Eur. Phys. J. B **61**, 217–223 (2008)
DOI: 10.1140/epjb/e2008-00066-4

THE EUROPEAN PHYSICAL JOURNAL B

Comparison between volatility return intervals of the S&P 500 index and two common models

I. Vodenska-Chitkushev[1,a], F.Z. Wang[1], P. Weber[1,4], K. Yamasaki[2], S. Havlin[3], and H.E. Stanley[1]

[1] Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215 USA
[2] Department of Environmental Sciences, Tokyo University of Information Sciences, Chiba 265-8501, Japan
[3] Minerva Center and Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
[4] Institute for Theoretical Physics, University of Cologne, 50937 Germany

Received 1st November 2007
Published online 16 February 2008 –

Abstract. We analyze the S&P 500 index data for the 13-year period, from January 1, 1984 to December 31, 1996, with one data point every 10 min. For this database, we study the distribution and clustering of volatility return intervals, which are defined as the time intervals between successive volatilities above a certain threshold q. We find that the long memory in the volatility leads to a clustering of above-median as well as below-median return intervals. In addition, it turns out that the short return intervals form larger clusters compared to the long return intervals. When comparing the empirical results to the ARMA-FIGARCH and fBm models for volatility, we find that the fBm model predicts scaling better than the ARMA-FIGARCH model, which is consistent with the argument that both ARMA-FIGARCH and fBm capture the long-term dependence in return intervals to a certain extent, but only fBm accounts for the scaling. We perform the Student's t-test to compare the empirical data with the shuffled records, ARMA-FIGARCH and fBm. We analyze separately the clusters of above-median return intervals and the clusters of below-median return intervals for different thresholds q. We find that the empirical data are statistically different from the shuffled data for all thresholds q. Our results also suggest that the ARMA-FIGARCH model is statistically different from the S&P 500 for intermediate q for both above-median and below-median clusters, while fBm is statistically different from S&P 500 for small and large q for above-median clusters and for small q for below-median clusters. Neither model can fully explain the entire regime of q studied.

PACS. 89.65.Gh Economics; econophysics, financial markets, business and management – 05.45.Tp Time series analysis – 89.75.Da Systems obeying scaling laws

1 Introduction

Correlations in stock price changes is a topic that has been an active field of study for many years [1–16]. For example, Fama [1] studied the behavior of stock prices and found that price changes are short-term correlated on time scales up to several minutes. Moreover, Dacorogna et al. [2], Ding and Granger [3,4], Liu et al. [6,10] and others have found that the absolute values of price changes, one definition of volatility, show correlations on time scales up to several years. Hence, long-term memory models such as the autoregressive moving average – fractionally integrated generalized autoregressive conditional heteroscedasticity (ARMA-FIGARCH) model have been developed in order to capture various characteristics of volatility [17–21]. An alternative approach to analyze long-term memory is the fractional Brownian motion (fBm) model [22]. Both models are discussed in the Appendix.

The volatility time series is known to be characterized by long-term power-law correlations [2–6,23–27]. To reveal more information about the temporal structure of the volatility time series, Yamasaki et al. [28] and Wang et al. [29] analyzed return intervals between events above a certain threshold q and found long-term power-law correlations in these intervals. Similar studies were done for climate and earthquake records by Bunde et al. [30,31] and Livina et al. [32].

Here, we compare the S&P 500 index data with the ARMA-FIGARCH and fBm models to explore how well the two models detect scaling and memory properties of the empirical data. We choose these models since they are commonly used to represent stock market dynamics [17,20–22,33,34]. Both ARMA-FIGARCH and fBm models are characterized by long-term memory, which also exists in the empirical data.

We define a return interval $\tau(q)$ as the time between successive volatilities above a certain threshold q (see

[a] e-mail: vodenska@bu.edu

Fig. 1a) and we examine the scaling and the memory properties of return intervals τ for both the empirical data and the two models [28–32]. To study the memory, we analyze how the conditional probability density function $P_q(\tau|\tau_0)$ depends on the preceding interval τ_0. We find in both the empirical data and the two models that short (long) return intervals τ_0 are more likely to be followed by short (long) return intervals, corresponding to clustering in volatility return intervals [35–37].

In addition, we perform Student's t-test for clusters of both above- and below-median return intervals. We compare the empirical data with the shuffled records, ARMA-FIGARCH and fBm. Our study shows that the empirical data are statistically different from the shuffled data for all thresholds q, supporting the finding of strong memory in return intervals. Furthermore, we observe that the ARMA-FIGARCH model is statistically different from S&P 500 for intermediate q for both above- and below-median return interval clusters, while fBm is statistically different from S&P 500 data for small and large q for above-median return interval clusters and for small q for below-median clusters.

Moreover, we demonstrate that the scaling properties of return intervals for different q values, which exist in the empirical data [15,28,29], are described better by the fBm model compared to the ARMA-FIGARCH model. We focus on the scaling characteristics of different intermediate volatility thresholds because the scaling property may help us estimate the statistical characteristics of extremely high volatility levels, where we do not have enough data for reliable analysis.

2 Dataset and definitions

We analyze the S&P 500 database for a 13-year period, from January 1, 1984 to December 31, 1996, with one data point every 10 minutes (total of 132 000 data points) [6]. The records are continuous in regular open hours for all trading days, and the times when the market is closed have been removed [10].

We define

$$R(t) \equiv \log \frac{p(t)}{p(t-1)} \approx \frac{p(t)}{p(t-1)} - 1 \tag{1}$$

as the change of the S&P 500 index level between t and $t-1$, where $p(t)$ is the value of the index at time t and $p(t-1)$ is the value 10 min earlier. The intraday volatility shows specific patterns [10] due to different behavior of traders at different times during the day, and different levels of order flow. For example, the largest number of trades are executed at opening and closing market hours in order to capture the opening or the closing price for the day. These observed trading patterns may induce potential artifacts in our data analysis. Hence, we remove the intraday patterns by dividing the absolute value of the return $|R(t)|$ by its average (over all days in the time series) for specific times during the day [10,29]. We use this normalized absolute value of the return, $|\hat{R}(t)|$, to define volatility $v(t)$ as $|\hat{R}(t)|$ divided by its standard deviation

$$v(t) \equiv \frac{|\hat{R}(t)|}{(\langle|\hat{R}(t)|^2\rangle - \langle|\hat{R}(t)|\rangle^2)^{1/2}}, \tag{2}$$

where $\langle ... \rangle$ represents the average over the entire time series. Consequently, the threshold q is measured in units of the standard deviation of $|\hat{R}(t)|$. We select volatilities $v(t)$ above a threshold q and obtain series of return intervals $\tau(q)$, as shown in Figure 1a. We then analyze these return interval series for different thresholds q.

3 Memory and scaling in return intervals

A. Memory

In order to illustrate the memory in return intervals, Figure 1b shows a typical sequence of return intervals for the S&P 500 index. Figure 1c displays the same sequence after the original volatility records have been randomly shuffled so that the memory in the volatility records is lost. We notice prominent "patches" of return intervals below and above their median value in Figure 1b, while there are no such patches in Figure 1c. The patches indicate clustering of the return intervals [28].

To investigate the effect of memory in the volatility on the return intervals, we analyze the conditional probability density function $P_q(\tau|\tau_0)$ conditioned on the preceding return interval τ_0. To this end, we arrange our return interval series $\tau(q)$ in increasing order and divide it into eight equally sized subsets ("octaves") [28,29,32]. Then, we plot $P_q(\tau|\tau_0)$ vs. $\tau/\bar{\tau}$ for both cases, where τ_0 belongs to the first and the last octaves, in Figure 2d. The finding that $P_q(\tau|\tau_0)$ depends on τ_0 demonstrates the existence of memory in return intervals. The probability for the shortest τ_0 to be followed by a short τ is larger than that for the longest τ_0 to be followed by a short τ. On the other end, the probability for the longest τ_0 to be followed by long τ is higher than that for the shortest τ_0 to be followed by a long τ.

B. Scaling

Recent studies have shown that financial time series exhibit complex behavior, and analysis of the scaling properties of the data may improve our understanding of the return interval statistics [38–43]. To better understand the origin of scaling and memory in the real data, we compare the S&P 500 Index return intervals with two models that are widely used for simulating volatility: (i) the ARMA-FIGARCH model, which is a combination of ARMA processes [21] and ARCH processes [33,44,45], and (ii) the fBm model, which has long been used to simulate different types of time series including financial market data [22,46]. We then test how well the simulation results obtained by these two models describe the S&P 500 return interval data. We perform simulations for different

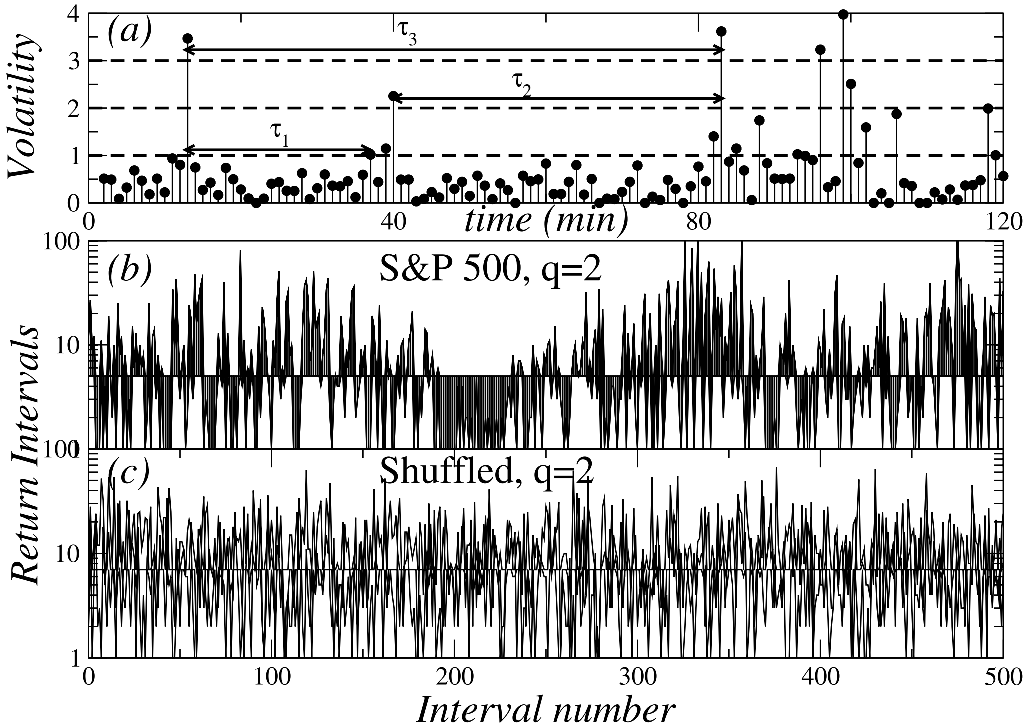

Fig. 1. (a) Illustration of the return intervals for the volatility time series of the S&P 500 stock index on December 26, 1985. Return intervals τ_1, τ_2, and τ_3 for three thresholds $q = 1$, 2, and 3 are displayed. (b) and (c) show a visual demonstration of return interval clustering using the method of Livina [32]. (b) Sequence of 500 typical return intervals for the S&P 500 index for $q = 2$. (c) Same as (b) except that the original volatilities have been shuffled. The median interval for the unshuffled data is 5 min and for the shuffled data is 7 min (horizontal lines).

parameters for both models and plot the simulation results choosing the parameters that best correspond to the data. For the ARMA-FIGARCH model, we perform simulations for $d = 0.2, 0.3, \ldots, 0.8$ and $\beta = 0.1, 0.2, 0.3, 0.4$ [47], where d and β are the parameters of the ARMA-FIGARCH model described in the Appendix. For the fBm model, we simulate return intervals for Hurst exponent $H = 0.1, 0.2, \ldots, 0.9$, and also for $H = 0.84, 0.85, \ldots, 0.95$ [22].

To test the scaling properties of the return interval records of the S&P 500 Index, we examine the probability density function (pdf) $P_q(\tau)$ for different thresholds q. Figure 2a shows that $P_q(\tau)$ scales well with the mean return interval $\bar{\tau}$ since all three curves collapse onto a single one when τ is divided by $\bar{\tau}$ and $P_q(\tau)$ is multiplied by $\bar{\tau}$. This means that after rescaling, the return interval time series for all studied thresholds are characterized by the same pdf.

The ARMA-FIGARCH simulation results shown in Figure 2b reveal that this model does not show good scaling because the curves for different thresholds $q = 2$, 3, and 4 do not seem to collapse onto a single curve for small values of $\tau/\bar{\tau}$. On the other hand, the fBm model shows good scaling, as illustrated in Figure 2c. In Figure 2d we can clearly see the scaling of the empirical results as the three curves – for different $q = 2$, 3, and 4 – collapse onto a single curve for τ_0 in the first octave as well as for τ_0 in the last octave. The lines in Figure 2d represent the fBm simulation results. We note that the fBm agrees very well with the empirical data, suggesting that scaling and memory exist in both the empirical results and the fBm model.

4 Clustering of return intervals

Next we examine the memory in return intervals by studying the probability $P(n)$ of obtaining above- or below-median return interval clusters (cluster size probability). We order the return intervals into two equally sized subsets: long (above-median) and short (below-median), and define clusters as consecutive above-median (or below-median) return intervals. We then record the cluster size n and study the statistics of the above- and below-median clusters separately. We study the cluster size probabilities in an attempt to test whether there exists memory in the return intervals which is longer than the short-term memory between the neighboring return intervals discussed in Section 3.

Figure 3 shows the cluster size probabilities $P(n)$ for clusters of above-median (Fig. 3a) and below-median (Fig. 3b) return intervals for $q = 2$. Similar results are obtained for other values of q. The five curves in Figure 3 correspond to S&P 500 Index (filled circles), shuffled S&P 500 (filled squares), ARMA-FIGARCH (open diamonds), fBm (open triangles), and simulated records with short-term

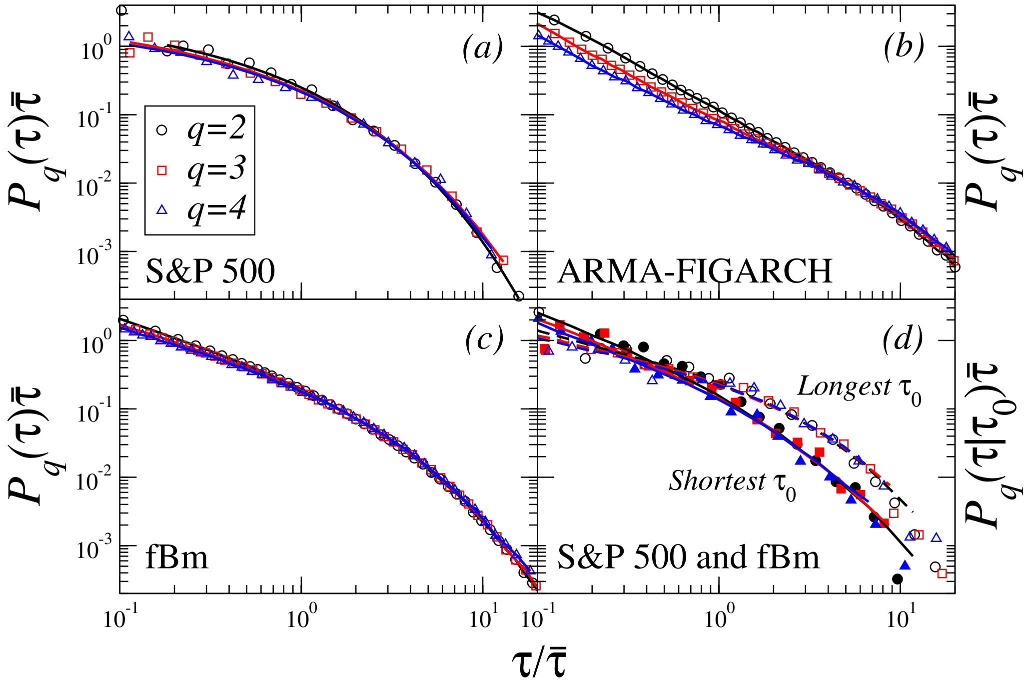

Fig. 2. (Color online) (a) Scaled return interval pdf for S&P 500 Index above three different thresholds $q = 2$, 3, and 4. (b) Scaled ARMA(1,1)-FIGARCH(1,d,0) simulation for thresholds $q = 2$, 3, and 4 and parameters $d = 0.33$ and $\beta = 0.15$. (c) Scaled fBm simulation with $H = 0.86$. (d) Scaled conditional probability $P_q(\tau|\tau_0)\bar{\tau}$ of shortest τ_0 (filled symbols), belonging to the first octave of the dataset (shortest return intervals), and longest τ_0 (open symbols), belonging to the last octave of the dataset (longest return intervals). The lines represent simulations for the fBm model. For details of the parameters d, β, and H, see the Appendix.

correlation (open circles). When comparing the S&P 500 records with the shuffled data, we find that the distribution of shuffled records has shorter tail, which indicates existence of memory in the empirical data.

In order to test whether the short-term memory discussed in Section 3 can explain the cluster size distribution, we generate a time series of return intervals containing only short-term correlations, and compare them with the empirical data. We obtain this simulated series by choosing return intervals from the two different distributions given in Figure 2d. If the observed τ is below the median, we choose the next τ from the distribution of return intervals occurring after the below-median return intervals. If the observed τ is above the median, we choose the next τ from the distribution of return intervals occurring after the above-median return intervals. We find that cluster size probabilities for both above- and below-median clusters in the S&P 500 Index exhibit longer tails compared to the simulated records with short-term correlations. This suggests the existence of long-term correlations in the return intervals of the empirical records.

In Figure 3 we also compare the empirical data with ARMA-FIGARCH and fBm models. We find that in the case of the above-median clusters, both ARMA-FIGARCH and fBm generally overestimate the probability of large clusters in the empirical data. In the case of the below-median clusters, ARMA-FIGARCH overestimates the distribution, while fBm is close to the empirical data. Furthermore, in the empirical data we find asymmetry in the above- and below-median cluster sizes, observing larger cluster sizes for below-median return intervals compared to that of the above-median ones. Since the above-median return interval clusters are comprised of longer time intervals, we can not say that below-median return intervals exhibit longer memory than the above-median return intervals. We observe similar asymmetry in ARMA-FIGARCH and fBm models as shown in Figure 3.

In addition, to compare the empirical data with the shuffled records, the ARMA-FIGARCH and fBm models, we perform the Student's t-test and obtain p-values for both clusters of above-median (Tab. 1a) and clusters of below-median (Tab. 1b) return intervals. The p-value is a measure of probability that a difference between two distributions is significant. The lower the p-value, the more likely is that the difference between two distributions is statistically significant. We observe that in the case of S&P 500 and shuffled records the difference between the two distributions is statistically significant, since all the p-values for different q are smaller than the usually-accepted statistical significance threshold 0.05. We then analyze the p-values for S&P 500 and ARMA-FIGARCH t-test and S&P 500 and fBm t-test, and observe that ARMA-FIGARCH is statistically different from S&P 500 for intermediate q for both above- and below-median clusters, while fBm is statistically different from S&P 500 for small and large q for above-median clusters and for small q for below-median clusters. Neither model can explain the entire regime of q values studied. When comparing

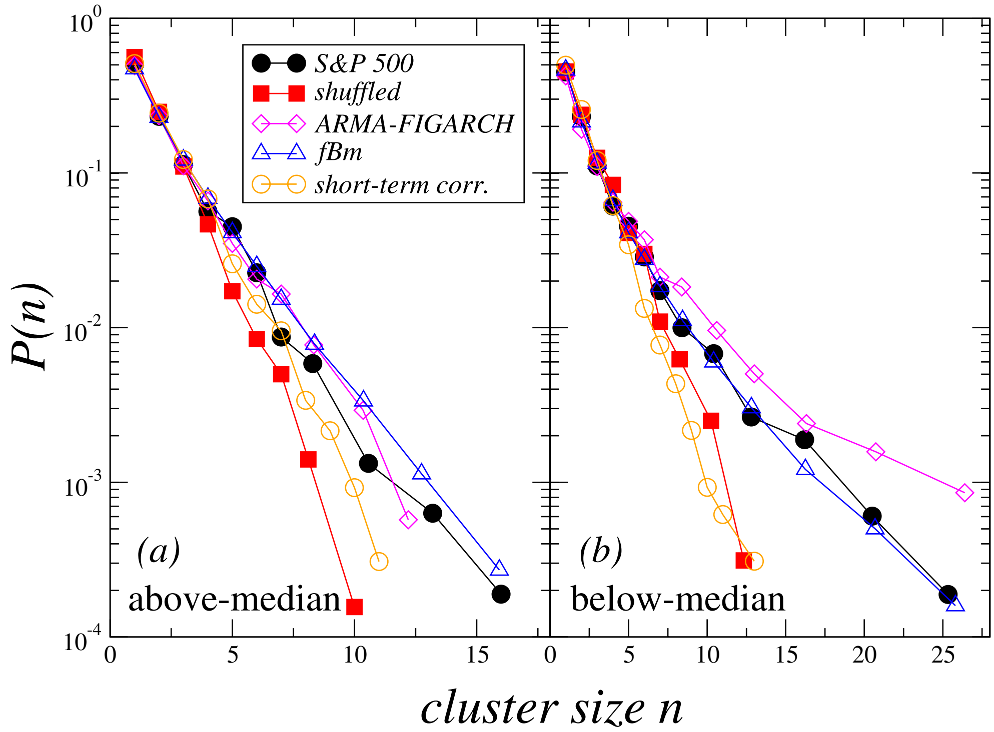

Fig. 3. (Color online) Probability $P(n)$ of obtaining (a) above-median and (b) below-median return interval clusters (cluster size probability) for threshold $q = 2$. The filled squares represent the shuffled return interval data, and show different behavior than the original data (filled circles) revealing that there is memory in the return intervals. The open circles represent the simulation results obtained by generating the return interval sequence with short-term memory. These simulation records have a thinner tail than the S&P data, showing that the S&P data cluster size records exhibit longer-term memory. The open diamonds show the ARMA(1,1)-FIGARCH (1,d,0) model for $d = 0.33$ and $\beta = 0.15$, while the open triangles represent the fBm model for $H = 0.86$. For details of the parameters d, β, and H, see the Appendix.

ARMA-FIGARCH and fBm, we note that the models are significantly different from one another for small thresholds q and they do not appear to be different for large q.

5 Discussion

We have studied the memory and scaling effects in volatility return intervals for the S&P 500 Index for a 13-year period, from January 1, 1984 to December 31, 1996, with one data point every 10 minutes. By comparing the empirical results with the ARMA-FIGARCH and the fBm models, we have found that fBm predicts scaling better than ARMA-FIGARCH, which is consistent with the argument that both ARMA-FIGARCH and fBm capture the long-term dependence in volatility return intervals to certain extent, but only fBm accounts for the scaling [48].

The scaling is important because it suggests that large and small financial market fluctuations are governed by the same laws, and it enables us to infer the statistical characteristics of rare events using the more common events where one has good statistics. In our analysis we have demonstrated that for different volatility thresholds q the scaled pdf $P_q(\tau)\bar{\tau}$ as a function of the scaled return intervals $\tau/\bar{\tau}$ collapses onto a single curve, showing the scaling property of the return intervals. Similarly, we have demonstrated that the scaled conditional pdf $P_q(\tau|\tau_0)\bar{\tau}$ for cases where τ_0 belongs to the subsets of either the shortest or the longest observed return intervals are different, but each collapses to a single curve for different q values, showing the existence of scaling in the conditional pdf.

We analyzed the above- and below-median return interval clusters and their cluster size distribution $P(n)$. We observe that large clusters of above- and below-median return intervals appear much more frequently than in non-correlated sequences. This supports the hypothesis that return interval sequences are also long-term correlated.

We performed Student's t-test to compare the empirical data to the shuffled records, ARMA-FIGARCH and fBm. The shuffled records show significant difference from the empirical data for all thresholds q. ARMA-FIGARCH is statistically different from S&P 500 for intermediate q for both above- and below-median clusters, while fBm is statistically different from S&P 500 for small and large q for above-median clusters and for small q for below-median clusters.

These differences in the empirical data distribution and the models may be due to volatility being more complex and not exhibiting pure long-term correlations while this is the nature of fBm, and on the other hand,

Table 1. Student's t-test results for clusters of return intervals. The values in the table represent the p-values obtained by Student's t-test, and q in the first row is the volatility threshold for return intervals. (a) Clusters of above-median return intervals; (b) Clusters of below-median return intervals.

	$q = 0.25$	$q = 0.5$	$q = 1$	$q = 2$	$q = 3$	$q = 4$
S&P500 and Shuffled	1.65E-03	1.08E-19	1.45E-28	2.02E-20	6.07E-10	1.13E-08
S&P500 and ARMA-FIGARCH	0.8109	9.19E-03	3.49E-04	0.3310	0.7544	0.5360
S&P500 and fBm	5.96E-15	6.49E-07	0.3401	0.1326	0.0871	0.0225
ARMA-FIGARCH and fBm	3.45E-16	8.78E-03	0.0286	0.0182	0.0611	0.11

(a)

	$q = 0.25$	$q = 0.5$	$q = 1$	$q = 2$	$q = 3$	$q = 4$
S&P500 and Shuffled	4.27E-05	1.04E-15	4.70E-21	8.12E-10	1.07E-03	3.06E-03
S&P500 and ARMA-FIGARCH	0.3173	7.63E-04	3.38E-03	0.1409	0.2857	0.5251
S&P500 and fBm	1.63E-12	3.66E-08	0.3635	0.7301	0.7062	0.2213
ARMA-FIGARCH and fBm	4.26E-16	8.49E-03	0.0837	0.0839	0.4812	0.5215

(b)

ARMA-FIGARCH is scale-inconsistent, while the empirical data are showing good scaling.

We thank X. Gabaix for helpful discussions, and NSF for financial support.

Appendix A: models

A.1 ARMA(l, m)-FIGARCH(k, d, e) model

A combination of Autoregressive Moving Average (ARMA) and Fractional Integrated Autoregressive Conditional Heteroscedasticity (FIGARCH) is a common model for the simulation of returns. As an univariate time series, returns r_t can be divided into two parts,

$$r_t = E(r_t|\Omega_{t-1}) + \epsilon_t, \tag{3}$$

where $E(.\mid.)$ denotes the conditional expectation, Ω_{t-1} is the information set at the previous time $t-1$, and ϵ_t is the disturbance term with zero mean, $E(\epsilon_t) = 0$ and no correlation, $E(\epsilon_t\epsilon_s) = 0$, for all $t \neq s$, where $E(.)$ denotes the expectation.

Equation (3) (the "mean equation") has been modeled by combining two of the most common specifications–autoregressive (AR) and moving average (MA)–and obtaining the ARMA(l, m) process,

$$\Phi(L)(r_t - \mu) = \Theta(L)\epsilon_t, \tag{4}$$

where L is the lag operator, μ is the mean value of r_t,

$$\Phi(L) \equiv 1 - \sum_{i=1}^{l} \phi_i L^i \tag{5}$$

and

$$\Theta(L) \equiv 1 + \sum_{j=1}^{m} \theta_j L^j \tag{6}$$

are AR and MA coefficients, which use the future and previous information respectively.

Following the classic ARCH(k) and GARCH(k, e) models [33,44], the unpredicted term ϵ_t in equation (3) is defined as

$$\epsilon_t = z_t\sigma_t, \tag{7}$$

where z_t is an $i.i.d.$ process with zero mean, $E(z_t) = 0$, unit variance, $E({z_t}^2) = 1$, and the conditional variance σ_t^2 varies with ARCH-type models. In the FIGARCH (k, d, e) model (here we use BBM method [45]), it follows ("variance equation"),

$$\sigma_t^2 = \sigma^2 + \lambda(L)(\epsilon_t^2 - \sigma^2) \tag{8}$$

where σ^2 is the unconditional variance of ϵ_t,

$$\lambda(L) = 1 - [1 - \beta(L)]^{-1}[1 - \alpha(L) - \beta(L)](1 - L)^{d-1}, \tag{9}$$

and

$$\alpha(L) = \sum_{i=1}^{e} \alpha_i L^i \tag{10}$$

are the ARCH parameters,

$$\beta(L) = \sum_{j=1}^{k} \beta_j L^j \tag{11}$$

are the GARCH parameters, and $d \in [0, 1]$ is the fractional differencing parameter. From the fractional differencing item $(1 - L)^d$, $\lambda(L)$ is an infinite summation which, in practice, has to be truncated. BBM suggested a truncated length with 1000 lags.

A key feature of the FIGARCH process is that it can capture the long-term dependence in volatilities of financial markets, which is connected to the fractional differencing parameter d. When d increases, the long-term memory effect will gradually vanish.

A.2 Fractional Brownian motion model

As a generalization of Brownian motion, fractional Brownian motion (fBm) [22] is a centered Gaussian process with stationary increments, and those increments have long-range dependence which can be characterized by the Hurst parameter $H \in (0, 1)$. When $H = 1/2$, the fBm process $B_H(t)$ reduces to a standard Brownian motion which has no dependence in their increments. When $H > 1/2$ ($H < 1/2$), the fBm process has positive (negative) correlations. An important feature of fBm is the scale invariance,

$$B_H(ct) = c^H B_H(t) \tag{12}$$

for all $c > 0$.

As noted above, the stock returns only have short-term correlations [1] while volatilities have long-term correlations [2–4,6,10]. To capture this feature, we simulate the returns r_t by

$$r_t = \eta_t \exp[B_H(t+1) - B_H(t)], \tag{13}$$

where η_t is a Gaussian noise.

References

1. E. Fama, J. Business **38**, 34 (1965)
2. M.M. Dacorogna, R.J. Muller, R.J. Nagler, R.B. Olsen, O.V. Pictet, J. Int. Money Finance **12**, 413 (1993)
3. Z. Ding, C.W.J. Granger, R.F. Engle, J. Empirical Finance **1**, 83 (1983)
4. C.W.J. Granger, Z. Ding, J. Econometrics **73**, 61 (1996)
5. T. Di Matteo, T. Aste, M.M. Dacorogna, J. Banking Finance **29**, 827 (2005)
6. Y. Liu, P. Cizeau, M. Meyer, C.-K. Peng, H.E. Stanley, Physica A **245**, 437 (1997); P. Cizeau, Y. Liu, M. Meyer, C.-K. Peng, H.E. Stanley, Physica A **245**, 441 (1997); Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng, H.E. Stanley, Phys. Rev. E **60**, 1390 (1999)
7. G.W. Schwert, J. Finance **44**, 1115 (1989); K. Chan, K.C. Chan, G.A. Karolyi, Rev. Financ. Stud. **4**, 657 (1991); T. Bollerslev, R.Y. Chou, K.F. Kroner, J. Econometr. **52**, 5 (1992); A.R. Gallant, P.E. Rossi, G. Tauchen, Rev. Financ. Stud. **5**, 199 (1992); B. Le Baron, J. Business **65**, 199 (1992)
8. R. Cont, Ph.D. thesis, Université Paris XI, 1998 (unpublished); see also e-print `arXiv:cond-mat/9705075`
9. M. Pasquini, M. Serva, Econ. Lett. **65**, 275 (1999)
10. V. Plerou, P. Gopikrishnan, X. Gabaix, L.A. Nunes Amaral, H.E. Stanley, Quant. Finance **1**, 262 (2001); V. Plerou, P. Gopikrishnan, H.E. Stanley, Phys. Rev. E **71**, 046131 (2005); X. Gabaix, P. Gopikrishnan, V. Plerou, H.E. Stanley, Nature **423**, 267 (2003); X. Gabaix, P. Gopikrishnan, V. Plerou, H.E. Stanley, Quarterly Journal of Economics **121**, 461 (2006)
11. H. Takayasu, H. Miura, T. Hirabayashi, K. Hamada, Physica A **184**, 127 (1992); H. Takayasu, A.H. Sato, M. Takayasu, Phys. Rev. Lett. **79**, 966 (1997); H. Takayasu, K. Okuyama, Fractals **6**, 67 (1998)
12. R. Mantegna, H.E. Stanley, *Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge Univ. Press, Cambridge, UK, 2000)
13. J.-P Bouchaud, M. Potters, *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management* (Cambridge Univ. Press, Cambridge, 2003)
14. N.F. Johnson, P. Jefferies, P.M. Hui, *Financial Market Complexity* (Oxford University Press, New York, 2003)
15. F. Wang, P. Weber, K. Yamasaki, S. Havlin, H.E. Stanley, Eur. Phys. J. B **55**, 123 (2007)
16. P. Weber, F. Wang, I. Vodenska-Chitkushev, S. Havlin, H.E. Stanley, Phys. Rev. E **76**, 016109 (2007)
17. T. Bollerslev, Rev. Economics and Statistics **69**, 542 (1987)
18. T. Bollerslev, R.F. Engle, Econometrica **61**, 166 (1993)
19. T. Bollerslev, H.O. Mikkelsen, J. Econometrics **92**, 75 (1999)
20. R.T. Baillie, T. Bollerslev, J. Econometrics **52**, 91 (1992)
21. G.E.P. Box, G.M. Jenkins, G.S. Reinsel, *Time Series Analysis: Forecasting and Control,* 3rd edn. (Prentice-Hall, Upper Saddle River NJ, 1994)
22. B.B. Mandelbrot, J.W. Van Ness, SIAM Rev. **10**, 422 (1968)
23. L. Harris, J. Financ. Econ. **16**, 99 (1986)
24. A. Admati, P. Pfleiderer, Rev. Financ. Stud. **1**, 3 (1988)
25. A. Pagan, J. Empirical Finance **3**, 15 (1996)
26. L. Bachelier, Annales Scientifiques de l'École Normale Supérieure *III-17*, 21 (1900). Translated and Reprinted in *The Random Character of Stock Market Prices*, edited by P. Cootner (MIT Press, Cambridge, Mass, 1964), p. 17
27. X. Gabaix, P. Gopikrishnan, V. Plerou, H.E. Stanley, J. Eur. Economic Association **5**, 564 (2007)
28. K. Yamasaki, L. Muchnik, S. Havlin, A. Bunde, H.E. Stanley, Proc. Natl. Acad. Sci. USA **102**, 9424 (2005)
29. F. Wang, K. Yamasaki, S. Havlin, H.E. Stanley, Phys. Rev. E **73**, 026117 (2006)
30. A. Bunde, J.F. Eichner, S. Havlin, J.W. Kantelhardt, Physica A **342**, 308 (2004)
31. A. Bunde, J.F. Eichner, J.W. Kantelhardt, S. Havlin, Phys. Rev. Lett. **94**, 048701 (2005)
32. V.N. Livina, S. Havlin, A. Bunde, Phys. Rev. Lett. **95**, 208501 (2005)
33. R.F. Engle, Econometrica **50**, 987 (1982)
34. R.F. Engle, T. Ito, W.L. Lin, Econometrica **58**, 525 (1990)
35. T. Lux, M. Marchesi, Int. J. Theor. Appl. Finance **3**, 675 (2000)
36. I. Giardina, J.-P. Bouchaud, Physica A **299**, 28 (2001)
37. T. Lux, M. Ausloos, in *The Science of Disasters: Climate Disruptions, Heart Attacks, and Market Crashes*, edited by A. Bunde, J. Kropp, H.J. Schellnhuber (Springer, Berlin, 2002), p. 373.
38. T. Di Matteo, Quant. Finan. **7**, 21 (2007); M. Bartolozzi, C. Mellen, T. Di Matteo, T. Aste, Eur. Phys. J.B. **58**, 207 (2007); T. Di Matteo, T. Aste, M.M. Dacorogna, see e-print `arXiv:cond-mat/0302434`
39. Z. Eisler, J. Kertész, Phys. Rev. E **73**, 046109 (2006)
40. Z. Eisler, J. Kertész, Eur. Phys. J. B **51**, 145 (2006)
41. P. Ch. Ivanov, A. Yuen, B. Podobnik, Y. Lee, Phys. Rev. E **69**, 056107 (2004)
42. V. Plerou, H.E. Stanley, Phys. Rev. E **76**, 046109 (2007)
43. F. Wang, K. Yamasaki, S. Havlin, H.E. Stanley, Phys. Rev. E **77**, 016109 (2008)
44. T. Bollerslev, J. Econometrics **31**, 307 (1986)
45. R.T. Baillie, T. Bollerslev, H.O. Mikkelsen, J. Econometrics **74**, 3 (1996)
46. H.A. Makse, S. Havlin, M. Schwartz, H.E. Stanley, Phys. Rev. E **53**, 5445 (1996)
47. S. Laurent, J.-P. Peters, J. Econ. Surveys **16**, 447 (2002)
48. B.B. Mandelbrot, A.J. Fisher, L.E. Calvet, A Multifractal Model of Asset Returns, *Discussion Paper #1164* (Cowles Foundation, Yale University, 1997)