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LETTER TO THE EDITOR

Generalisation of the Sinai anomalous diffusion law

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Abstract. Sinai has considered a novel one-dimensional walk with a random bias field E on each site. He has shown that when the field is taken with equal probability to be $+E_0$ or $-E_0$ the RMS displacement $R \equiv [\langle x^2 \rangle]^{1/2}$ increases with time t by the anomalously slow law $R \sim (\log t)^2$. Here we introduce long-range correlation between the random fields on each site by choosing a 'string' of k sites to have the same value of E, where k is chosen from the power law distribution $P(k) = k^{-\beta}$. We find that the Sinai law is generalised to the form $R \sim (\log t)^y$, where y sticks at the Sinai value $y = 2$ for $\beta > 2$. However, for $1 < \beta < 2$, y varies continuously with β as $y = \beta/(\beta - 1)$. We interpret this result physically in terms of a novel crossover between the physics underlying the Sinai effect and the physics of biased diffusion.

The physics of diffusion on random media has been an active topic of recent interest involving many disciplines ranging from mathematical physics to materials science (see, e.g., the recent review by Havlin and Ben Avraham (1987)). Much of this interest has focused on the conditions under which the familiar laws of diffusion break down, and what form of laws are needed to replace them. Sinai (1982) has recently discovered conditions under which the usual law that the RMS displacement R is proportional to $t^{1/2}$ is replaced by a logarithmically slow diffusion law

$$R \sim (\log t)^2. \qquad (1a)$$

The Sinai model is as follows: a one-dimensional linear chain for which a random walker at each site experiences a random bias field $E = p_+ - p_-$. Here $p_+(p_-)$ is the transition probability for a step to the right (left), with $p_+ + p_- = 1$; $p_\pm$ are taken from a distribution that satisfies the 'Sinai condition' $\langle \ln(p_+/p_-) \rangle = 0$. In particular, one may choose the bias field E to have an equal probability of being $+E_0$ or $-E_0$, with $0 < E_0 < 1$. The physical basis for this dramatic slowing down of ordinary power law diffusion is that there is an uncorrelated disorder in the bias field; a uniform bias field would produce the simple law $R \sim t$. Derrida and Pomeau (1982) found that $(1a)$ is replaced by a 'faster' power law diffusion of the form $R \sim t^\nu$ if the Sinai condition does not hold.

Our purpose here is to demonstrate that a simple generalisation of the Sinai model permits one to obtain logarithmically slow diffusion, but with a tunable exponent y

$$R \sim (\log t)^y \qquad (1b)$$

with the property that one can pass continuously out of the logarithmically slow domain as $y \to \infty$. The model is similar to the Sinai model in that each site experiences a random bias field $E = p_+ - p_-$ that can take only the two values $+E_0$ and $-E_0$. However there is now a long-range correlation between the bias fields on each site. Instead of making a new selection of bias for each new site, we randomly select the same bias ($\pm E_0$) for an entire string of k sites, where k is a random variable chosen from the power law distribution

$$P(k) \equiv k^{-\beta} \qquad (\beta > 1). \tag{2}$$

Typical configurations of the Sinai model and the present model are shown in figures 1(a) and (b) respectively. Consider a finite segment of the lattice consisting of S strings and a total of l sites, and let E_i be the field acting on all k_i sites belonging to string i. As with the Sinai model, E_i takes the values $+E_0$ and $-E_0$ with equal probability. Thus the walker executes a normal random walk, with steps of unit length occurring at each time interval. The correlation is in the values of the random bias fields, and this introduces *indirectly* a long-range correlation in the steps of the random walker. Since the sites in each string can be randomly biased in either direction, the net bias in the S-string segment is given by

$$E(S) \equiv \sum_{i=1}^{S} E_i k_i \sim \begin{cases} S^{1/\beta} & (1 < \beta < 2) \\ S^{1/2} & (\beta > 2) \end{cases}. \tag{3}$$

Note that $E(S)$ is analogous to the net displacement $R(S)$ of a Lévy flight after S time steps—a Lévy flight (Mandelbrot 1982, Shlesinger and Klafter 1985) is a correlated walk in which at each time step the walker moves k steps in a random direction, with k chosen from the distribution (2).

The total time for the random walker to exit this region of l sites scales exponentially with the net bias field is

$$t_{\text{exit}} \sim e^{E(S)}. \tag{4}$$

To obtain the diffusion law, we shall eliminate the variable S in favour of the variable

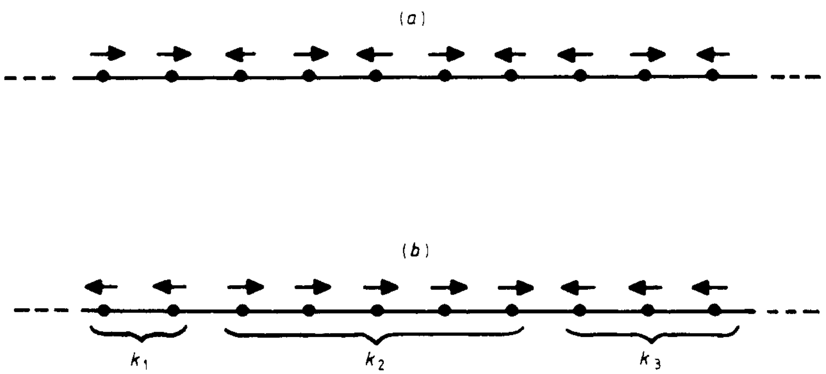

Figure 1. Comparison between (a) the Sinai model, and (b) the present model. Shown is a segment of $l = 10$ sites of the linear chain lattice. For both models, the bias field $\{E_i\}$ takes the values $+E_0$ and $-E_0$ with equal probability. However in the Sinai model, a new selection is made at each site, while for the present model there is a long-range spatial correlation of the random fields: a new selection is made only after k sites, where k is a random number chosen from the power law distribution (2). Thus the sites fall into S strings, with k_i sites in string $i (i = 1, \ldots, S)$.

l and find that $S \sim l^{\beta-1}$ for $1<\beta<2$ and $S \sim l$ for $\beta>2$. To see this, we first note that

$$l \equiv \sum_{i=1}^{S} k_i \sim S \int_1^{k_{\max}} kP(k)\,\mathrm{d}k \tag{5a}$$

will be dominated by its upper integration limit if $\beta<2$ and by its lower integration limit if $\beta>2$:

$$l \sim \begin{cases} Sk_{\max}^{2-\beta} & (1<\beta<2) \\ S & (\beta>2) \end{cases}. \tag{5b}$$

To see how $k_{\max}$ scales with S, we introduce a uniform distribution $p(u)$ and equate $P(k)\,\mathrm{d}k$ to $\mathrm{d}(u)$, with the result that $\mathrm{d}u/\mathrm{d}k = P(k) = k^{-\beta}$, or $u \sim k^{-(\beta-1)}$. Now $u_{\min} = 1/S$ so that

$$k_{\max} = u_{\min}^{-1/(\beta-1)} = S^{1/(\beta-1)}. \tag{6}$$

Substituting (6) and (5b) into (3), we find

$$E \sim \begin{cases} l^{(\beta-1)/\beta} & (1<\beta<2) \\ l^{1/2} & (\beta>2) \end{cases}. \tag{7}$$

Finally, we combine (7) and (4), and note that t_{exit} is the time to cover a region $l \equiv R$ so that the Sinai anomalous diffusion law (1a) is replaced by (1b) with

$$y = \begin{cases} \beta/(\beta-1) & (1<\beta<2) \\ 2 & (\beta>2) \end{cases}. \tag{8}$$

The case $\beta=2$ is 'marginal' in the sense that the integral in (5a) involves the logarithm of $k_{\max}$. Accordingly, (5b) becomes $l \sim S \log S$. It would be interesting to further explore the consequences of the logarithmic factors appearing for $\beta=2$.

It was noted above that the present walk is not the same as a Lévy flight: for the Lévy flight, the relation between the range R and the time t is a power law, $R \sim t^{1/d_w}$, not logarithmic. However, the present walk is also a correlated walk and our results bear some intriguing parallels to the results for the Lévy flight. For example, the exponent d_w of the present model has the same feature as the exponent y for the Lévy flight. For $\beta>2$, y sticks at its value $y=2$ for a Sinai random walk, while for $\beta<2$, y varies continuously with β. Similarly, in the Lévy flight, the exponent d_w sticks at its 'classical' random walk value $d_w=2$ for $\beta>2$, while for $\beta<2$, d_w varies continuously with β.

As β approaches its limiting value of unity, the frequency of very long strings of sites whose random fields are oriented in the same direction increases without limit and the exponent y diverges. When the length of a string is longer than l, the size of a region, the walk is a biased random walk in this region ($d_w=1$). Hence as $\beta \to 1$ we expect that the diffusion approaches the limit of a biased random walk. Indeed, a logarithmic anomaly of the form (8) will pass to a power law if the exponent approaches infinity.

In summary, we have introduced a generalisation of the Sinai random walk, which has the feature that the random field of the Sinai model is replaced by a random field with long-range spatial correlations whose range is parametrised by β. For all allowable values of β, we find that the diffusion law is not a power law but rather is a power of $\log t$, $R \sim (\log t)^y$. For $\beta>2$, the correlations in the random fields are of such short range that the exponent y sticks at the Sinai value $y=2$. For $\beta<2$, these correlations

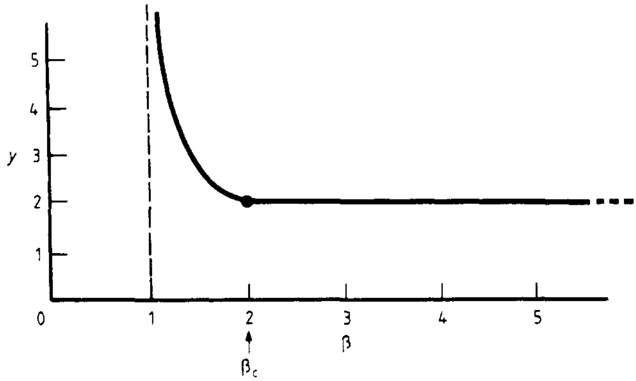

Figure 2. Dependence of the anomalous diffusion exponent y on β. The curve shows $\beta/(\beta-1)$. The marginal case $\beta=\beta_c=2$ separates the two cases of interest, $\beta>2$ and $\beta<2$. In the latter case, it is possible for one to find sufficiently long strings of identically oriented random bias fields that the Sinai form (1a) passes over to a new form (1b).

are of increasingly longer range and y increases as $y=\beta/(\beta-1)$. As β approaches its limiting value $\beta=1$, y becomes large without limit and the logarithmic anomaly becomes indistinguishable from a more conventional power law relation.

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