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Renormalization-group calculation of the critical-point exponent η for a critical point of arbitrary order*

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The critical-point exponent η for a critical point of order $\mathcal{O}$ in dimensions less than $d_\mathcal{O} \equiv 2\mathcal{O}/(\mathcal{O}-1)$, is calculated to leading nonvanishing order in the parameter $\epsilon_\mathcal{O} \equiv d_\mathcal{O} - d$. The result is given for n-component isotropically interacting magnetic systems. For Ising systems, $n=1$, the result is $\eta_\mathcal{O} = \epsilon_\mathcal{O}^2 4(\mathcal{O}-1)^2/\binom{2\mathcal{O}}{\mathcal{O}}^3$. As $\mathcal{O}$ increases, the coefficient of $\epsilon_\mathcal{O}^2$ rapidly becomes very small, varying as $2^{-6\mathcal{O}}$ for $\mathcal{O}$ large. In the limit of large n, $\eta_\mathcal{O}$ for odd order points approaches a constant and, for even order points, is proportional to $1/n$.

The classification and study of critical points of "higher order" has been of recent interest.[1–7] The order of a critical point is defined by some authors to be the number of phases simultaneously critical at the critical point.[1,2] Thus, an ordinary critical point is an $\mathcal{O}=2$ point; tricritical points are $\mathcal{O}=3$ points. Although there are many kinds of higher-order points,[3] much of the work has concentrated on systems that in the mean-field approximation could be represented by a Landau-Ginzberg form for the Hamiltonian density,

$$H(\vec{s}) = -\int d^d\vec{x}\left(\tfrac{1}{2}\left|\nabla\vec{s}(\vec{x})\right|^2 + \sum_{k=1}^{\mathcal{O}} \frac{u_{2k}}{(2k)!}(\vec{s}\cdot\vec{s})^k\right), \quad (1)$$

where we have specialized to the "magnetic" case of an isotropically interacting n-component spin $\vec{s}(\vec{x})$.

The renormalization-group approach to such systems was introduced by Wilson[8] for the case $\mathcal{O}=2$. Corrections to mean-field behavior are calculated in a perturbation expansion in $\epsilon_2 \equiv 4-d$. The tricritical case, $\mathcal{O}=3$, has been studied by Riedel and Wegner[4] at $d=3$. Chang, Tuthill, and Stanley and Stephen and McCauley calculated exponents below three dimensions in an expansion in $\epsilon_3 \equiv 3-d$.[5,6] Reference 5 also gave explicit exponents to first order in $\epsilon_4 \equiv \frac{8}{3} - d$ for the $\mathcal{O}=4$ case. The critical-point exponents for the general $\mathcal{O}$ case were given in Nicoll, Chang, and Stanley[7] to first order in $\epsilon_\mathcal{O} \equiv 2\mathcal{O}/(\mathcal{O}-1)-d$. The critical-point exponent η was shown in Ref. 7 to be at most $O(\epsilon_\mathcal{O}^2)$. In this work, we complete the calculation of all critical-point exponents to leading order by calculating η to $O(\epsilon_\mathcal{O}^2)$.

The $\epsilon_\mathcal{O}$ calculations of Ref. 7 were based on the differential renormalization-group generator of Wegner and Houghton.[9] The calculation of η by this method is difficult and, therefore, through most of this work we will adopt a field-theoretic approach utilizing Feynman diagrams. However, we will extract the dependence of η on the number of spin components n by combining graph counting with the solutions of Ref. 7.

Following the method used to locate fixed points,[5–9] we assume u_{2k} to be $O(\epsilon_\mathcal{O})$ for $k \le \mathcal{O}$. It is then possible to carry out a self-consistent perturbation expansion in the parameters $u_4, u_6, \ldots, u_{2\mathcal{O}}$ while applying a "mass counterterm"[8] so that the bare propagator is $(p^2+r)^{-1}$, with r^{-1} the zero-ordering-field susceptibility. The exponent $\eta_\mathcal{O}$ is defined by a proportionality relation for the Fourier transform G of the spin-spin correlation function for small wave number,

$$G^{-1}(p) \sim p^{2-\eta_\mathcal{O}} \sim p^2(1-\eta_\mathcal{O}\ln p \cdots), \quad (2)$$

at the order-$\mathcal{O}$ point ($r=0$). We will now show that to $O(\epsilon_\mathcal{O}^2)$, the calculation of $\eta_\mathcal{O}$ involves only two Feyman graphs to be evaluated in dimension $d_\mathcal{O}$.

In the perturbation expansion for G^{-1} we may write

$$G^{-1}(p,r) = p^2 + r + \Sigma(p, r), \quad (3)$$

where Σ represents the sum of all remaining graphs (with counterterm) displayed schematically in Fig. 1. The mass counterterm $u_2 - r$ cancels all p-independent terms in (3) (in particular, all single-vertex diagrams). The series may be further simplified by formally eliminating closed loops that include only one vertex and introducing r-dependent generalized vertices $\bar{u}_{2k}(r)$, defined by

$$\bar{u}_{2k}(r) = u_{2k} + \sum_{l=1}^{\mathcal{O}-k} \frac{u_{2k+2l}}{l!\,2^l}\,[F(r)]^l. \quad (4)$$

Here, as in Ref. 6, $F(r)$ represents the loop integral $\int d^d p G(p,r)/(2\pi)^d$. With this change in notation and to $O(\epsilon_\mathcal{O}^2)$, the set of graphs in Σ is reduced to those shown in Fig. 2.

Next, we note that $\bar{u}_{2k}(r=0)=0$ for all $k<\mathcal{O}$. This follows from Wilson's scaling theorem[8] for $2k$-point vertex functions

$$\Gamma_{2k}(p=0) \sim r^{k-(k-1)d/(2-\eta)}. \quad (5)$$

For $d \le d_\mathcal{O}$, Eq. (5) requires that Γ_{2k} vanish at $r=0$ for all $k<\mathcal{O}$. Since the first-order perturba-

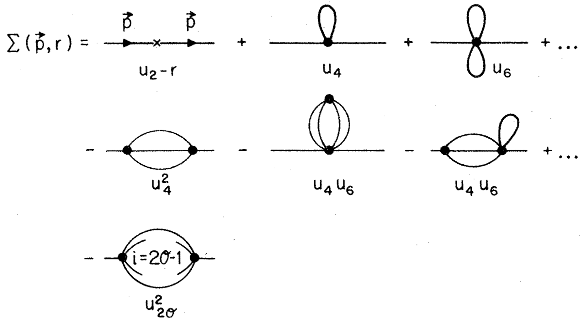

FIG. 1 Perturbation series to $O(\epsilon_\Theta^2)$ for the function $\Sigma(\vec{p},\ r)$, defined by Eq. (3). Each diagram carries net momentum $\vec{p}$.

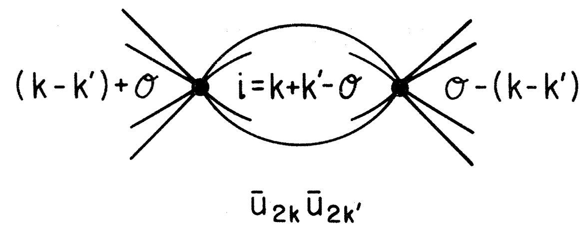

FIG. 3. Typical second-order contribution to $\Gamma_{2\Theta}$. The 2Θ external lines carry no momenta.

tion expansion for Γ_{2k} is just $\bar{u}_{2k}$, $\bar{u}_{2k}$ must vanish as well. At the critical point, therefore, all the graphs except the last shown in Fig. 2 are zero.

The combinatorial factor for this diagram may be evaluated by considering first the Ising case, in which it is simply $1/(2\Theta-1)!$. To determine the n dependence, it suffices to note that a factor of $n+2N-2$ is associated with the connection of two legs of a single $2N$-leg vertex. Thus, the n dependence of the $\bar{u}_{2\Theta}^2$ diagram is given by $f_1(n)/(2\Theta-1)!$, where

$$f_1(n)=\prod_{l=1}^{\Theta-1}\frac{n+2l}{2l+1}\ . \tag{6}$$

With this factor and denoting the $\bar{u}_{2\Theta}^2$ integral by I_1, the correspondence between (2) and (3) gives

$$p^2(1-\eta_\Theta\ln p)=p^2-\frac{u_{2\Theta}^2 f_1(n)}{(2\Theta-1)!}\times[I_1(p,\ r=0)]_{p^2\,\ln p\ \text{part}}\ . \tag{7}$$

Since $u_{2\Theta}$ is $O(\epsilon_\Theta)$, η_Θ is clearly $O(\epsilon_\Theta^2)$.

The fixed point value of $u_{2\Theta}$ remains to be determined; it is chosen so that the vertex functions satisfy scaling laws. For $k=\Theta$ in (5) this gives

$$\Gamma_{2\Theta}\sim r^{\epsilon_\Theta(\Theta-1)/2}\sim 1+\epsilon_\Theta[(\Theta-1)/2]\ln r\cdots\ . \tag{8}$$

The constant of proportionality must also be expanded as a series in ϵ_Θ, so that

$$\Gamma_{2\Theta}=A+\epsilon_\Theta\{[A(\Theta-1)/2]\ln r+B\}, \tag{9}$$

with A and B constants.

In first order, $\Gamma_{2\Theta}$ is $u_{2\Theta}$, so that $A=u_{2\Theta}$. Second-order terms all involve two-vertex diagrams $\bar{u}_{2l}\,\bar{u}_{2l'}$ (with $l,\ l'\le\Theta$) and graphs with internal lines numbering $l\le\Theta$, (cf. Fig. 3). The r dependence of $\bar{u}_{2l}$ is given by the integral $F(r)-F(0)$, since by the remarks above $\bar{u}_{2l}(0)=0$ for $l<\Theta$:

$$F(r)-F(0)=\int\frac{d^d\vec{k}}{(2\pi)^d}\left(\frac{1}{k^2+r}-\frac{1}{k^2}\right)=-\frac{\Omega_d}{(2\pi)^d}\,r\int_0^\infty\frac{k^{-3+d}\,dk}{k^2+r}\ , \tag{10}$$

where $\Omega_d=2(\pi)^{d/2}/\Gamma(\frac{1}{2}d)$ is the area of the unit sphere in dimension d. Changing variables, we have

$$F(r)-F(0)=-\frac{\Omega_d}{(2\pi)^d}\,r^{(d-2)/2}\int_0^\infty\frac{dx\,x^{d-3}}{1+x^2}\ . \tag{11}$$

The integral converges for $2<d<4$ so that all r dependence is in the prefactor; no $\ln r$ factors are present.

Next, we examine the r dependence of the graph of Fig. 3. By power counting, this integral diverges like $r^{(i-\Theta)/(\Theta-1)}$ for small r. [For $i<\Theta$, the integral converges at large k without a momentum cutoff, and a change of variables similar to that in (11) shows that the diagram gives a prefactor of $r^{(i-\Theta)/(\Theta-1)}$ multiplied by a convergent integral.] Only for $i=\Theta$ will $\ln r$ terms arise; the integral for this case is denoted as $I_2(r)$.

To compare with the scaling form (9), we note that the perturbation expansion gives

$$\Gamma_{2\Theta}=u_{2\Theta}-\frac{(2\Theta)!\,I_2(r)}{2(\Theta!)^2}\,u_{2\Theta}^2\cdots\ . \tag{12}$$

The resulting value for $u_{2\Theta}$ to first order is

$$u_{2\Theta}=\frac{-(\Theta-1)(\Theta!)^3\epsilon_\Theta}{(2\Theta!)[I_2(r)]_{\ln r\ \text{part}}}\ . \tag{13}$$

Combining (13) with (7), the expression for the exponent η_Θ (for $n=1$; n dependence will be discussed below) to leading order is

$$\eta_\Theta=\epsilon_\Theta^2\frac{(\Theta-1)^2(\Theta!)^6[I_1(p)]_{p^2\,\ln p}}{(2\Theta-1)!\,[(2\Theta!)]^2[I_2(r)]_{\ln r}^2}\ . \tag{14}$$

All that remains is the calculation of the two integrals

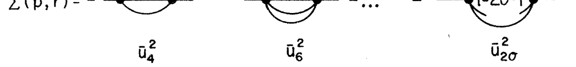

FIG. 2. Mass counterterm u_2-r cancels all $\vec{p}$-independent terms in Σ, and the use of the generalized vertices $\bar{u}_{2k}(r)$ eliminates all closed single-vertex loops.

$$I_1 = \int d^d\vec{R}\, e^{i\vec{p}\cdot\vec{R}} \left(\int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{R}}}{k^2} \right)^{2\mathcal{O}-1} \tag{15}$$

and

$$I_2 = \int d^d\vec{R} \left(\int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{R}}}{k^2+r} \right)^{\mathcal{O}}, \tag{16}$$

where d and $\mathcal{O}$ are, of course, related by $d = d_{\mathcal{O}} = 2\mathcal{O}/(\mathcal{O}-1)$. Both integrals are divergent as written; I_1 diverges quadratically and I_2 diverges logarithmically. To extract the finite terms desired, we cut off the R integrations by integrating over $|R| > \Lambda^{-1}$.

From Bateman[10] we note that

$$\int d^d\vec{x}\, e^{i\vec{q}\cdot\vec{x}} = \int_0^\infty dx\, x^{d-1}\, \Omega_d \Gamma(\tfrac{1}{2}d)\, J_\nu(xq)\, (\tfrac{1}{2}xq)^{-\nu}, \tag{17}$$

where $\nu \equiv \frac{1}{2}(d-2)$. Therefore, applying (17) to (15) we have

$$I_1 = \frac{[\Omega_d \Gamma(d/2)]^{2\mathcal{O}}}{(2\pi)^{d(2\mathcal{O}-1)}} \int_{1/\Lambda}^\infty R^{d-1}\, dR\, J_\nu(Rp)\, (\tfrac{1}{2}Rp)^{-\nu} \times \left[\int_0^\infty k^{d-3}\, dk\, J_\nu(kR)\, (\tfrac{1}{2}kR)^{-\nu} \right]^{2\mathcal{O}-1}. \tag{18}$$

The inner integral can be evaluated exactly; after a change of variable (18) becomes

$$I_1 = \left(\frac{\Omega_d \Gamma(\frac{1}{2}d)\, 2^{2\nu-1} \Gamma(\nu)}{(2\pi)^d} \right)^{2\mathcal{O}-1} \Omega_d \Gamma(\tfrac{1}{2}d)\, p^2 \times \int_{p/\Lambda}^\infty dx\, x^{-3} J_\nu(x)\, (\tfrac{1}{2}x)^{-\nu}. \tag{19}$$

The integral over the interval $[1, \infty)$ gives a finite contribution to the p^2 term. The integral over the interval $[p/\Lambda, 1]$ can be evaluated by expanding the Bessel function in its Taylor series. We find that

$$I_1 = \left(\frac{\Omega_d \Gamma(\frac{1}{2}d)\, 2^{2\nu-1} \Gamma(\nu)}{(2\pi)^d} \right)^{2\mathcal{O}-1} \times \frac{\Omega_d \Gamma(\frac{1}{2}d)}{4\Gamma(\frac{1}{2}d+1)}\, p^2 \ln p + O(\Lambda^2). \tag{20}$$

The I_2 integral can be handled in the same way except that $r \neq 0$. Although the inner integral with nonzero r can be performed exactly, it is not necessary to do so explicitly. We merely note that

$$\int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{R}}}{k^2+r} = \frac{2^{\nu-1}\Gamma(\nu)}{(2\pi)^{d/2} R^{d-2}} C(rR^2), \tag{21}$$

where $C(x)$ is analytic at $x=0$, $C(0)=1$, and $C(x) \sim 1/x$ for x large. The I_2 integral is therefore

$$I_2 = \Omega_d \left(\frac{\Omega_d \Gamma(\frac{1}{2}d)\, 2^{2\nu-1}}{(2\pi)^d} \Gamma(\nu) \right)^{\mathcal{O}} \times \int_{\sqrt{r}/\Lambda}^\infty \frac{dx}{x} C^{\mathcal{O}}(x^2). \tag{22}$$

The integral over $[1, \infty)$ is a finite constant which we may discard. For the integral over $[\sqrt{r}/\Lambda, 1]$, $C(x)$ may be expanded in its Taylor series. Thus,

$$I_2 = \frac{\Omega_d}{2} \left(\frac{\Omega_d \Gamma(\frac{1}{2}d)\, 2^{2\nu-1}}{(2\pi)^d} \Gamma(\nu) \right)^{\mathcal{O}} \ln r + O(\ln \Lambda). \tag{23}$$

Combining (14), (20), and (23) we have for $n=1$ the following simple expression

$$\eta_{\mathcal{O}} = 4(\mathcal{O}-1)^2 \epsilon_{\mathcal{O}}^2 / \binom{2\mathcal{O}}{\mathcal{O}}^3. \tag{24}$$

For the general n case, more combinatorial factors must be computed. The n dependent factor for the numerator is $f_1(n)$, given in (6). The combinatorial factor for the fixed-point vertex $u_{2\mathcal{O}}$ is more complicated. In the differential-equation formulation of Ref. 7, these same combinatorial complications determine the n dependence of the fixed point. In Ref. 7, this n dependence is given as an integral involving three generalized Laguerre polynomials. Performing this integral we find[11] the combinatorial factor for the numerator is $f_2(n)$, where

$$f_2(n) = \frac{\langle \mathfrak{D}^n(\mathcal{O},\mathcal{O}) | \mathcal{O} \rangle}{\langle \mathfrak{D}^1(\mathcal{O},\mathcal{O}) | \mathcal{O} \rangle} \tag{25a}$$

and the inner product $\langle \mathfrak{D}^n(\mathcal{O},\mathcal{O}) | \mathcal{O} \rangle$ is given by the double summation

$$\langle \mathfrak{D}^n(\mathcal{O},\mathcal{O}) | \mathcal{O} \rangle = \sum_{i,j} \binom{\mathcal{O}+\frac{1}{2}n-1}{\mathcal{O}-i} \binom{\mathcal{O}+\frac{1}{2}n-1}{\mathcal{O}-j} [n-1+(2i-1)(2j-1)] \binom{i+j-2}{i-1} \binom{i+j-1+\frac{1}{2}n}{i+j-2-\mathcal{O}} (-1)^{i+j}. \tag{25b}$$

With these combinatorial factors, the result for general n and general $\mathcal{O}$ is

$$\eta_{\mathcal{O}} = \frac{4(\mathcal{O}-1)^2}{\binom{2\mathcal{O}}{\mathcal{O}}^3} \frac{f_1(n)}{f_2^2(n)} \epsilon_{\mathcal{O}}^2. \tag{26}$$

It is easy to check that (26) reduces to the previously calculated results for $\mathcal{O}=2^8$ and $\mathcal{O}=3^6$,

$$\eta_2 = \epsilon_2^2 \frac{n+2}{2(n+8)^2},$$

$$\eta_3 = \epsilon_3^2 \frac{(n+2)(n+4)}{12(3n+22)^2}. \tag{27}$$

We note that as $\mathcal{O}$ increases the coefficient of $\epsilon_{\mathcal{O}}^2$ rapidly becomes very small, $\sim 2^{-6\mathcal{O}}$ for $\mathcal{O}$ large. In the limit of large n, $\eta_{\mathcal{O}}$ for odd order points approaches a constant and, for even order points, is proportional to $1/n$.

For all $\mathcal{O} \geq 3$ we have $d_{\mathcal{O}} \leq 3$, and the mean-field result, $\eta_{\mathcal{O}} = 0$, therefore applies in three-dimensional systems. However, these results and those of Refs. 5–7 may apply to higher-order critical points in two-dimensional systems. In any event, the previously obtained results[8] for ordinary critical points are placed in a broader theoretical context by the extension to general $\mathcal{O}$.

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