Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Physica A 346 (2005) 174–182
PHYSICA A
www.elsevier.com/locate/physa

Transition between strong and weak disorder regimes for the optimal path

Sameet Sreenivasan[a,*], Tomer Kalisky[b], Lidia A. Braunstein[a,c], Sergey V. Buldyrev[d], Shlomo Havlin[a,b], H. Eugene Stanley[a]

[a]*Department of Physics, Center for Polymer Studies, Boston University, Boston, MA 02215, USA*
[b]*Department of Physics, Minerva Center, Bar-Ilan University, 52900 Ramat-Gan, Israel*
[c]*Departamento de Física, Universidad Nacional de Mar del Plata, Funes 3350, 7600 Mar del Plata, Argentina*
[d]*Department of Physics, Yeshiva University, 500 W 185 Street, New York, NY 10033, USA*

Available online 17 September 2004

Abstract

We study the transition between the strong and weak disorder regimes in the scaling properties of the average optimal path $\ell_{\rm opt}$ in a disordered Erdős–Rényi (ER) random network and scale-free (SF) network. Each link i is associated with a weight $\tau_i \equiv \exp(ar_i)$, where r_i is a random number taken from a uniform distribution between 0 and 1 and the parameter a controls the strength of the disorder. We find that for any finite a, there is a crossover network size $N^*(a)$ such that for $N \ll N^*(a)$ the scaling behavior of $\ell_{\rm opt}$ is in the strong disorder regime, while for $N \gg N^*(a)$ the scaling behavior is in the weak disorder regime. We derive the scaling relation between $N^*(a)$ and a with the help of simulations and also present an analytic derivation of the relation.

PACS: 89.75.Hc

Keywords: Networks; Optimal path; Strong disorder

*Corresponding author.
E-mail address: sameet@buphy.bu.edu (S. Sreenivasan).

doi:10.1016/j.physa.2004.08.064

1. Introduction

In a real-world network, whether it be a communication network or transport network, the time τ_i taken to traverse a link i may not be the same for all the links. In other words, there is a "cost" or a "weight" τ_i associated with each link, and the larger the weight on a link, the harder it is to traverse this link. In such a case, the network is said to be disordered. Consider two nodes A and B on such a disordered network. In general, there will be a large number of paths connecting A and B. Among these paths, there is usually a single path for which the sum of the costs $\sum \tau_i$ along the path is minimum and this path is called the "optimal path." When most of the links on the path contribute to the sum, the system is said to be "weakly disordered" (WD). In some cases, however, the cost of a single link along the path dominates the sum. In this case each path between two nodes can be characterized by a value equal to the maximum cost along that path, and the path with the minimal value of the maximum cost is the optimal path between the two nodes. This limit of disorder is called the *strong disorder* (SD) limit ("ultrametric" limit) [1] and we refer to the optimal path in this limit as the *min–max path*. We implement disorder on a network as follows [2–4]. We assign to each link i of the network a random number r_i, uniformly distributed between 0 and 1. The cost associated with link i is then $\tau_i \equiv \exp(ar_i)$, where a is the parameter which controls the breadth of the distribution of link costs. The parameter a represents the strength of disorder. The limit $a \to \infty$ is the strong disorder limit, since for this case only one link dominates the cost of the path. The parameter a can be regarded as an inverse temperature. There are distinct scaling relationships between the length of the average optimal path $\ell_{\rm opt}$ and the network size (number of nodes) N depending on whether the network is strongly or weakly disordered [4]. For SD [4], $\ell_{\rm opt} \sim N^{v_{\rm opt}}$, where $v_{\rm opt} = \frac{1}{3}$ for Erdős–Rényi (ER) random networks [5] and for scale-free (SF) [6] networks with $\lambda > 4$, where λ is the exponent characterizing the power law decay of the degree distribution. For SF networks with $3 < \lambda < 4$, $v_{\rm opt} = (\lambda - 3)/(\lambda - 1)$. For WD ER networks and for SF networks with $\lambda > 3$, $\ell_{\rm opt} \sim \ln N$. Here we show that similar to regular lattices [2], there exists for any finite a, a crossover network size $N^*(a)$ such that for $N \ll N^*(a)$, the scaling properties of the optimal path are in the SD regime while for $N \gg N^*(a)$, the network is in the WD regime. We obtain the functional dependence of $N^*(a)$ on a.

2. Scaling approach

In general, the average optimal path length $\ell_{opt}(a)$ in a disordered network depends on a as well as on N. In the following, we use instead of N the min–max path length ℓ_∞ which is related to N as $\ell_\infty \equiv \ell_{\rm opt}(\infty) \sim N^{v_{\rm opt}}$ and hence $N \sim \ell_\infty^{1/v_{\rm opt}}$. Thus, for finite a, $\ell_{\rm opt}(a)$ depends on both a and ℓ_∞. We expect that there exists a crossover length $\ell^*(a)$, corresponding to the crossover network size $N^*(a)$, such that (i) for $\ell_\infty \ll \ell^*(a)$, the scaling properties of $\ell_{\rm opt}(a)$ are those of the strong disorder regime, and (ii) for $\ell_\infty \gg \ell^*(a)$, the scaling properties of $\ell_{\rm opt}(a)$ are those of the weak disorder regime.

We propose the following scaling Ansatz for $\ell_{\rm opt}(a)$:

$$\ell_{\rm opt}(a) = \ell_\infty F\left(\frac{\ell_\infty}{\ell^*(a)}\right) , \tag{1}$$

where

$$F(u) \sim \begin{cases} \text{const.}, & u \ll 1 , \\ \ln(u)/u, & u \gg 1 . \end{cases} \tag{2}$$

3. Discussion

We now develop arguments to obtain the dependence of the crossover length $\ell^*(a)$ on the disorder strength a. We begin by making a few observations about the min–max path. In Fig. 1 we plot the average value of the random numbers r_n on the min–max path as a function of their rank n ($1 \leqslant n \leqslant \ell_\infty$) for ER networks and SF networks. This can be done for a min–max path of any length but in order to get good statistics we use the most probable min–max path length. We call links with $r \leqslant p_c$ "black" links, and links with $r > p_c$ "gray" links, following the terminology of Ioselevich and Lyubshin [7] where p_c is the percolation threshold of the network [8]. We make the following observations regarding the min–max path:

(i) For $r_n < p_c$, the values of r_n decrease linearly with rank n, implying that the values of r for black links are uniformly distributed between 0 and p_c, consistent with the results of Ref. [9]. This is shown in Fig. 1.
(ii) The average number of black links, $\langle \ell_b \rangle$, along the min–max path increases linearly with the average path length ℓ_∞. This is shown in Fig. 2a.
(iii) The average number of gray links $\langle \ell_g \rangle$ along the min–max path increases logarithmically with the average path length ℓ_∞ or, equivalently, with the network size N. This is shown in Fig. 2b.

The simulation results presented in Fig. 2 pertain to ER networks; however, we have confirmed that observations (ii) and (iii) also hold for SF networks. Observations (ii) and (iii) indicate that the dominant portion of the min–max path lies along the giant component of the network at percolation [10].

Now we will discuss the implications of our findings for the crossover from strong to weak disorder. From observations (i) and (ii), it follows that for the portion of the path belonging to the giant component, the distribution of random values r is uniform. Hence we can approximate the sum of weights by an integral.

$$\sum_{k=1}^{\ell_b} \exp(ar_k) \approx \frac{\ell_b}{p_c} \int_0^{p_c} \exp(ar)\,\mathrm{d}r = \frac{\ell_b}{ap_c}\left(\exp(ap_c) - 1\right) \equiv \exp(ar^*) , \tag{3}$$

where $r^* \approx p_c + (1/a)\ln(\ell_\infty/ap_c)$ since $\langle \ell_b \rangle \approx \ell_\infty$. Thus, restoring a short-cut link between two nodes on the optimal path with $p_c < r < r^*$ may drastically reduce the length of the optimal path. When $ap_c \gg \ell_\infty$, $r^* < p_c$ and such a link does not exist, but

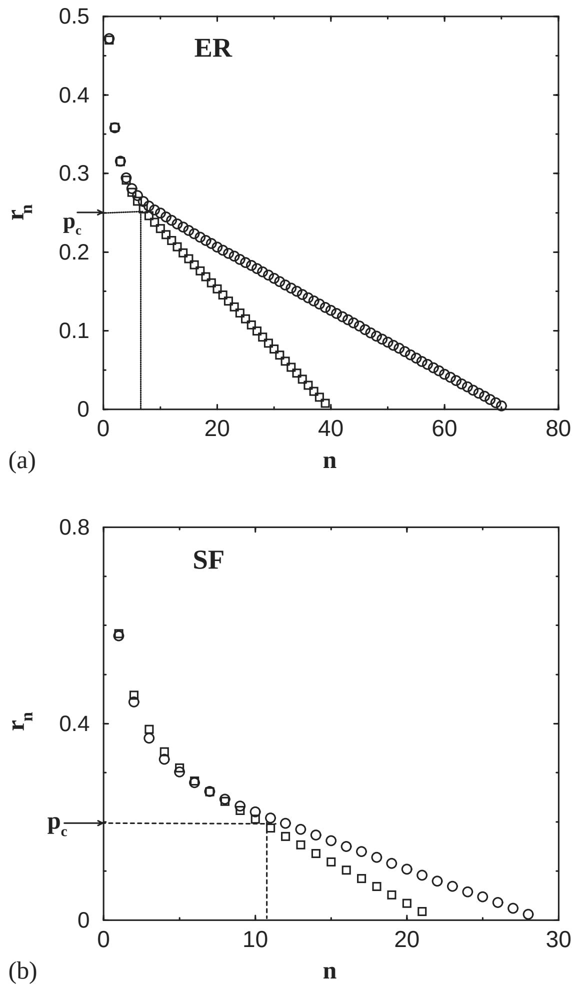

Fig. 1. Dependence on rank n of the average values of the random numbers r_n along the most probable optimal path for (a) ER random networks with $\langle k \rangle = 4$ of two different sizes $N = 4096$ (□) and $N = 16384$ (◯) and, (b) SF random networks with $\lambda = 3.5$ for the same network sizes as in (a).

there begins to be a finite probability for such a link to exist if $\ell_\infty > ap_c$. Hence when the min–max path is of length $\ell_\infty \approx ap_c$, the optimal path starts deviating from the min–max path. The length of the min–max path at which the deviation first occurs is precisely the crossover length $\ell^*(a)$, and therefore $\ell^*(a) \sim ap_c$. Note that in the case of SF networks, as $\lambda \to 3^+$, p_c approaches zero and consequently $\ell^*(a) \to 0$. This suggests that for any finite value of disorder strength a, a SF network with $\lambda \leqslant 3$ is in the weak disorder regime.

We perform numerical simulations and show that the results agree with our theoretical predictions. The details of our simulation methods are published elsewhere [10]. From our theoretical arguments, $\ell^*(a) \sim a$ and therefore, from Eq. (1), $\ell_{\rm opt}(a)/\ell_\infty$ must be a function of ℓ_∞/a. In Fig. 3 we show the ratio $\ell_{\rm opt}(a)/\ell_\infty$ for different values of a plotted against $\ell_\infty/\ell^*(a) \equiv \ell_\infty/a$ for ER networks and SF networks. The excellent data collapse is consistent with the scaling relations Eq. (1) (see also [10]).

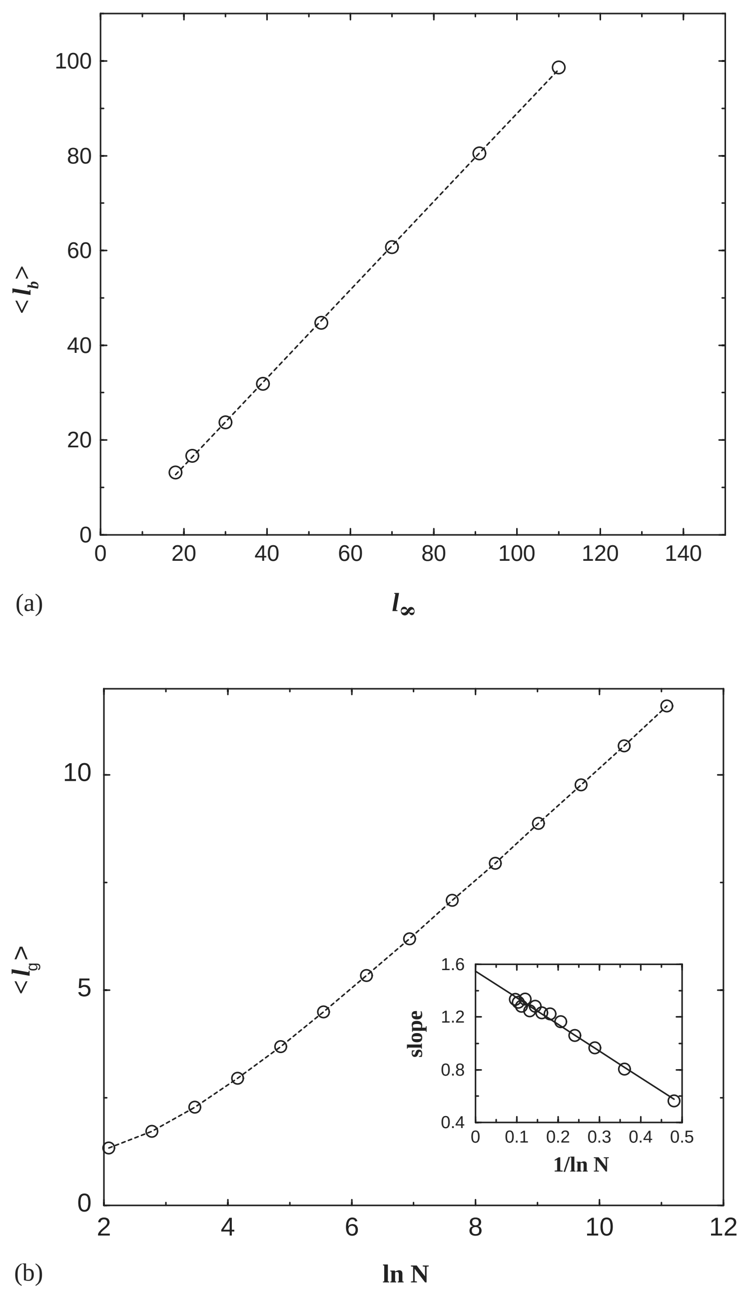

Fig. 2. The average number of links (a)$\langle \ell_b \rangle$ with random number values $r \leqslant p_c$ on the min–max path plotted as a function of its length ℓ_∞ for an ER network, showing that $\langle \ell_b \rangle$ grows linearly with ℓ_∞. (b) $\langle \ell_g \rangle$ with random number values $r > p_c$ on the min–max path versus $\ln N$ for an ER network, showing that $\langle \ell_g \rangle \sim \ln N$. The inset shows the successive slopes, indicating that in the asymptotic limit $\langle \ell_g \rangle \approx 1.55 \ln N$.

4. Analytic derivation

An analytic derivation for obtaining the crossover length $\ell^*(a)$ is as follows. Consider a disordered network with disorder strength a. Let us define for its

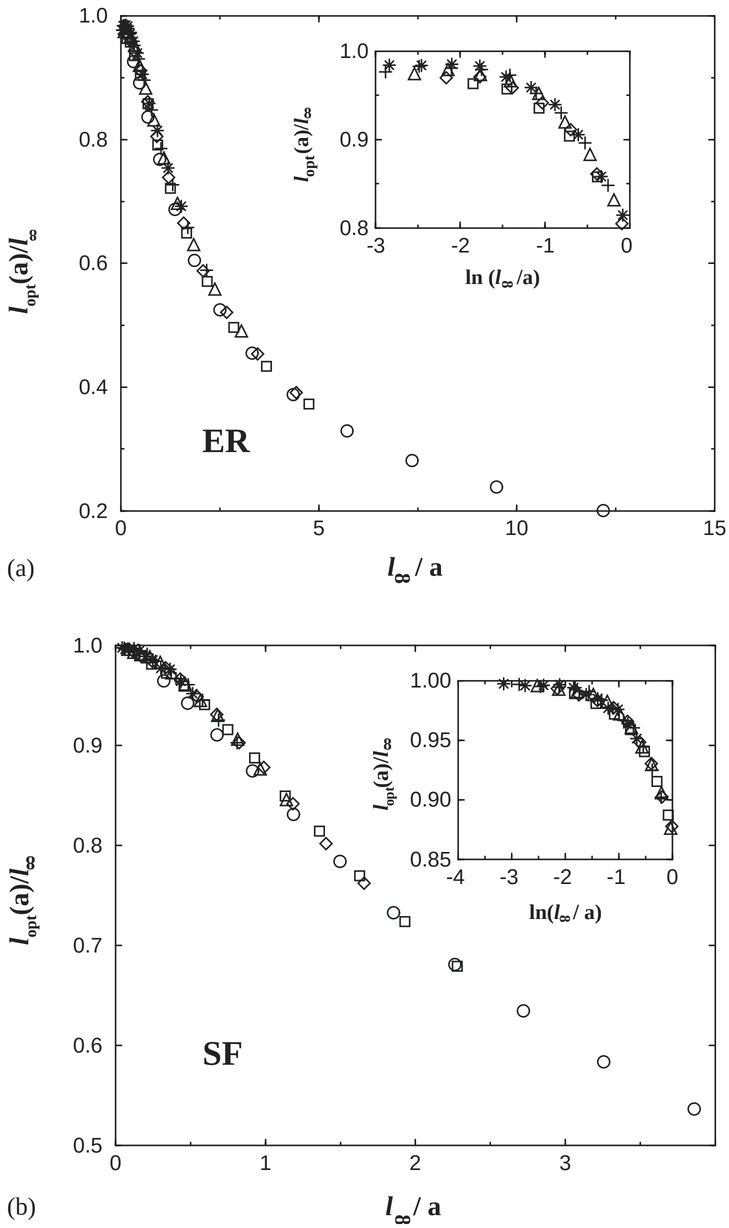

Fig. 3. Test of Eqs. (1) and (2). (a) $\ell_{\rm opt}(a)/\ell_\infty$ plotted as a function of ℓ_∞/a for different values of a for ER networks with $\langle k\rangle = 4$. The different symbols represent different a values: 8(◯), 16(□), 22(◇), 32(△), 45(+), and 64(∗). (b) Same for SF networks with $\lambda = 3.5$. The symbols correspond to the same values of disorder as in (a). The insets show $\ell_{\rm opt}(a)/\ell_\infty$ plotted against $\log(l_\infty/a)$, and indicate for $\ell_\infty \ll a$, $\ell_{\rm opt}(a)/\ell_\infty$ approaches a constant in agreement with Eq. (2).

min–max path of average length ℓ_∞,

$$R^*(a, \ell_\infty) = \left\langle \log\left(\frac{\sum_i \tau_i}{\tau_{max}}\right)\right\rangle, \tag{4}$$

where τ_{max} is the maximal weight encountered along the path, $\sum_i \tau_i$ is the total weight of the path and $\langle\quad\rangle$ denotes an average over network configurations and disorder realizations. $R^*(a, \ell_\infty)$ serves as a measure of the disorder because when we are in the ultrametric limit, $\sum_i \tau_i/\tau_{max} = 1$, and therefore $R^*(a, \ell_\infty) = 0$, and as the disorder becomes weak, $R^*(a, \ell_\infty)$ takes non-zero positive values. In the vicinity of strong disorder $R^*(a, \ell_\infty)$ is determined significantly by τ_2, the second highest link weight on the path. Let the random numbers corresponding to τ_{max} and τ_2 be r_{max} and r_2, respectively. We now use the result of the previous section that the min–max path (for the dominant portion of its length) can be identified as a path along the giant component of the network at percolation [11]. The probability distribution $\rho(r_{max})$ of the maximal random number r_{max} on a path of length ℓ_∞ on the giant component [12], takes significant values only in an interval $\Delta p_c = |p - p_c| \sim \ell_\infty^{-1/\nu}$ around p_c [13], where ν is the correlation length critical exponent for percolation. The probability distribution of the second highest random number r_2 given a maximal random number r_{max} obeys the same distribution but the distribution is truncated at r_{max}. By a similar reasoning the probability distribution of the remaining random numbers can also be obtained from $\rho(r_{max})$. We can write [13]

$$R^*(a, \ell_\infty) \equiv \left\langle \log \frac{\sum_i e^{ar_i}}{e^{ar_{max}}} \right\rangle \approx \left\langle \log(1 + e^{a(r_i - r_{max})}) \right\rangle \approx \left\langle e^{a(r_i - r_{max})} \right\rangle \approx P(|r_{max} - r_2| < 1/a) \, , \quad (5)$$

where $P(|r_{max} - r_2| < 1/a)$ is the probability that the highest and the second highest random numbers on the path (within the giant component) differ at least by $1/a$. Since, the distributions of r_{max} and r_2 take significant values only within an interval Δp_c of p_c, we have (see Fig. 4)

$$P(|r_{max} - r_2| < 1/a) = \begin{cases} 1, & \frac{1}{a} > \Delta p_c, \\ \frac{(1/a)}{\Delta p_c}, & \frac{1}{a} < \Delta p_c. \end{cases} \quad (6)$$

In the context of the transition, Eq. (6) can be interpreted as follows. When the average min–max path length of the network ℓ_∞ is such that $1/a > \ell_\infty^{-1/\nu}$, we have $a\Delta r < 1$, where $\Delta r = r_{max} - r_2$. This means that the maximal link cost and the second highest link cost along the min–max path are comparable in order of magnitude. In

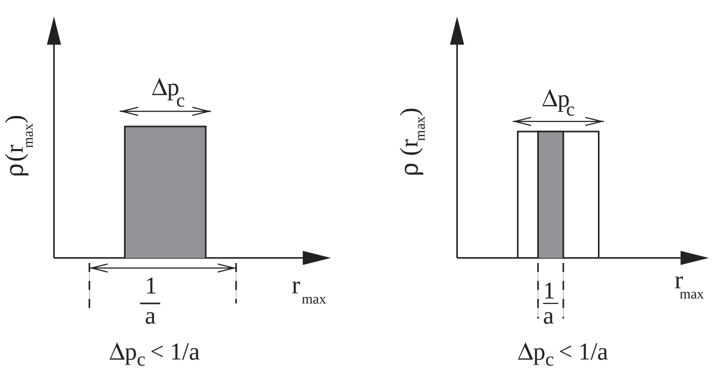

Fig. 4. Schematic representation of the probability distribution for the maximal random number r_{max} along a path of length ℓ_∞ along the percolation cluster. Relations between Δp_c and $1/a$ are the same as the cases presented in Eq. (6).

general, this will hold for any two consecutive link costs along the min–max path when the costs are ranked in descending order. This comes from the fact the probability distributions of all the random numbers along the path are of a similar form as the $\rho(r_{max})$, but with the distribution truncated at a certain value. Thus when $1/a > \ell_\infty^{-1/\nu}$, all link costs are typically the same order of magnitude and therefore contribute equally to the total cost of the path. Hence, the total cost of the path is proportional to its length. In such a case, it is not advantageous to follow the long min–max path to optimize the cost, and we can find a shortcut or deviation which reduces the cost. Therefore when $\ell_\infty < a^\nu \equiv \ell^*(a)$, the optimal path deviates from the min–max path and this deviation causes the scaling of the optimal path to crossover from the strong disorder regime into the weak disorder regime. Thus the crossover length at which the transition occurs is $\ell^*(a) \sim a$, since for infinite dimensional networks $\nu = 1$ [14]. A justification of the scaling ansatz proposed in Section 2 using this approach will be discussed elsewhere [15].

In summary, for both ER random networks and SF networks we obtain a scaling function for the crossover from weak disorder characteristics to strong disorder characteristics. We show that the crossover occurs when the min–max path reaches a crossover length $\ell^*(a)$ with $\ell^*(a) \sim a$. Equivalently, the crossover occurs when the network size N reaches a crossover size $N^*(a)$, where $N^*(a) \sim a^3$ for ER networks and for SF networks with $\lambda \geqslant 4$ and $N^*(a) \sim a^{(\lambda-1)/(\lambda-3)}$ for SF networks with $3 < \lambda < 4$. A theory for the transition in the case of SF networks with $\lambda \leqslant 3$ is still pending [16–18].

Acknowledgements

We thank the Office of Naval Research and ONR-Global, the Israel Science Foundation, and the Israeli Center for Complexity Science for financial support.

References

[1] M. Cieplak, A. Maritan, J.R. Banavar, Phys. Rev. Lett. 72 (1994) 2320;
M. Cieplak, A. Maritan, J.R. Banavar, Phys. Rev. Lett. 76 (1996) 3754.
[2] M. Porto, N. Schwartz, S. Havlin, A. Bunde, Phys. Rev. E 60 (1999) R2448.
[3] L.A. Braunstein, S.V. Buldyrev, S. Havlin, H.E. Stanley, Phys. Rev. E 65 (2001) 056128.
[4] L.A. Braunstein, S.V. Buldyrev, R. Cohen, S. Havlin, H.E. Stanley, Phys. Rev. Lett. 91 (2003) 168701.
[5] P. Erdős, A. Rényi, Publ. Math. 6 (1959) 290.
[6] R. Albert, A.-L. Barabási, Rev. Mod. Phys. 74 (2002) 47.
[7] A.S. Ioselevich, D.S. Lyubshin, JETP 79 (2004) 286.
[8] R. Cohen, K. Erez, D. ben-Avraham, S. Havlin, Phys. Rev. Lett. 85 (2000) 4626.
[9] G.J. Szabó, M. Alava, J. Kertész, Physica A 330 (2003) 31.
[10] S. Sreenivasan, T. Kalisky, L.A. Braunstein, S.V. Buldyrev, S. Havlin, H.E. Stanley, Phys. Rev. E (2004), in press.
[11] A. Bunde, S. Havlin, eds., Fractal and Disordered Systems, second ed., Springer, Berlin, 1996.
[12] A. Coniglio, J. Phys. A 15 (1982) 3829;
P. Grassberger, Y.C. Zhang, Physica A 224 (1996) 169.

[13] E. Perlsman, S. Havlin, (2004) unpublished.
[14] R. Cohen, D. ben-Avraham, S. Havlin, Phys. Rev. E 66 (2002) 036113.
[15] T. Kalisky et al., unpublished.
[16] M. Molloy, B. Reed, Random struct. Algor. 6 (1995) 161;
M. Molloy, B. Reed, Combin. Probab. Comput. 7 (1998) 295.
[17] S.V. Buldyrev, L.A. Braunstein, R. Cohen, S. Havlin, H.E. Stanley, Physica A 330 (2003) 246.
[18] L.A. Braunstein, S.V. Buldyrev, S. Sreenivasan, R. Cohen, S. Havlin, H.E. Stanley, in: E. Ben-Naim, H. Frauenfelder, Z. Toroczkai(Eds.), Lecture Notes in Physics, Springer, Berlin, 2004.