \rightarrow \infty$) [21], where a single bond dominates the optimal path (and conductance as we see below). The strong disorder limit only has been related to critical percolation [1–3,19]. Here we show that for exponential disorder, the flow paths for all values of $a$ are controlled by critical percolation and by the scaling properties of the optimal path in the strong disorder limit. Indeed, the resistance of each path is equal to the sum of its resistances. When $a \rightarrow \infty$ the resistance of each path is dominated by the largest resistance on this path $\exp(a_{\text{max}})$. Almost all currents must go along the path which minimizes $a_{\text{max}}$. We denote this min-max value of disorder as $x_{ij} = \min_{\text{all paths}}a_{\text{max}}$. Almost all currents which go through the bond with $x_{ij}$, the maximum current goes along the path which minimizes the second largest value of disorder $x_{ij}^2$, and so on. Thus the algorithm of selecting the path with the maximum current is equivalent to selecting the optimal path in the strong disorder limit (ultrametric algorithm [16]). As $a \rightarrow \infty$ the maximum-current path coincides with the optimal path in the strong disorder limit. On the other hand, since all values $x_{ij}$ on the maximum-current path are below $x_{ij}$, this path must belong to the percolation backbone with concentration $p$ equal to the fraction of bonds whose $x_{ij} < x_{ij}$ [22]. The value of $p$ at which percolation between two edges of the system does occur has a narrow distribution with a mean of $p = p_c$ and a standard deviation that scales as $L^{-1/\nu}$ [23], where $p_c$ is the critical percolation threshold, $L$ is the linear system size, and $\nu$ is the connectedness length exponent. Thus the value of $x_{ij}$ also must have a narrow distribution of width $L^{-1/\nu}$.

Next we estimate the value $a$ at which the maximum-current path starts to bifurcate. Consider the paths which do not pass through bond $x_{ij}$ as if this bond has been cut [2]. The maximum-current will then pass through bond $x_{ij} < x_{ij}$, which is characterized by the same narrow distribution. Hence $(x_{ij} - x_{ij})$ is of the order of $L^{-1/\nu}$. These paths become competitive with the true optimal path if its resistance $\exp(ax_{ij})$ becomes of the same order as $\exp(ax_{ij})$ or if $a(x_{ij} - x_{ij}) = aL^{-1/\nu} = 1$. This condition determines the crossover from weak to strong
disorder. If $L \ll a^*$, the disorder is strong and the maximum-current path does not bifurcate. If $L \gg a^*$ the disorder is weak and the maximum-current path can bifurcate. Moreover, the value $\xi \sim a^*$ determines the connectedness length below which the disorder is strong and the maximum-current path is determined by the unique optimal path and above which the maximum-current path bifurcates.

To confirm these analytical predictions, we study the problem numerically. If we define the electric potential at node $i$ of the lattice as $V_i$, and set the potentials at source and sink as $V_A=1$ and $V_B=0$, we numerically solve the set of Kirchhoff equations for all $V_i$ [24]. We begin by building an intuitive understanding of the effect of changing the strength of disorder on current flow. Figures 1(a)–1(c) show, for different values of $a$, the magnitudes of the bond currents represented by the density of dots on each bond. We see that the set of bonds carrying most of the current decreases as $a$ increases, so that only a few current paths dominate. This confirms earlier findings that for large $a$, one or very few paths dominate the current flow [1–3.8]. In Fig. 1(d), we plot the optimal path for the same disorder realization. The similarity between the path of the current carrying bonds in Fig. 1(c) and the optimal path in Fig. 1(d) exhibits how these two quantities are related in the strong disorder limit and supports the argument above that the maximum-current path coincides with the optimal path.

Figure 1 illustrates that the paths used by the current are intimately related to the disorder of the system. Therefore, we study the ensemble of current paths on the lattice by performing tracer dynamics with the particle launching algorithm [25]. For a given realization, all bond currents are determined by Kirchhoff equations and then tracers are injected into node $A$ and extracted at node $B$. At a given node, the tracer follows the bond from node $i$ to $j$ with probability

$$\omega_{ij} = \frac{J_{ij}}{\sum_j J_{ij}},$$

where $j$ runs over all the neighbor bonds of node $i$, $J_{ij}=I_i$ if $I_{ij}>0$, and $J_{ij}=0$ if $I_{ij}<0$, so that only “out” currents are taken into account.

To understand the behavior of the current flow in the presence of disorder in all ranges of disorder, we calculate the length distribution of all tracer paths, $P(\ell | L, a)$, from $A$ to $B$ for a system of linear size $L$ and disorder strength $a$. We first fix $u = L/a^*$ and calculate the distribution $P(\ell | L, a)$ for different system sizes $L$ and the corresponding values of $a=(L/u)^{1/\nu}$. We obtain weak disorder when $u \approx 1$ and strong disorder when $u \ll 1$, as found for the optimal path in networks [26] and as shown below for current flow. Moreover, we find that $u$ is the only parameter that characterizes the disorder and thus determines $P(\ell)$.

In Fig. 2(a) we show three normalized distributions $P(\ell)$ with $u=10$ (weak disorder), which collapses to a single curve as shown in Fig. 2(b). Figure 2(b) Also shows two other peaked curves with $u=1.26$ (close to the crossover) and $u=0.25$ (strong disorder). Each curve shows the collapse of three distributions with different system sizes $L$ but the same value of $u$. This collapse implies that $P(\ell | L, a)$ is controlled by a single parameter $u$,

$$P(\ell | L, a) \sim \frac{1}{\bar{L}_{opt}} \int_a \left( \frac{\ell}{\bar{L}_{opt}} \right)^{1/\nu - 1}.$$  

(3)

We confirmed this scaling numerically for values of $u$ between $u=10$ (weak disorder) and $u=0.25$ (strong disorder) for $a>10$ and $L>15$ [27].
The arguments leading to Eq. (5) for weak disorder, also imply that the standard deviation $\sigma$ of $\ell$ scales as

$$\sigma \sim \sqrt{u} \xi_{\text{opt}} = L^{d_{\text{opt}}-1-d_{\text{opt}}} g_\ell(u),$$

and for all values of $u$,

$$\sigma \sim L^{d_{\text{opt}}-1-d_{\text{opt}}} g_\ell(u),$$

with

$$g_\ell(u) \sim \begin{cases} u^{1-d_{\text{opt}}}, & u \gg 1 \\ 1, & u \ll 1. \end{cases}$$

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In summary, we find that the tracer path length $\ell$ in flow in the presence of exponential disorder behaves similarly to the optimal path length $\ell_{\text{opt}}$, and even has the same scaling exponents ($d_{\text{opt}}$ for $u \ll 1$ and one for $u \gg 1$). Moreover, we also find that when the disorder is weak and $\ell \sim L$, there is a connectedness length $\xi \sim a^r$, where strong disorder and critical percolation exist for regimes smaller than $\xi$. As a result, the probability distribution of $\ell$ is determined by the ratio $L/a^r$, which is the number of units of size $\xi \sim a^r$ in a linear size $L$.

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[15] This class corresponds to a distribution of resistances, which behaves as $1/r$ below a cutoff value $r \leq e^\alpha$, and zero for $r > e^\alpha$.
[21] The values of the exponent $d_{\text{opt}}$ are $d_{\text{opt}} = 1.22$ ($d=2$) and $1.43$ ($d=3$), see Ref. [16]. References [18,26] show that when $L/a^r \ll 1$ these results hold (strong disorder) while for $L/a^r \gg 1$, $\ell_{\text{opt}} \sim L$ which is the weak disorder regime.
[22] In Ref. [14] similar considerations are used to calculate the average conductance of this model, but not the properties of flow paths. The latter have been mostly ignored in the literature. However, recent studies regarding the optimal path in strong disorder and our own insight allow us to explain the properties of the flow paths.
[24] The current between nodes $i$ and $j$ is $I_{ij} = (V_i - V_j)/r_{ij}$. The Kirchoff equations become $\Sigma I_{ij} = I(\delta_{iA} - \delta_{jB})$, where $\delta$ is the Kronecker delta and $I$ is the total current from $A$ to $B$.
[27] For lower values of $a$ and $L$, the discreteness of the lattice becomes relevant yielding deviations from scaling.
[28] In the strong disorder $\xi = a^r > L$ and therefore the whole system is in strong disorder.
[29] Because the whole distribution scales the same way according to Eq. (3), the mean value $\bar{\ell}$ and the most probable value $\ell^*$ have the same scaling.
[30] According to the central limit theorem, for $a \rightarrow \infty$ the distribution of $\ell$ converges to a Gaussian with mean value $\bar{\ell}$ in Eq. (4) and standard deviation $\sigma$ in Eq. (7).