Flow in porous media: The "backbone" fractal at the percolation threshold

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We show that for all Euclidean dimensions d $\tilde{\zeta}=\bar{d}_w-\bar{d}_f$, where $L_R\sim\xi^{\tilde{\zeta}}$ is the effective resistance between two points separated by a distance comparable with the correlation length ξ, $\bar{d}_f$ is the fractal dimension of the backbone, and $\bar{d}_w$ is the fractal dimension of a random walk on the same backbone. We also find a relation between the backbone and the *full* percolation cluster, $\bar{d}_w-\bar{d}_f=d_w-d_f$. Thus the Alexander-Orbach conjecture ($d_f/d_w=2/3$ for $d\geq 2$) fails numerically for the backbone.

How can one describe the flow of fluid in a random porous media? This important question has long eluded explanation, yet is of general interest since it is related to the "propagation of order" at a critical point of any sort.[1-3] Recently considerable attention has focused on the utility of the percolation "backbone" as a useful model of the actual path that this fluid flow might take.[3-8] To define the backbone, consider two points i and j separated by a distance comparable to the correlation length ξ on a large bond percolation cluster just below the threshold p_c. The $L_{\rm BB}$ backbone bonds between i and j are the bonds that belong to at least one self-avoiding walk between i and j. The remaining bonds in the cluster are "dangling ends" (Fig. 1). If the system is just above p_c, then the backbone is defined in a similar fashion.[6]

We define the fractal dimension $\bar{d}_f$ of the backbone through[8]

$$L_{\rm BB}\sim\xi^{\bar{d}_f} \ . \tag{1a}$$

We shall introduce the fractal dimension of a random walk on the backbone, $\bar{d}_w$, which relates the number of steps N_w in a random walk on the backbone fractal to the range ξ_w of the walk,

$$N_w\sim(\xi_w)^{\bar{d}_w} \ . \tag{1b}$$

The utility of the backbone in describing the onset of

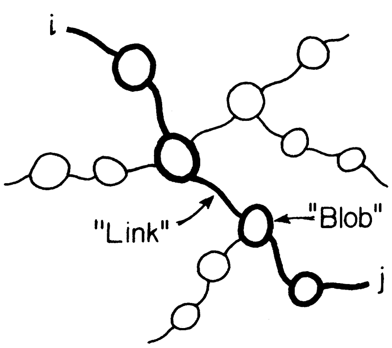

FIG. 1. Typical percolation cluster just below the percolation threshold, indicating backbone bonds (heavy line) and dangling ends (light lines); both are similar in that they are composed of "links and blobs" (Ref. 3).

fluid flow in randomly porous media is perhaps most physically presented in terms of the Einstein relation for the dc conductivity $\sigma_{\rm dc}\propto nD$, where n is the density of carriers and D the diffusion constant. In contrast to the usual treatment,[9-12] we shall interpret $n\sim\bar{P}_\infty\sim\xi^{-\bar{\beta}/\nu}$ as the fraction of bonds belonging to the *backbone*. Since $D=d\xi_w^2/dN_w=(\xi_w)^{2-\bar{d}_w}$, and $\sigma_{\rm dc}\sim\xi^{-\tilde{t}}$ (with $\tilde{t}=t/\nu$), we have for $\xi_w\sim\xi$

$$\tilde{t}=(d-2)+(\bar{d}_w-\bar{d}_f) \ . \tag{2}$$

In writing (2) we have used the fact that the order-parameter exponent is the codimension, $\bar{\beta}/\nu=d-\bar{d}_f$.

It is convenient to introduce an effective length L_R giving the resistance between points i and j, where $L_R\sim\xi^{\tilde{\zeta}}$ and $\tilde{\zeta}=\zeta/\nu=\tilde{t}-(d-2)$. Thus (2) becomes simply

$$\tilde{\zeta}=\bar{d}_w-\bar{d}_f \ . \tag{3}$$

We have confirmed the validity of (3) for the Cayley tree, which is thought to accurately describe the statistics of percolation behavior above $d=6$. We find $\bar{d}_f=2$, and $\bar{d}_w=4$,[13] which agrees with the known result $\tilde{\zeta}=2$.

We have also directly verified (3) on the Sierpinski gasket model of the backbone, since[10] $\bar{d}_w=\ln(d+3)/\ln 2$ and[5] $\bar{d}_f=\ln(d+1)/\ln 2$ are both known exactly, as is[5] $\tilde{\zeta}=\ln[(d+3)/(d+1)]/\ln 2$. Although (3) *is* confirmed by the gasket model, we note that this model does not perfectly describe the percolation backbone; e.g., $\tilde{\zeta}$ increases with d up to $d=6$, while for the gasket $\tilde{\zeta}$ decreases with d for *all d* (Fig. 2).

Equation (3) is new. However, in the usual treatment one applies the Einstein relation to the *full* cluster and obtains[9-12]

$$\tilde{\zeta}=d_w-d_f \ . \tag{4}$$

Here d_f is the fractal dimension of the *full* cluster, defined by $s^*\sim\xi^{d_f}$ where s^* is the number of sites in the incipient infinite cluster. Similarly, d_w is the fractal dimension of a random walk on the *full* cluster, defined in analogy with Eq. (1b), $N_w^*=(\xi_w)^{d_w}$. If we combine (3) and (4), we find

$$\bar{d}_w-\bar{d}_f=d_w-d_f \ . \tag{5}$$

Note that (5) relates the full cluster to the backbone only. This is a consequence of the intriguing property of the Ein-

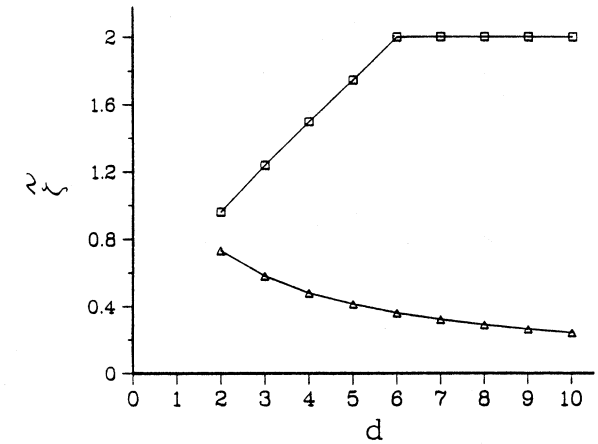

FIG. 2. Dependence on system dimensionality d of the effective resistance exponent $\tilde{\zeta}=\zeta/\nu$. The triangles are exact results for the Sierpinski gasket (Ref. 5), while the squares are from direct calculations on the percolation backbone (Refs. 16–20). This figure for the resistivity exponent $\tilde{\zeta}$ is analogous to Fig. 2 of Ref. 10 for the localization exponent β_L.

stein relation, which connects the dc conductivity (defined via an external electric field that fixes the paths along which the electrons may move) to the diffusivity (which is independent of the external forces).

Although (5) is quite surprising at first sight, an intuitive explanation follows from examination of Fig. 1. Suppose we view the incipient infinite cluster as one "backbone chain," from which emanate $k-1$ additional "dangling end chains." Since the structure of *all* k chains is similar, we expect for ξ_w

$$s^* \sim kL_{\mathrm{BB}} \sim k\xi^{\bar{d}_f} , \tag{6a}$$

$$N_w^* \sim k\bar{N}_w \sim k\xi^{\bar{d}_w} . \tag{6b}$$

Since $s^* \sim \xi^{d_f}$ and $N_w^* \sim \xi^{d_w}$, Eq. (5) follows from (6). On the Cayley tree, (5) can be tested exactly since $d_w=6$ (Ref. 14) and $\bar{d}_f=4$. To test (4), it would be very interesting to calculate $\bar{d}_w$ and d_w for other dimensions, and work along these lines is underway.

We conclude with three remarks:

(i) Alexander and Orbach (AO) suggested[9] that the "spectral" dimension $d_s \equiv 2d_f/d_w=\frac{4}{3}$ for $d \geq 2$; from Eq. (5) it would then follow that $\bar{d}_w-\bar{d}_f=\frac{1}{2}d_f$. We might ask if an analogous relation holds for the backbone. Were the AO conjecture valid for the backbone, then $\bar{d}_w-\bar{d}_f=\frac{1}{2}\bar{d}_f$, while we know that $\bar{d}_f<d_f$. Moreover, we find that the backbone function $\bar{d}_s=2\bar{d}_f/\bar{d}_w=2\bar{d}_f/(\tilde{\zeta}+\bar{d}_f)$ *depends* on d, varying from $\frac{91}{72}$ for $d=2$ to 1 for $d=6$. Although the AO conjecture clearly fails for the backbone[15] we find that all available numerical data[16–20] are consistent (to ± 1–2%) with the relation

$$\bar{d}_s \cong \zeta \tag{7}$$

for any d. Note that (7) hold as an exact equality for $d=1$ and $d \geq 6$, suggesting the possibility that a reasonable backbone conjecture to complement the AO conjecture $d_s=\frac{4}{3}$ could be $\bar{d}_s=\zeta$.

(ii) It has recently been proposed that[10,21] $\beta_L=d_w-d_f$, where β_L is the exponent characterizing the localization problem.[22] If one applies parallel reasoning to the *backbone* of a percolation cluster, one finds

$$\beta_L=\bar{d}_w-\bar{d}_f . \tag{8}$$

(iii) Equation (3) suggests a simple geometric relation between various effective lengths in the percolation problem. We can think of the two points i and j as being connected via three separate effective paths. Path 1 has an effective length equal to the total *number* of backbone bonds, $L_{\mathrm{BB}} \sim \xi^{\bar{d}_f}$. Path 2 is a chain of length equal to the resistance $L_R \sim \xi^{\tilde{\zeta}}$ between i and j, while path 3 is a chain of length $L_w=N_w^{1/2}$ equal to the number of bonds that would be necessary to connect i and j *if* the walk was one dimensional. Thus Eq. (3) states that L_w is the geometric mean of L_R and L_{BB}.

$$L_w=(L_R L_{\mathrm{BB}})^{1/2} . \tag{9}$$

The backbone bonds are made of links and blobs (Fig. 1), where the number of links L in any dimension diverges as $L \sim \xi^{1/\nu}$ (Ref. 6). Hence we can separate the contribution of the links from that of the blobs by writing $L_R=L+\delta L_R$, $L_W=L+\delta L_W$, and $L_{\mathrm{BB}}=L+\delta L_{\mathrm{BB}}$. Since $L_{\mathrm{BB}}>L_R$, we have from (4)

$$L \leq L_R \leq L_w \leq L_{\mathrm{BB}} . \tag{10a}$$

From the definitions $L_R \sim \xi^{\zeta/\nu}$, $L_w \sim \xi^{\bar{d}_w/2}$, and $L_{\mathrm{BB}} \sim \xi^{\bar{d}_f}$, (10a) is equivalent to the exponent inequalities

$$1 \leq \zeta \leq \tfrac{1}{2}\nu\bar{d}_w \leq \nu\bar{d}_f . \tag{10b}$$

The more these exponents differ from 1, the more important are the blobs (Fig. 3). They assume their maximum values for $d=2$ (where the blobs are most important), while they approach unity for $d \rightarrow 1$ and also for $d \geq 6$

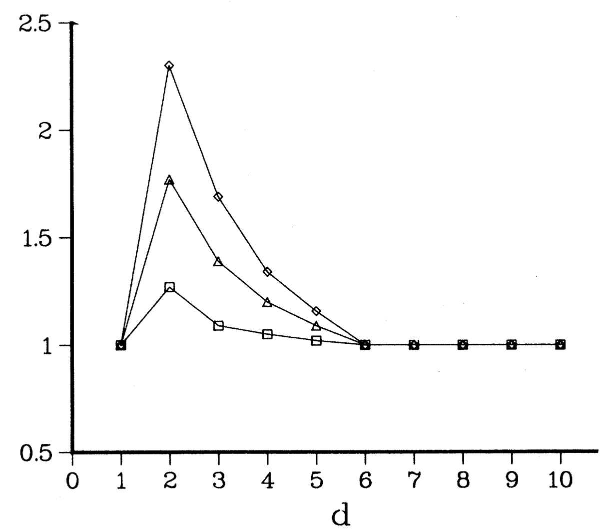

FIG. 3. Dependence on d of three critical exponents characterizing the percolation backbone: $\nu\bar{d}_f(\Diamond)$, $\frac{1}{2}\nu\bar{d}_w$ (Δ), and ζ ($\square$). The exponents clearly satisfy the inequalities (5b), which become strict equalities only where the blobs do not contribute ($d=1$ and $d \geq 6$). The data for $\bar{d}_f$ and ζ are from Refs. 5 and 16–20, while $\frac{1}{2}\bar{d}_w$ is from Eq. (3).

(where the blobs do not contribute).

In summary, we have considered the problem of flow in porous media by focusing on the backbone bonds of a percolation cluster rather than on the entire cluster. We have found that the resistivity exponent $\tilde{\zeta}=\zeta/\nu$ is given by an extremely simple relation, Eq. (3), between two fractal dimensions pertaining to the backbone. If one applies the same argument to the full cluster,[9-12] then we can relate the backbone to full cluster as in (5). This relation is the consequence of a deep feature of the Einstein relation, which could be exploited to relate properties of a wide range of different systems with the same dc conductivity. More generally, we expect that analogous statements apply to other fluctuation dissipation relations.

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