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Diffusion in the presence of quenched random bias fields: A two-dimensional generalization of the Sinai model

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We report a theoretical and numerical study of diffusion in two dimensions in the presence of quenched random bias fields. The local bias field is taken to be the gradient of a random scalar potential $V(i,j)$. We consider the special case $V(i,j)=V_1(i)+V_2(j)$, where the gradients of V_1 and V_2 are chosen to be randomly $\pm\epsilon_0$ with $0<\epsilon_0\leq 1$. We find that asymptotically ($t\to\infty$) the mean-square displacement grows with the time t as $(\ln t)^4$, just as in the one-dimensional Sinai model.

Perhaps the simplest example of diffusion in a nonuniform medium is the model studied by Sinai.[1] Sinai considered the motion of a diffusing particle on a one-dimensional lattice, where the motion of the particle is discretized in both time and space. At each site i in the lattice there is a randomly chosen "bias field" ϵ_i, with $-1\leq\epsilon_i\leq+1$. A value of $\epsilon_i>0$ indicates a bias to the right, while $\epsilon_i<0$ indicates a bias to the left. Thus if $\omega(i\to j)$ denotes the transition probability per unit time from site i to j,

$$\omega(i\to i+1)=\frac{1+\epsilon_i}{2},\qquad \omega(i\to i-1)=\frac{1-\epsilon_i}{2}. \tag{1}$$

Sinai showed that if $\langle\ln(1+\epsilon_i)/(1-\epsilon_i)\rangle=0$, then the mean-square displacement of the diffusing particle $\overline{\langle x^2\rangle_t}$ behaves as

$$\overline{\langle x^2\rangle_t}\sim(\ln t)^4. \tag{2}$$

Here the overbar denotes averaging over walks and the brackets denote averaging over configurations of the bias fields ϵ_i. These bias fields may be chosen according to any probability distribution function which satisfies the above condition. In particular, we may choose the probability distribution function $P(\epsilon)=\frac{1}{2}\delta(\epsilon-\epsilon_0)+\frac{1}{2}\delta(\epsilon+\epsilon_0)$. Thus each ϵ_i takes values $\pm\epsilon_0$ with equal probability, with $0\leq\epsilon_0\leq 1$. An example of such a configuration is shown in Fig. 1. The limit $\epsilon_0\to 0$ represents a uniform medium, in which one finds the usual diffusion law

$$\langle x^2\rangle\sim t. \tag{3}$$

The master equation for the probability density $\rho(i,t)$ of a diffusing particle on a one-dimensional lattice is

$$\rho(i,t+1)=\rho(i-1,t)\omega(i-1\to i)+\rho(i+1,t)\omega(i+1\to i). \tag{4}$$

Substituting the transition probabilities from Eq. (1) and rearranging yields

$$\rho(i,t+1)-\rho(i,t)=\tfrac{1}{2}\{[\rho(i+1,t)-\rho(i,t)]-[\rho(i,t)-\rho(i-1,t)]\}-\tfrac{1}{2}[\rho(i+1,t)\epsilon_{i+1}-\rho(i-1,t)\epsilon_{i-1}]. \tag{5}$$

This is a discretization of the equivalent Fokker-Planck equation,

$$\frac{d\rho}{dt}=D\frac{d^2\rho}{dx^2}-\frac{d}{dx}(\rho F), \tag{6}$$

where F is a continuous local drift force which corresponds to the discrete bias fields ϵ_i, and the diffusion constant $D=\frac{1}{2}$.

The one-dimensional Sinai model has been well studied.[1-6] Several investigators have also studied models of diffusion in two dimensions with quenched random bias fields. The two-dimensional problem is fundamentally

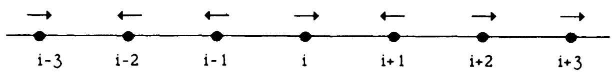

FIG. 1. A one-dimensional lattice with local bias fields ϵ_i indicated by arrows. A right arrow indicates $\epsilon_i = +\epsilon_0$; a left arrow indicates $\epsilon_i = -\epsilon_0$.

more complicated, because walkers may circumvent barriers and wells instead of going through them. Marinari *et al.*[7] introduced a two-dimensional model of diffusion with tunable random bias fields. To each site on a square lattice they assigned four random variables Q_j, and then defined π_j, the transition probability in each direction j, as

$$\pi_j = \frac{(Q_j)^K}{\sum_{m=1}^{4} (Q_m)^K} . \qquad (7)$$

Here Q_j are random variables in the range $0 \le Q_j \le 1$, and K is an adjustable parameter which tunes between the simple random walk ($K=0$) and the deterministic walk ($K \to \infty$). In the latter case, at each site one direction j will have $\pi_j = 1$, and the walk is therefore deterministic. Marinari *et al.*[7] carried out numerical simulations which showed that for $K=1$ and 2 there is normal diffusion with $\overline{\langle x^2 \rangle} \sim t$, and for $K=7$, anomalous diffusion with $\overline{\langle x^2 \rangle} \sim t^{0.6}$. For even higher K, the results appeared to be consistent with the one-dimension Sinai-model result $\overline{\langle x^2 \rangle} \sim (\ln t)^4$. Thus they suggested that this two-dimensional model obeyed the same scaling law as the one-dimensional model. However, the authors warned that the calculations were not conclusive and that analytic investigation was necessary.

Fisher[8] and Luck[9] argued that these results were perhaps not entirely correct. They showed that two is the upper critical dimension for random walks in the presence of a nonsingular spatially random bias field, or drift force. That is, above two dimensions, they predicted convergence to normal diffusive behavior, $\overline{\langle x^2 \rangle} \sim t$. In two dimensions they predicted universal logarithmic corrections. These conclusions were based on a renormalization-group expansion about the limit of weak disorder. Fisher noted that these results might not be valid in the limit of a singular distribution of hopping probabilities, which could, in principle, explain the discrepancy between the theoretical predictions and the numerical simulations of Marinari *et al.*[7] discussed above.

Later, Fisher *et al.*[10] extended these calculations to consider spatially random drift forces of different types. They found a variety of different behaviors. For a purely transverse, or divergence-free drift force, they found superdiffusive behavior,

$$\overline{\langle x^2 \rangle} \sim t (\ln t)^{1/2} . \qquad (8)$$

For a purely longitudinal, or curl-free drift force, they predicted subdiffusive behavior, and conjectured that it might take the form

$$\overline{\langle x^2 \rangle} \sim t \exp[-c(\ln t)^{(n-2)/(n-1)}] . \qquad (9)$$

Here c is a universal constant, and n is the order of the unknown next term in the recursion relation calculated in Ref. 10. They predicted that if the drift force has independent divergence-free and curl-free parts, then the behavior would be diffusive with logarithmic corrections. They found subdiffusive behavior if the two components of the force are parallel and perpendicular to the gradients of a scalar potential.

An example of diffusion with a divergence-free drift force is the diffusion of light particles in an incompressible fluid which flows through a disordered porous medium.[11] Diffusion with a curl-free drift force is perhaps more common in nature; an example is diffusion of a particle on a rough surface in the presence of gravity or the diffusion of a charged particle in a disordered electric potential.

The situation in more than one dimension is less clear than the one-dimensional case, where the Sinai result is exact and its generalizations can be obtained.[12] Exactly soluble models may therefore play an important role in understanding the origin of the physical behaviors that may be encountered in higher dimensions.

Here we introduce a model of diffusion in two dimensions on a discrete lattice in the presence of a random potential $V(i,j)$ which has the symmetry

$$V(i,j) = V_1(i) + V_2(j) . \qquad (10)$$

$V_1(i)$ is a one-dimensional random potential with gradients ϵ_i randomly distributed according to

$$P(\epsilon) = \tfrac{1}{2}\delta(\epsilon - \epsilon_0) + \tfrac{1}{2}\delta(\epsilon + \epsilon_0), \quad 0 \le \epsilon_0 \le 1 . \qquad (11)$$

That is, the gradients of $V_1(i)$ take on the values $\pm\epsilon_0$ with equal probability. $V_2(j)$ is equivalently constructed.

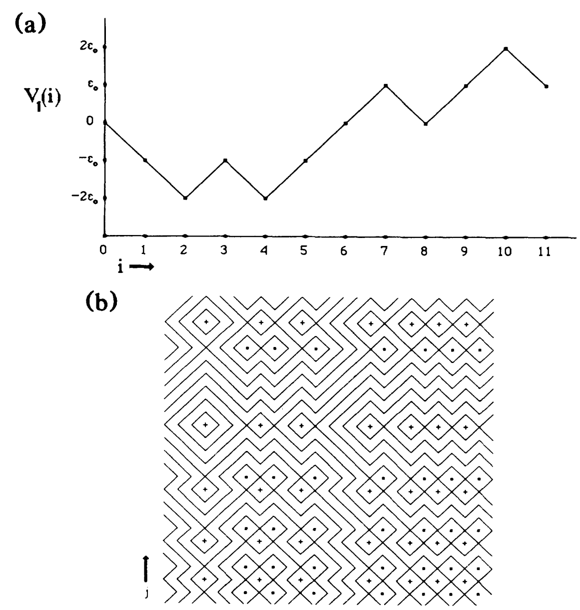

FIG. 2. (a) A realization of the one-dimensional random potential $V_1(i)$. (b) A realization of the two-dimensional random potential $V(i,j) = V_1(i) + V_2(j)$, shown as a contour map. Local maxima are marked by a plus sign and local minima by a square. The maxima and minima lie in rows and columns. A saddle point occurs where contour lines cross. Saddle points are found at the intersection of a row of maxima (minima) with a column of minima (maxima).

A typical realization of $V_1(i)$ is shown in Fig. 2(a), and a typical realization of the two-dimensional potential $V(i,j)$ is shown in Fig. 2(b) as a contour map. This potential, visualized as a surface, has several unusual properties. Because the local gradient is everywhere $\pm\epsilon_0$, even and odd values of $V(i,j)/\epsilon_0$ fall in a checkerboard pattern. Local maxima fall in unevenly spaced rows and columns, as do local minima. A saddle point is found at each intersection of a row of maxima with a column of minima, and at each intersection of a row of minima with a column of maxima. Because of the way in which the surface is constructed, the spacing m between a row of minima and a row of maxima is randomly distributed according to the probability distribution $P(m)=(\frac{1}{2})^m$.

A diffusing particle on this surface is assigned the transition probabilities

$$\omega(i,j\rightarrow k,l)=\tfrac{1}{8}\{1+[V(i,j)-V(k,l)]\}\,, \tag{12}$$

where (i,j) and (k,l) are nearest-neighbor sites. The particle has a nonzero "staying probability" $\omega(i,j\rightarrow i,j)$ when the sum of the transition probabilities to the nearest-neighbor sites is less than unity.

Note that the limit $\epsilon_0=0$ corresponds to the simple random walk. In the limit $\epsilon_0=1$, all "uphill" steps are forbidden and all "downhill" steps have equal probability. Thus a diffusing particle will inevitably become permanently trapped in a local minimum.

The master equation describing the evolution of the probability density $\rho(i,j)$ according to Eq. (12) is

$$\rho_{t+1}(i,j)=\sum_{(k,l)}\rho_t(k,l)\tfrac{1}{8}[1+V(k,l)-V(i,j)]+\rho_t(i,j)\left[1-\sum_{(k,l)}\tfrac{1}{8}[1+V(i,j)-V(k,l)]\right], \tag{13}$$

where the sums on (k,l) are over nearest neighbors of (i,j). The first term on the right-hand side represents the inflow of particles from nearest-neighbor sites, while the second term represents the particles that stayed at (i,j) from time t to $t+1$. Rearrangement yields

$$\rho_{t+1}(i,j)-\rho_t(i,j)=\tfrac{1}{8}\sum_{(k,l)}[\rho_t(k,l)-\rho_t(i,j)]+\tfrac{1}{4}\sum_{(k,l)}\frac{[\rho_t(k,l)+\rho_t(i,j)]}{2}[V(k,l)-V(i,j)]\,. \tag{14}$$

Thus in two dimensions we again find a discretized version of the Fokker-Planck equation for diffusion with drift force $\mathbf{F}$,

$$\frac{d\rho}{dt}=D\nabla^2\rho-\nabla\cdot(\mathbf{F}\rho)\,. \tag{15}$$

Here $D=\frac{1}{8}$ is the diffusion constant, and $\mathbf{F}=-\frac{1}{4}\nabla V$ is the local drift force. In higher dimensions d, $D=2^{-(d+1)}$ and $\mathbf{F}=-(2^{-d})\nabla V$.[13]

To prove that $\overline{\langle x^2\rangle}\sim(\ln t)^4$, we first carry out a separation of variables for Eq. (15). The calculation may be carried out in either the discretized or the continuum language. For convenience we employ the continuum notation. Recall that the two-dimensional potential $V(x,y)$ has the symmetry $V(x,y)=V_1(x)+V_2(y)$. The drift force $\mathbf{F}$ may be written as

$$\mathbf{F}=F_1(x)\hat{x}+F_2(y)\hat{y}\,, \tag{16}$$

where

$$F_1(x)=-\frac{1}{4}\frac{dV_1}{dx},\quad F_2(y)=-\frac{1}{4}\frac{dV_2}{dy}\,. \tag{17}$$

The probability density $\rho(x,y)$ may be written as

$$\rho(x,y)=\rho_1(x)\rho_2(y)\,. \tag{18}$$

Substituting Eqs. (16) and (18) into Eq. (15) we find

$$\rho_1\frac{d\rho_2}{dt}+\rho_2\frac{d\rho_1}{dt}=D\left[\rho_2\frac{d^2\rho_1}{dx^2}+\rho^2\frac{d^2\rho_2}{dy^2}\right]-\rho_2\left[\frac{d\rho_1}{dx}F_1+\rho_1\frac{dF_1}{dx}\right]-\rho_1\left[\frac{d\rho_2}{dy}F_2+\rho_2\frac{dF_2}{dy}\right]. \tag{19}$$

Dividing both sides by $\rho_1\rho_2$ and further rearrangement yields the separated equations

$$\frac{d\rho_1}{dt}=D\frac{d\rho_1}{dx^2}-\frac{d(\rho_1F_1)}{dx}\,,$$
$$\frac{d\rho_2}{dt}=D\frac{d\rho_2}{dy^2}-\frac{d(\rho_2F_2)}{dy}\,. \tag{20}$$

Thus, the two dimensions in the problem have been decoupled; both ρ_1 and ρ_2 obey the one-dimensional Fokker-Planck equation, Eq. (6). Therefore, we conclude that

$$\overline{\langle y^2\rangle}=\overline{\langle x^2\rangle}\sim(\ln t)^4\,. \tag{21}$$

Thus the mean-square displacement $\overline{\langle x^2+y^2\rangle}\sim(\ln t)^4$, and we see that this two-dimensional system has the same scaling behavior as the original one-dimensional Sinai model.

Monte Carlo simulations were carried out to confirm this result. Realizations of the potential $V(i,j)$ such as the one shown in Fig. 2(b) were generated, and then random walkers were released and allowed to diffuse according to Eq. (1). The system size was chosen to be large enough so that no walker reached the system boundaries. This process was carried out for different values of ϵ_0. In each simulation 600 configurations of $V(i,j)$ were generated, and on each configuration 512 walks of 100 000 steps each were carried out. Thus the total number of walks generated for each value of ϵ_0 is $512\times600=307\,200$.

Figure 3 shows the mean-square displacement $\overline{\langle x^2+y^2\rangle}$ vs $(\ln t)^4$ for $\epsilon_0=0.7$ and $\epsilon_0=0.9$. In the latter case, convergence to a straight line clearly occurs. In the former, the numerical evidence as it stands does not permit an unambiguous conclusion. However, it is possible to see convergence of higher moments of the displacement. Fig-

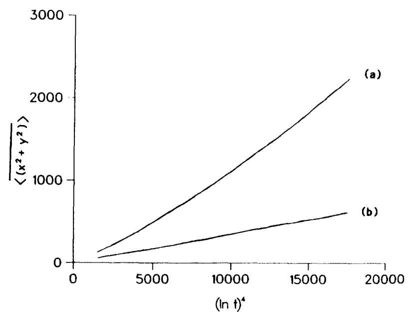

FIG. 3. Simulation results for the mean-square displacement $\langle x^2+y^2\rangle$ vs $(\ln t)^4$. Shown is $\epsilon_0=0.7$ (curve *a*) and $\epsilon_0=0.9$ (curve *b*). The predicted convergence to linear behavior is observed.

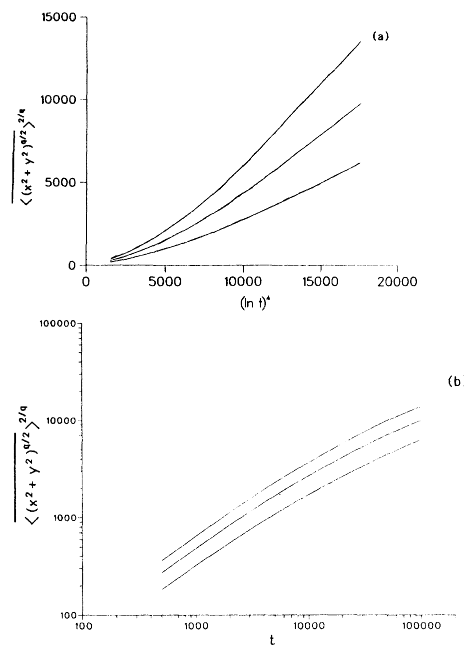

FIG. 4. (a) Simulation results with $\epsilon_0=0.5$ for several different moments of the displacement $\overline{\langle (x^2+y^2)^{q/2}\rangle}^{2/q}$ vs $(\ln t)^4$, for $q=6$ (top curve), $q=4$ (middle curve), and $q=2$ (bottom curve). Convergence to the expected linear behavior is faster for higher values of q. (b) Same calculations plotted on double logarithmic paper.

ure 4(a) shows the square of the characteristic length for several different moments of the displacement for $\epsilon_0=0.5$. Plotted is $\overline{\langle (x^2+y^2)^{q/2}\rangle}^{2/q}$ vs $(\ln t)^4$ for $q=2$, 4, and 6. Convergence to the expected linear behavior is found for $q=4$ and 6, but not for $q=2$, in good agreement with the previous findings. Note that all the figures are biased to the hypothesis of a $(\ln t)^4$ behavior. This result is expected since it was shown[14] that for the one-dimensional Sinai model (without correlation), all moments scale as $\langle x^q\rangle^{1/q}\sim\ln^4 t$. This method of studying moments is quite sensitive to deviations from the expected behavior. We have also replotted the identical set of data on double-logarithmic scale to test for possible power-law behavior. The results are shown in Fig. 4(b) and clearly do not support a power-law behavior.

The results of Marinari *et al.*[7] leading to Sinai-type behavior of the mean-square displacement at large K are in contradiction with the results of Fisher *et al.*[10] This contradiction arises because the model studied in Ref. 7 falls into the class of models discussed in Ref. 10. In the present article we have presented a model that has exactly a Sinai-type behavior, although we assume curl-free forces. The difference between our results and the results of Ref. 10 is due to the long-range correlations between the local fields in our model, as opposed to the short-range correlations appearing in Ref. 10.

In summary, we have studied diffusion in two dimensions in the presence of a disordered potential $V(x,y)$ with the symmetry $V(x,y)=V_1(x)+V_2(y)$. Here $V_1(x)$ and $V_2(y)$ are random one-dimensional surfaces. In the discretized version of the problem, diffusion proceeds according to Eq. (12), while in the continuum version, it is governed by the Fokker-Planck equation, Eq. (15). The mean-square displacement $\overline{\langle (x^2+y^2)\rangle}$ scales as $(\ln t)^4$. Hence this model represents a two-dimensional analog of the one-dimensional Sinai model.

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