Effect of disorder strength on optimal paths in complex networks

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We study the transition between the strong and weak disorder regimes in the scaling properties of the average optimal path $\ell_{\rm opt}$ in a disordered Erdős-Rényi (ER) random network and scale-free (SF) network. Each link i is associated with a weight $\tau_i \equiv \exp(ar_i)$, where r_i is a random number taken from a uniform distribution between 0 and 1 and the parameter a controls the strength of the disorder. We find that for any finite a, there is a crossover network size $N^*(a)$ at which the transition occurs. For $N \ll N^*(a)$ the scaling behavior of $\ell_{\rm opt}$ is in the strong disorder regime, with $\ell_{\rm opt} \sim N^{1/3}$ for ER networks and for SF networks with $\lambda \geqslant 4$, and $\ell_{\rm opt} \sim N^{(\lambda-3)/(\lambda-1)}$ for SF networks with $3<\lambda<4$. For $N \gg N^*(a)$ the scaling behavior is in the weak disorder regime, with $\ell_{\rm opt} \sim \ln N$ for ER networks and SF networks with $\lambda>3$. In order to study the transition we propose a measure which indicates how close or far the disordered network is from the limit of strong disorder. We propose a scaling ansatz for this measure and demonstrate its validity. We proceed to derive the scaling relation between $N^*(a)$ and a. We find that $N^*(a) \sim a^3$ for ER networks and for SF networks with $\lambda \geqslant 4$, and $N^*(a) \sim a^{(\lambda-1)/(\lambda-3)}$ for SF networks with $3<\lambda<4$.

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I. INTRODUCTION

The subject of complex networks has been widely explored in the past few years due in part to its broad range of applications to social, biological, and communication systems [1–6]. In a real world network, whether it be a communication network or transport network, the time τ_i taken to traverse a link i may not be the same for all the links. In other words, there is a "cost" or a "weight" τ_i associated with each link, and the larger the weight on a link, the harder it is to traverse this link. In such a case, the network is said to be disordered.

Consider two nodes A and B on such a disordered network. In general, there will be a large number of paths connecting A and B. Among these paths, there is usually a single path for which the sum of the costs $\Sigma \tau_i$ along the path is minimum and this path is called the "optimal path." The problem of optimal paths on networks is of importance since the purpose of many real networks is to provide an efficient traffic route between its nodes.

When most of the links on the path contribute to the sum, the system is said to be "weakly disordered" (WD). In some cases, however, the cost of a single link along the path dominates the sum. In this case, every path between two nodes can be characterized by a value equal to the maximum cost along that path, and the path with the minimal value of the maximum cost is the optimal path between the two nodes. This limit of disorder is called the *strong disorder* (SD) limit ("ultrametric" limit) [7] and we refer to the optimal path in this limit as the *min-max path*.

The procedure to implement disorder on a network is as follows [7–10]. One assigns to each link i of the network a random number r_i, uniformly distributed between 0 and 1. The cost associated with link i is then

$$\tau_i \equiv \exp(ar_i), \tag{1}$$

where a is the parameter which controls the broadness of the distribution of link costs. The parameter a represents the strength of disorder. The limit $a \to \infty$ is the strong disorder limit, since for this case only one link dominates the cost of the path.

There are distinct scaling relationships between the length of the average optimal path $\ell_{\rm opt}$ and the network size (number of nodes) N depending on whether the network is strongly or weakly disordered [10]. For strong disorder [10], $\ell_{\rm opt} \sim N^{\nu_{\rm opt}}$, where $\nu_{\rm opt}=1/3$ for Erdős-Rényi (ER) random networks [11] and for scale-free (SF) [1] networks with $\lambda>4$, where λ is the exponent characterizing the power law decay of the degree distribution. For SF networks with $3<\lambda<4$, $\nu_{\rm opt}=(\lambda-3)/(\lambda-1)$. For weakly disordered ER networks and for SF networks with $\lambda>3$, $\ell_{\rm opt} \sim \ln N$. Porto *et al.* [8] considered the optimal path transition from weak to strong disorder for two-dimensional and three-dimensional lattices, and found a crossover in the scaling properties of the optimal path that depends on the disorder strength a, as well as on the lattice size L.

Here we show that similar to regular lattices, there exists for any finite a, a crossover network size $N^*(a)$ such that for $N \ll N^*(a)$, the scaling properties of the optimal path are in the strong disorder regime while for $N \gg N^*(a)$, the network is in the weak disorder regime. We evaluate the function $N^*(a)$. The structure of the paper is as follows. In Sec. II we derive a scaling approach for the transition from weak disorder to strong disorder of the optimal path. In Sec. III we present simulation results which support the scaling ansatz.

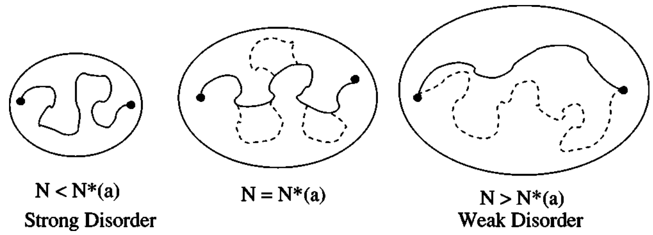

FIG. 1. Schematic representation of the transition in the topology of the optimal path with system size N for a given disorder strength a. The solid line shows the optimal path at a finite value of a connecting two nodes indicated by the filled circles. The portion of the min-max path that is distinct from the optimal path is indicated by the dashed line. (a) For $N \ll N^*(a)$ [i.e., $\ell_\infty \ll \ell^*(a)$], the optimal path coincides with the min-max path, and we expect the statistics of the SD limit. (b) For $N=N^*(a)$ [i.e., $\ell_\infty=\ell^*(a)$], the optimal path starts deviating from the min-max path. (c) For $N \gg N^*(a)$ [i.e., $\ell_\infty \gg \ell^*(a)$], the optimal path has almost no links in common with the min-max path, and we expect the statistics of the WD limit.

Finally, in Sec. IV we conclude with an analytic justification for the scaling of the transition.

II. SCALING APPROACH

In general, the average optimal path length $\ell_{\rm opt}(a)$ in a disordered network depends on a as well as on N. In the following we use instead of N the min-max path length ℓ_∞ which is related to N as $\ell_\infty \equiv \ell_{\rm opt}(\infty) \sim N^{\nu_{\rm opt}}$. Hence, N can be expressed as a function of ℓ_∞,

$$N \sim \ell_\infty^{1/\nu_{\rm opt}}. \tag{2}$$

Thus, for finite a, $\ell_{\rm opt}(a)$ depends on both a and ℓ_∞. We expect that there exists a crossover length $\ell^*(a)$, corresponding to the crossover network size $N^*(a)$, such that (i) for $\ell_\infty \ll \ell^*(a)$, the scaling properties of $\ell_{\rm opt}(a)$ are those of the strong disorder regime, and (ii) for $\ell_\infty \gg \ell^*(a)$, the scaling properties of $\ell_{\rm opt}(a)$ are those of the weak disorder regime. In Fig. 1, we show a schematic representation of the change of the optimal path as the network size increases.

In order to study the transition from strong to weak disorder, we introduce a measure which indicates how close or far the disordered network is from the limit of strong disorder. A natural measure is the ratio

$$W(a) \equiv \frac{\ell_{\rm opt}(a)}{\ell_\infty}. \tag{3}$$

Using the scaling relationships between $\ell_{\rm opt}(a)$ and N in both regimes, and $\ell_\infty \sim N^{\nu_{\rm opt}}$ (see Sec. I), we get

$$\ell_{\rm opt}(a) \sim \begin{cases} \ell_\infty \sim N^{\nu_{\rm opt}} & [\text{SD}] \\ \ln \ell_\infty \sim \ln N & [\text{WD}]. \end{cases} \tag{4}$$

From Eqs. (3) and (4) it follows:

$$W(a) \sim \begin{cases} \text{const.} & [\text{SD}] \\ \ln \ell_\infty/\ell_\infty & [\text{WD}]. \end{cases} \tag{5}$$

We propose the following scaling ansatz for $W(a)$:

$$W(a) = F\left(\frac{\ell_\infty}{\ell^*(a)}\right), \tag{6}$$

where

$$F(u) \sim \begin{cases} \text{const.} & u \ll 1 \\ \ln(u)/u & u \gg 1, \end{cases} \tag{7}$$

with

$$u \equiv \frac{\ell_\infty}{\ell^*(a)}. \tag{8}$$

The dependence of $\ell^*(a)$ on a can be estimated as follows. In the strong disorder limit, the cost on the links for any path on the network typically differ by at least an order of magnitude. This means that for a min-max path of ℓ link (or length ℓ), if we arrange the costs of the links in descending order, then two consecutive costs typically differ at least by an order of magnitude. If r_1 and r_2 are the random numbers associated with two such consecutive links, with $r_1 > r_2$, then the ratio of the costs on the links is

$$\frac{\tau_1}{\tau_2} = \exp(a\Delta r), \tag{9}$$

where $\Delta r \equiv r_1 - r_2$. Thus, in the case of strong disorder we must have $a\Delta r \gg 1$. Consequently the transition to weak disorder occurs when all the links become equivalent in order of magnitude, i.e., when $a\Delta r \sim 1$. The value of Δr depends on the length of the path. If the distribution of random numbers on the min-max path is uniform, then $\Delta r \sim 1/\ell$ for a min-max path of length ℓ. The condition for the transition, $a\Delta r \sim 1$ is satisfied at the crossover length $\ell^*(a)$ which implies that

$$\ell^*(a) \sim a. \tag{10}$$

Therefore, from Eq. (6), $W(a)$ must be a function of ℓ_∞/a.

III. SIMULATION RESULTS

Next we describe the details of our numerical simulations and show that the results agree with our theoretical predictions. To construct an ER network of size N with average node degree $\langle k \rangle$, we start with $\langle k \rangle N/2$ edges and randomly pick a pair of nodes from the total possible $N(N-1)/2$ pairs to connect with each edge. The only condition we impose is that there cannot be multiple edges between two nodes.

In order to generate SF networks, we use the Molloy-Reed algorithm [12]. Each node is assigned a random integer k taken from a power-law distribution

$$P(k) = \left(\frac{k}{k_0}\right)^{-\lambda}, \tag{11}$$

where k_0 is the minimal possible number of links that a node possesses. Next, we randomly select a node and attempt to connect each of its k links with randomly selected k nodes that still have free positions for links. The disorder in the link costs is then implemented using the procedure described in Ref. [9].

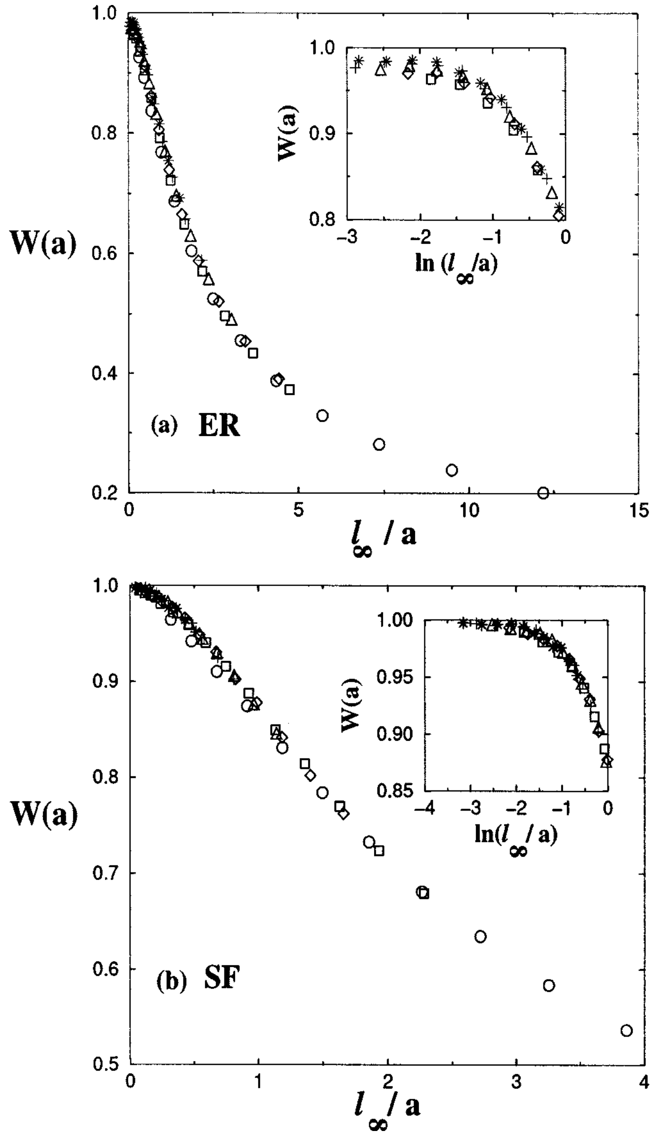

FIG. 2. Test of Eqs. (6) and (7). (a) $W(a)$ plotted as a function of ℓ_∞/a for different values of a for ER networks with $\langle k\rangle=4$. The different symbols represent different a values: $a=8(\bigcirc)$, $a=16(\square)$, $a=22(\diamond)$, $a=32(\triangle)$, $a=45(+)$, and $a=64(*)$. (b) Same for SF networks with $\lambda=3.5$. The symbols correspond to the same values of disorder as in (a). The insets show $W(a)$ plotted against $\log(l_\infty/a)$, and indicate for $\ell_\infty \ll a$, $W(a)$ approaches a constant in agreement with Eq. (7).

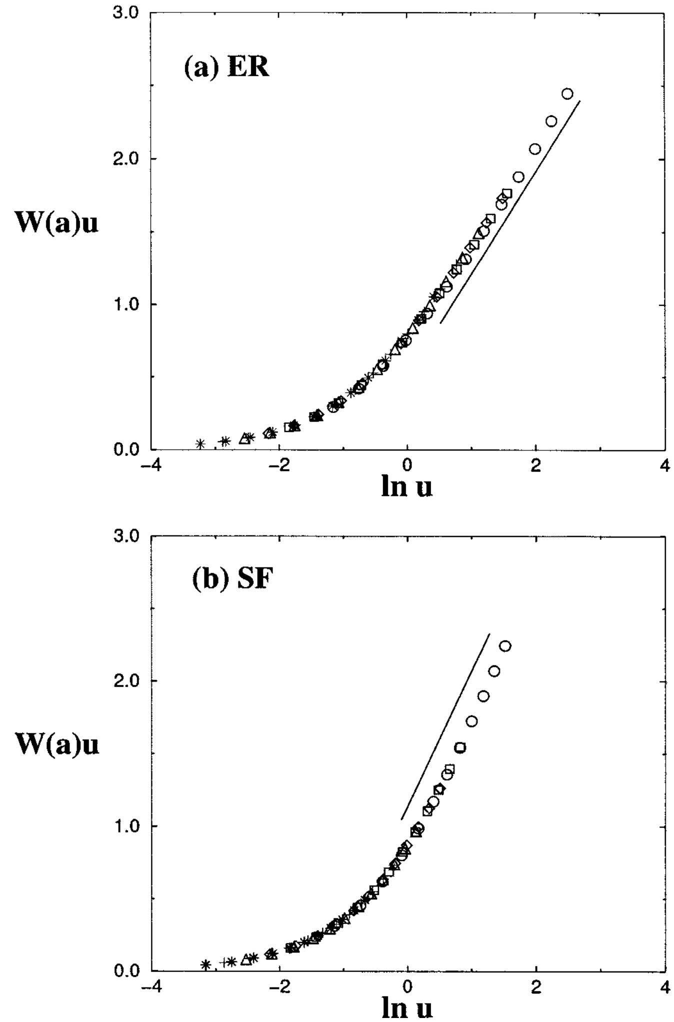

FIG. 3. (a) Plot of $W(a)u=\ell_{\rm opt}(a)/\ell^*(a)=\ell_{\rm opt}(a)/a$ vs $\ln u \equiv \ln[\ell_\infty/\ell^*(a)]=\ln(\ell_\infty/a)$ for ER networks with $\langle k\rangle=4$ for different values of a. (b) Plot of $W(a)u=\ell_{\rm opt}(a)/\ell^*(a)=\ell_{\rm opt}(a)/a$ vs $\ln u =\ln[\ell_\infty/\ell^*(a)]=\ln(\ell_\infty/a)$ for SF networks with $\lambda=3.5$. The values of a represented by the symbols in (a) and (b) are the same as in Fig. 2.

For the ER networks generated we use $\langle k\rangle=4$ and for the SF networks we use $k_0=2$. These parameter values ensure that the networks generated are almost surely fully connected [13].

To obtain ℓ_∞, we use the algorithm proposed by Cieplak *et al.* [7], modified as described in Ref. [14]. With this modification we reach system sizes of $N=2^{16}=65\,536$. In order to obtain the optimal path for a given realization, we use the Dijkstra algorithm [8]. We calculate the average optimal path $\ell_{\rm opt}(a)$ by taking the average of the optimal paths over 10^6 pairs of nodes.

In Fig. 2 we show the ratio $W(a)$ for different values of a plotted against $\ell_\infty/\ell^*(a)\equiv\ell_\infty/a$ for ER networks with $\langle k\rangle=4$ and for SF networks with $\lambda=3.5$. The excellent data collapse is consistent with the scaling relations Eq. (6). Figure 3 shows the scaled quantities $W(a)u=\ell_{\rm opt}(a)/\ell^*(a)$ vs $\ln u \equiv \ln[\ell_\infty/\ell^*(a)]\equiv\ln(\ell_\infty/a)$, for both ER networks with $\langle k\rangle=4$ and for SF networks with $\lambda=3.5$. The curves are linear at large $u\equiv\ell_\infty/\ell^*(a)$, supporting the validity of the logarithmic term in Eq. (7) for large u.

IV. DISCUSSION

We next develop analytic arguments that support Eq. (10). These arguments will lead to a clearer picture about the nature of the transition of the optimal path with disorder strength.

We begin by making a few observations about the min-max path. In Fig. 4 we plot the average value of the random numbers r_n on the min-max path as a function of their rank n $(1\leqslant n\leqslant\ell_\infty)$ for ER networks with $\langle k\rangle=4$ and for SF net-

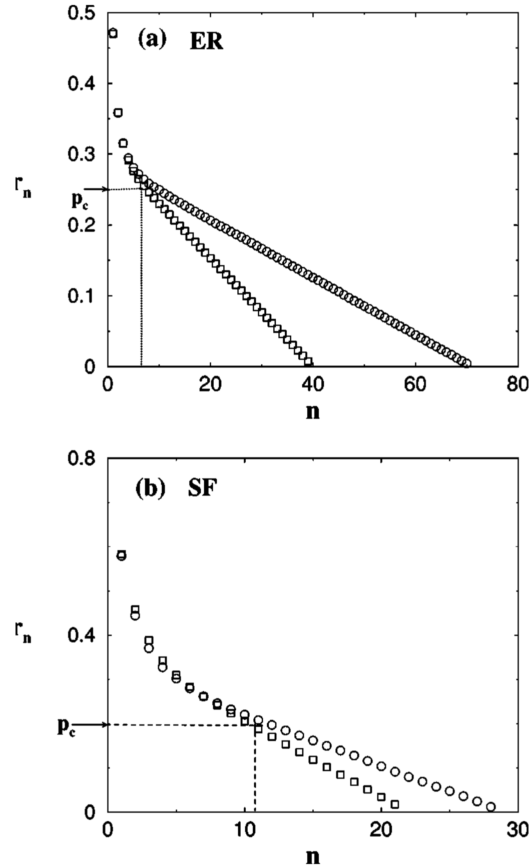

FIG. 4. Dependence on rank n of the average values of the random numbers r_n along the most probable optimal path for (a) ER random networks of two different sizes $N=4096$ (□) and $N=16384$ (○) and (b) SF random networks.

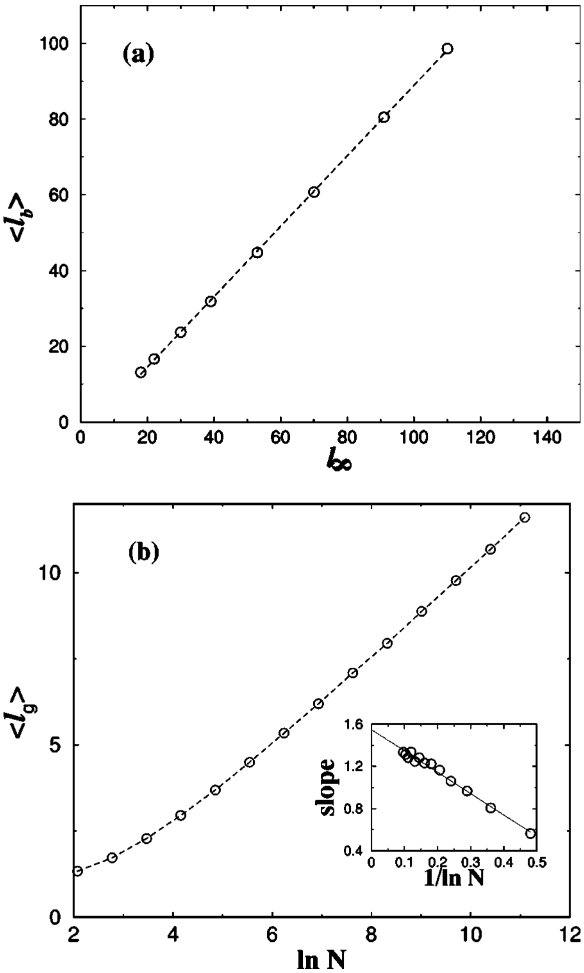

FIG. 5. (a) The average number of links $\langle \ell_b \rangle$ with random number values $r \leqslant p_c$ on the min-max path plotted as a function of its length ℓ_∞ for an ER network with $\langle k \rangle = 4$, showing that $\langle \ell_b \rangle$ grows linearly with ℓ_∞. (b) The average number of links $\langle \ell_g \rangle$ with random number values $r > p_c$ on the min-max path vs $\ln N$ for an ER network with $\langle k \rangle = 4$, showing that $\langle \ell_g \rangle \sim \ln N$. The inset shows the successive slopes, indicating that in the asymptotic limit $\langle \ell_g \rangle \approx 1.55 \ln N$.

works with $\lambda = 3.5$. This can be done for a min-max path of any length but in order to get good statistics we use the most probable min-max path length [15]. We call links with $r \leqslant p_c$ "black" links, and links with $r > p_c$ "gray" links, following the terminology of Ioselevich and Lyubshin [16] where p_c is the percolation threshold of the network [13].

We make the following observations regarding the min-max path.

(i) For $r_n < p_c$, the values of r_n decrease linearly with rank n, implying that the values of r for black links are uniformly distributed between 0 and p_c, consistent with the results of Ref. [17]. This is shown in Fig. 4.

(ii) The average number of black links, $\langle \ell_b \rangle$, along the min-max path increases linearly with the average path length ℓ_∞. This is shown in Fig. 5(a).

(iii) The average number of gray links $\langle \ell_g \rangle$ along the min-max path increases logarithmically with the average path length ℓ_∞ or, equivalently, with the network size N. This is shown in Fig. 5(b).

The simulation results presented in Fig. 5 pertain to ER networks; however, we have confirmed that the observations (ii) and (iii) also hold for SF networks.

Next we will discuss our observations using the concept of the optimal spanning tree. The *optimal spanning tree* (OST) is a subset of links of a connected graph which provides an optimal path from node A (which serves as the root of the tree) to any other node on the graph. When the total weight of this path is dominated by the largest weight of the links along the path (strong disorder limit), the OST does not depend on the root and is determined only by the structure of the original graph and a particular realization of the disorder. In this limit, the OST becomes identical to the *minimal spanning tree* (MST) [17,18]. The path on the MST between any two nodes A and B, is the optimal path between the nodes in the strong disorder limit—i.e, the min-max path.

To construct the MST, we remove links in the descending order of their costs τ_i. If removal of a link destroys the connectivity of the graph, we restore that link. This procedure is continued until there are exactly $N-1$ links remaining. At this point the number of remaining black links is

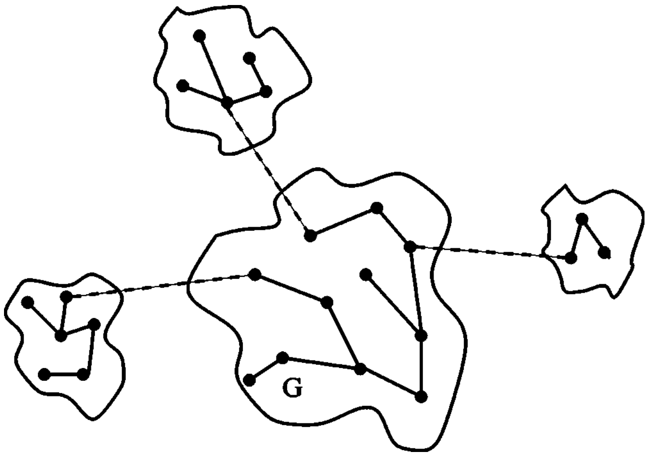

FIG. 6. Schematic representation of the structure of the minimal spanning tree, at the percolation threshold, with G being the giant component. Inside each cluster, the nodes are connected by black links to form a tree. The dotted lines represent the gray links which connect the finite clusters to form the gray tree. In this example $N_c=4$ and the number of gray links equals $N_c-1=3$.

$$N_b = \frac{N\langle k\rangle p_c}{2}, \tag{12}$$

where $\langle k\rangle$ is the average degree of the original graph and p_c is given by [13]:

$$p_c = \frac{\langle k\rangle}{\langle k^2 - k\rangle}. \tag{13}$$

The black links give rise to N_c disconnected clusters. One of these is a spanning cluster, called the *giant component*. The N_c clusters are linked together into a connected tree by exactly N_c-1 gray links (see Fig. 6). Each of the N_c clusters is itself a tree, since a random graph can be regarded as an infinite dimensional system, and at the percolation threshold in an infinite dimensional system the clusters can be regarded as trees. Thus the N_c clusters containing N_b black links, together with N_c-1 gray links form a spanning tree consisting of N_b+N_c-1 links.

Thus the MST provides a min-max path between any two points on the graph. Since the MST connects N nodes, the number of links on this tree must be equal to $N-1$, so

$$N_b + N_c = N. \tag{14}$$

From Eqs. (12) and (14), it follows that

$$N_c = N\left(1 - \frac{\langle k\rangle p_c}{2}\right). \tag{15}$$

Therefore, N_c is proportional to N.

The path between any two nodes on the MST consists of ℓ_b black links. Since the black links are the links that remain after removing all links with $r>p_c$, the random number values r on the black links are uniformly distributed between 0 and p_c, in agreement with observation (i) and Ref. [17].

Since there are N_c clusters which include clusters of nodes connected by black links as well as isolated nodes, the MST can be described as an effective tree of N_c nodes, each representing a cluster, and N_c-1 gray links. We call this tree the "gray tree" (see Fig. 6). This tree is in fact a scale-free tree [19,20] with degree exponent $\lambda_g=2.5$ for ER networks and for scale-free networks with $\lambda \geqslant 4$, and $\lambda_g=(2\lambda-3)/(\lambda-2)$ for SF networks with $3<\lambda<4$. If we take two nodes A and B on our original network, they will most likely lie on two distinct effective nodes of the gray tree. The number of gray links encountered on the min-max path connecting these two nodes will therefore equal the number of links separating the effective nodes on the gray tree. Hence, the average number of gray links $\langle \ell_g\rangle$ encountered on the min-max path between an arbitrary pair of nodes on the network is simply the average diameter of the gray tree. Our simulation results [see Fig. 5(b)] indicate that

$$\langle \ell_g\rangle \sim \ln N. \tag{16}$$

Since $\langle \ell_g\rangle \sim \ln \ell_\infty \ll \ell_\infty$, the average number of black links $\langle \ell_b\rangle$ on the min-max path scales as ℓ_∞ in the limit of large ℓ_∞, in agreement with observation (ii) as shown in Fig. 5(a).

Now we will discuss the implications of our findings for the crossover from strong to weak disorder. From observations (i) and (ii), it follows that for the portion of the path belonging to the giant component, the distribution of random values r is uniform. Hence, we can approximate the sum of weights by an integral

$$\sum_{k=1}^{\ell_b} \exp(ar_k) \approx \frac{\ell_b}{p_c}\int_0^{p_c} \exp ar\, dr$$
$$= \frac{\ell_b}{ap_c}[\exp(ap_c) - 1]$$
$$\equiv \exp(ar^*), \tag{17}$$

where $r^* \approx p_c + (1/a)\ln(\langle \ell_b\rangle/ap_c)$. Since $\langle \ell_b\rangle \approx \ell_\infty$:

$$r^* \approx p_c + \frac{1}{a}\ln\left(\frac{\ell_\infty}{ap_c}\right). \tag{18}$$

Thus restoring a short-cut link between two nodes on the optimal path with $p_c<r<r^*$ may drastically reduce the length of the optimal path. When $ap_c \gg \ell_\infty$, $r^*<p_c$ and such a link does not exist, but there starts to be a finite probability for such a link to exist if $\ell_\infty > ap_c$. Hence, when the min-max path is of length $\ell_\infty \approx ap_c$, the optimal path starts deviating from the min-max path. The length of the min-max path at which the deviation first occurs is precisely the crossover length $\ell^*(a)$, and therefore $\ell^*(a) \sim ap_c$. In the case of a network with an arbitrary degree distribution we can write using Eq. (13), $\ell^*(a) \sim a\langle k\rangle/\langle k^2-k\rangle$. Note that in the case of SF networks, as $\lambda \to 3^+$, p_c approaches zero and consequently $\ell^*(a) \to 0$. This suggests that for any finite value of disorder strength a, a SF network with $\lambda \leqslant 3$ is in the weak disorder regime.

In summary, for both ER random networks and SF networks we obtain a scaling function for the crossover from weak disorder characteristics to strong disorder characteristics. We show that the crossover occurs when the min-max path reaches a crossover length $\ell^*(a)$ and $\ell^*(a) \sim a$. Equivalently, the crossover occurs when the network size N reaches a crossover size $N^*(a)$, where $N^*(a) \sim a^3$ for ER networks and for SF networks with $\lambda \geqslant 4$ and for SF networks with $3 < \lambda < 4$.

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