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Percolation phenomena: a broad-brush introduction with some recent applications to porous media, liquid water, and city growth

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Abstract

This brief overview is designed to introduce some of the advances that have occurred in our understanding of percolation phenomena. We organize our presentation around three simple questions: (i) What are percolation phenomena? (ii) Why do we care? (iii) What do we actually do? To answer the third question, we will briefly review some recent applications of percolation that have been the subject of research in the Boston University group. © 1999 Elsevier Science B.V. All rights reserved.

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Dedicated to the memory of Shlomo Alexander

1. What are percolation phenomena?

Percolation addresses questions that arise when considering geometric connectivity, connectivity of virtually any kind of object. An example of such a question is how many clusters exist [1–3]. A typical clustering phenomena arises if we occupy randomly a fraction p of the sites of an $L \times L$ square lattice. Two neighboring occupied sites are

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said to belong to the same cluster. When p is small, the mean cluster size is small, but as p increases there appears a threshold value $p = p_c$ at which a cluster (at least one) suddenly spans the lattice [4].

2. Why to care about percolation phenomena: scientific reasons

2.1. Prototype threshold phenomenon

One reason that percolation is interesting is because it displays a threshold phenomenon. If we plot on the y-axis the probability of finding a spanning cluster P_{span} for $L \times L$ system as a function of p, the fraction of occupied sites, we find that in the $L \to \infty$ limit, P_{span} approaches a Heavyside step function – i.e., as $p \to p_c$, there is a genuine discontinuous switch between two different macroscopic phases: "no infinite cluster" and "infinite cluster." This switching effect is remarkable, since as one smoothly changes a *microscopic* control parameter p, one discovers a discontinuous change in a relevant *macroscopic* function P_{span}.

The switch in a finite system is not sudden, but is broadened over a region of p values. The thickness of this transition region for finite L gives information on the critical exponent – more precisely, on the exponent d_{red}, the fractal dimension of the singly-connected "red" bonds in the cluster.

Threshold phenomena bring to mind phase transition and critical phenomena problems. Why? Because threshold phenomena are a kind of switch. Phase transitions are also switches. As the field across a membrane increases, nothing happens until a threshold is reached where the membrane conducts electricity. Similarly, this transition has the properties of a critical point. Percolation phenomena share with critical phenomena the hallmark features of *scaling*, *universality*, and *renormalization group* [5]. So the percolation threshold has the properties of a critical point but, because the only concept involved is geometry, the system is less difficult. There is no temperature: the problem is purely geometric. In principle, Archimedes could have studied percolation, and using percolation he could have invented scaling, universality, and renormalization group.

2.2. Prototype fractal

Another reason percolation is scientifically interesting is that it is a prototype fractal – in an era when the ubiquity of fractals in nature is being challenged by practitioners in the field [6,7]. We can experience the striking self-similarity of a fractal when we examine a series of pictures of a large percolation cluster created for some fixed value of $p = p_c$. A little box is cut out of the first picture, blown up, and used as the second picture (Fig. 1). The same little box procedure can be repeated in the second picture, creating the third picture, and in the third, creating the fourth. The untrained eye recognizes that the statistical properties in all four pictures are the same, and to confirm this by a simple experiment, we can remove the labels, mix the pictures up, and

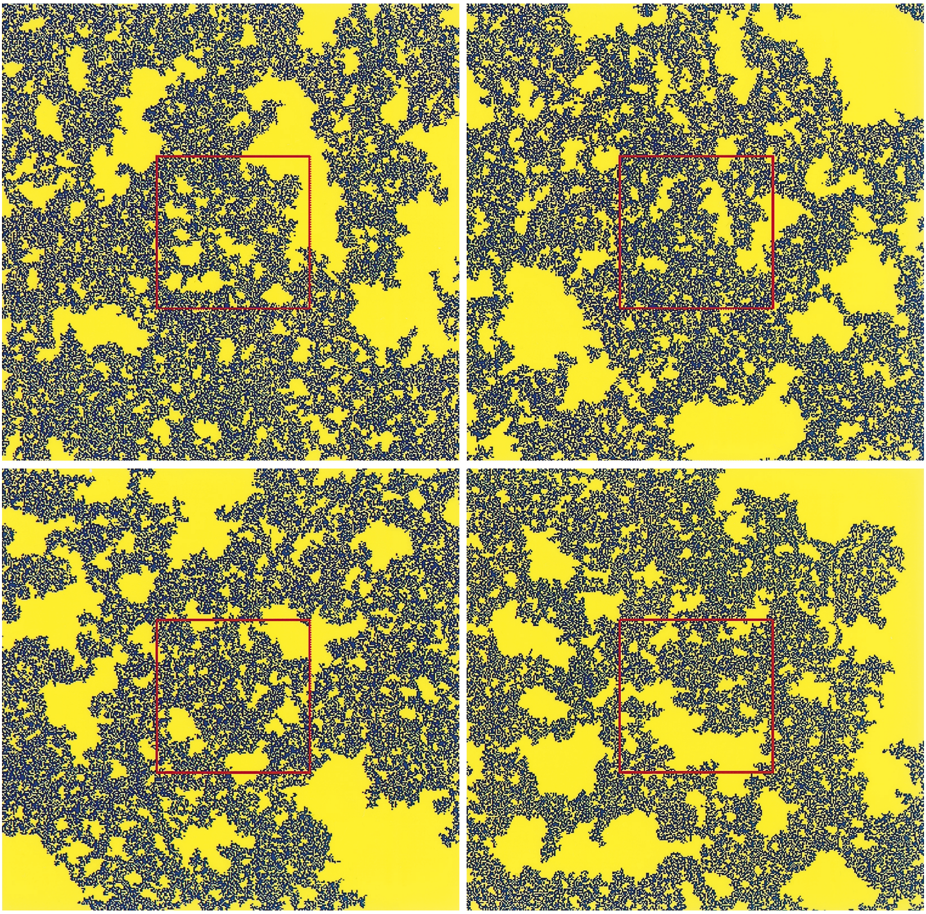

Fig. 1. Four successive magnifications of the incipient infinite cluster that forms at the percolation threshold. In the figure that you see, however, the labels of the four panels have been removed and the panels have been scrambled. Attempt to put them back into sequence by eye – it is extremely difficult if the system is at the percolation threshold ($p = p_c$). An educational game is to time how long it takes each player to detect by eye which of the 24 possible panel orderings is the correct one that arranges the four panels in increasing order of magnification. (This figure is courtesy of J. Kantelhardt and A. Bunde.)

then see how long it takes to put them back into sequence. It takes a remarkably long time, and more significantly can be done only by looking for *nonstatistical* features of the patterns, such as specific invaginations of a specific part of the cluster.

As a control, one can repeat the same experiment examining not the incipient infinite cluster that forms right at the percolation threshold ($p = p_c$) (Fig. 1), but rather the infinite cluster that appears above the percolation threshold ($p > p_c$) (Fig. 2). While it is essentially impossible to order the four panels of Fig. 1 using only statistical information, it is trivial to order the four panels of Fig. 2.

In Fig. 1 we saw that the incipient infinite cluster itself is self-similar. A second form of "self-similarity" arises: the entire collection of clusters at $p = p_c$ obeys a

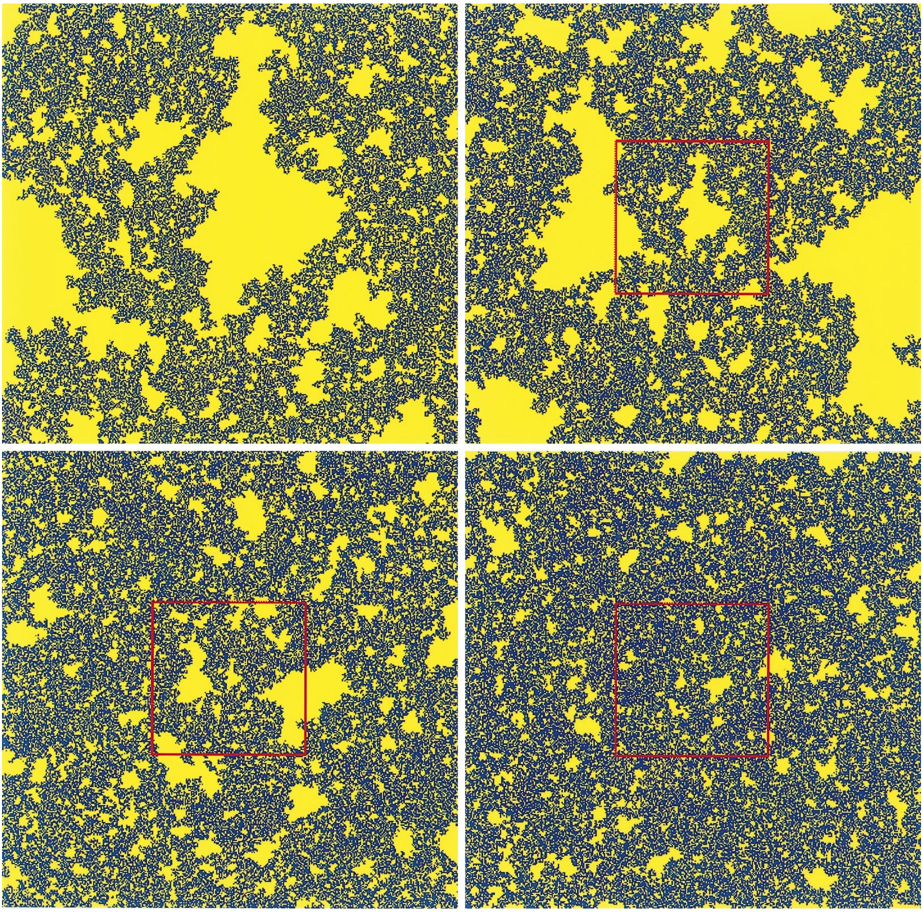

Fig. 2. The same as Fig. 1 except that now the system is slightly above the percolation threshold. In the figure that you see, the labels of the four panels are removed and the panels are scrambled. Attempt to put them back into sequence by eye – it is extremely easy since the system is now above the percolation threshold ($p > p_c$). Because self-similarity is lost above the percolation threshold, it does not take long to detect by eye which of the 24 possible orderings is the correct one in which the four panels are arranged in increasing order of magnification. (This figure is courtesy of J. Kantelhardt and A. Bunde.)

hierarchical relationship that can be uncovered by making a histogram on log–log paper of the number of s-site clusters. Remarkably, this histogram is linear, over as much as eight decades for the four terasite systems now studied [8,9]. The slope τ of this straight line (a "scaling exponent") depends on the system dimension but not on the lattice type (e.g., square vs. triangular), a property that is a hallmark of the universality principle in critical phenomena. The scaling exponent τ shows the remarkable feature of critical phenomena, elucidated first in momentum space renormalization group studies, of sticking at a fixed "chemical" value above a critical dimension d_c (which equals 6 for percolation phenomena). The hierarchical feature of percolation phenomena (quantified by τ) is related to the self-similarity (quantified by d_{fractal}). Indeed, the two parameters are easily related, with $\tau = 1 + d/d_{\text{fractal}}$, so d_{fractal} has also the property of sticking at a fixed value $d_{\text{fractal}} = 4$ above d_c.

2.3. Connection with thermal critical phenomena

Thirty years ago at Kubo's 1968 Statphys-9 meeting, the late Pieter Kastelyn presented a remarkable demonstration that percolation, a purely geometric problem, could be derived from a statistical mechanical model called the Potts model, which models the interaction of objects that can be in one of s states [10]. Kastelyn found that if one forms very carefully the limit in which the number of states approaches unity, then one recovers the purely geometric percolation problem. This important work mathematically links pure connectivity problems with a statistical mechanical model (the Potts model, a special case of which is the Ising model). This connection is deep and not fully understood. Indeed, the parameters that enter into modern theories of critical phenomena – e.g., the field-like scaling power y_H and the temperature-like scaling power y_T – can be related to purely geometric objects. The field-like scaling power y_H is exactly equal to the fractal dimension d_{fractal} of the percolation cluster, while the temperature-like scaling power y_T is exactly equal to the fractal dimension d_{red} of the singly-connected bonds – i.e., the "red" bonds – that tenuously hold together the incipient infinite cluster that forms just below the percolation threshold [11,12].

The geometric interpretation of thermal critical phenomena is even more profound: the correlations that form in a cooperative system (and the entire world constitutes a cooperative system) arise from only one thing – connectivity. One can trace these correlations out with a pencil on a piece of paper. This is what makes it profound: it links all of nature to something geometrical. Perhaps sometime in the not-so-near future scientists will understand very deeply the nature of reality. If and when that happens, we conjecture that connectivity concepts will play a role in that understanding.

Another reason for our interest in percolation phenomena is the relevance to real-world problems [13–15]. For example, the clusters that form in percolation by themselves are uniquely associated with a class of geometric objects called "lattice animals." These lattice animals are not compact objects – like human animals – but rather are stringy, ramified, fractal objects. They have many remarkable properties. For example, Parisi and Sourlas rigorously connect the exponent θ_{LA} characterizing the statistics of lattice animals in d dimensions with the exponent θ_{LY} characterizing the Lee–Yang singularity in the Ising model partition function zeros in $d-2$ dimensions [16]. This is useful in that if we uncover some property in the Ising model in, say, two dimensions we automatically know some property in the lattice animal problem in four dimensions [17]. In particular, the critical dimension of the lattice animal problem is 8 and the critical dimension of the Lee–Yang Ising model is 6.

Another variant is to impose direction. Instead of connectivity just being joining hands, we impose the rule that you can only join hands in a subset of the directions. For example, on a square lattice we might allow only connections to the "south" or to the "west" (but not to the "north" or the "east"). This gives us the "directed" percolation

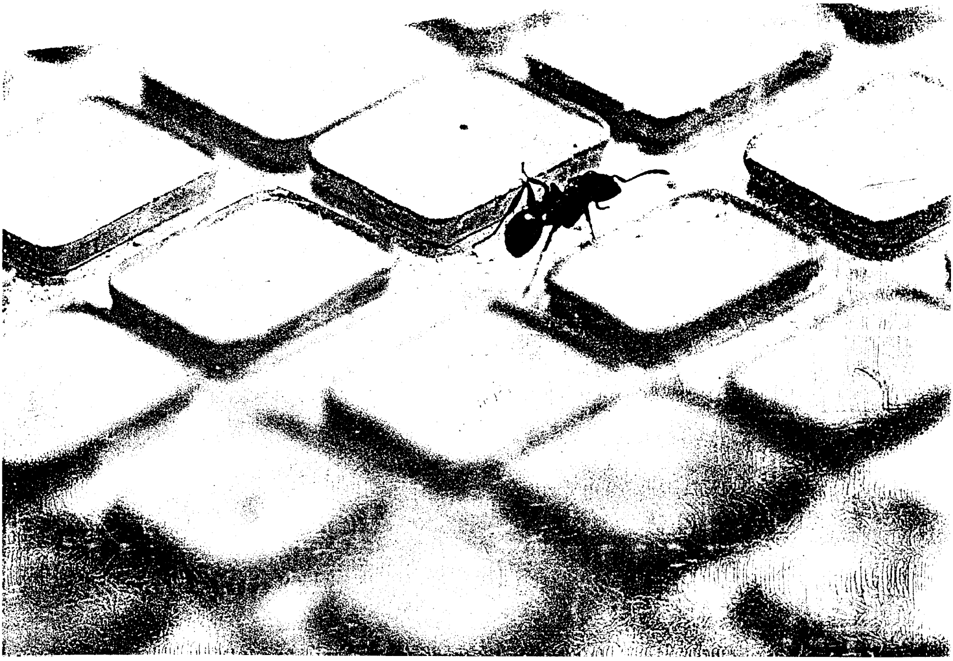

Fig. 3. Photograph of a drunken ant placed on a square lattice, a fraction of whose bonds are too narrow for the ant to pass. This artificial experiment – christened the "ant in the labyrinth" by de Gennes – is a prototype model for investigating how the laws of diffusion are changed when the substrate is a fractal object. (This photograph is courtesy of S. Alexander.)

problem, which has a critical dimension one lower than normal percolation, or the "directed" lattice animal problem, which has a critical dimension also one lower than the original lattice animal problem. This directed percolation is fascinating. Suppose we notice that the bottom of our necktie has strayed into our full coffee cup. We know that this table disaster is not a complete disaster because the coffee stops rising up the necktie when a pinning path emerges, and the statistics of this pinning path are identical to those of a directed percolation cluster (see Ref. [18] and references therein).

3. Why to care about percolation phenomena: practical reasons

3.1. The problem of anomalous diffusion

Imagine we make an ant drunk; this is illegal in the US – only people, not animals, can be legally encouraged to get drunk. What happens if we place this drunken ant on a percolation cluster (Fig. 3) and it executes the classic drunkard's walk? How is Fick's law of diffusion modified? Fick's law predicts that the mean-square displacement of this drunk animal is linear in time (certainly true in percolation if the fraction of occupied sites is one). On the other hand, if the fraction of occupied sites is exactly at the threshold, and the drunk animal is stumbling around on the fractal incipient infinite cluster, then much of the time this animal reaches the edge of the incipient infinite cluster and must turn back – making the "diffusion" less rapid. Its diffusion is characterized by an exponent d_{walk}, the fractal dimension of its walk, which is found also to stick at the value $d_{\text{walk}} = 6$ for all lattice dimensions down to 6, whereupon d_{walk} breaks free, and actually takes on its Fick value, $d_{\text{walk}} = 2$ only at one dimension (where the incipient infinite cluster is a "Euclidean" one-dimensional non-fractal object).

When we consider any dynamical process occurring on a percolation substrate, we must remember that this process is taking place on a fractal. Fractals have the feature that their density is actually decreasing as we move away from the origin. Thus the effective density seen by the drunken ant decreases as a function of time, so as it stumbles away from where it started it goes slower and slower than it would if the substrate were non-fractal. That is why this fractal dimension d_{walk} is greater than 2.

3.2. *The problem of localized vibrations*

If we tap a three-dimensional percolation cluster with $p = 1$, it will vibrate – at the normal mode frequencies. If we tap a percolation cluster at $p = p_c$ we will have localized vibrations, and their statistical properties can be described by means of a spectral dimension d_{spectral} [19] that is related to the original fractal dimensions of the fractal object and the random walk on the fractal object, thereby connecting vibrations and random walks. This number sticks at the value $\frac{4}{3}$ down to six dimensions, below which it may or may not stay at that value.

4. Three recent applications of percolation phenomena

4.1. *Fluid flow in porous media*

Half of the world's oil is still underneath the earth, and a major problem is how to extract it. A number of methods are used. One is to drill not one but two holes and then push water down one of them so that oil can be extracted from the other. The oil resides in some kind of connected cluster underneath the earth and the water pushes the oil up so that it can be more easily extracted out of the second hole some distance away. It is therefore of interest to oil people to understand the statistics of this connected cluster underneath the earth.

The *chemical distance or minimal path*, ℓ, between two sites is defined as the shortest path on a percolating cluster connecting the two sites [20]. If we look at a large percolation cluster we see that the minimum path is traced in white from the top to the bottom (see Fig. 4), and it is evident to the eye that this minimum path is a

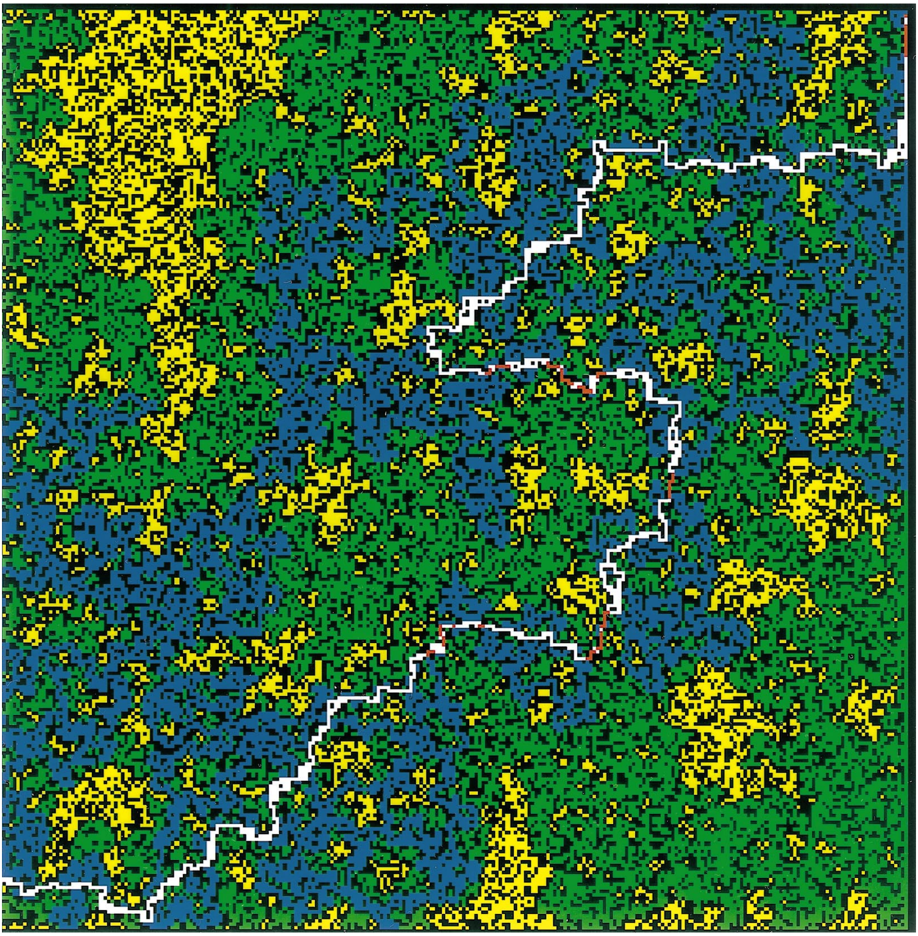

Fig. 4. The incipient infinite percolation cluster that forms at the percolation threshold, highlighting the path that connects one point to another using the minimum number of bonds. This "minimum path" is a fractal object whose dimension is not known exactly even for the case of a two-dimensional cluster (but is approximately 1.13). The average chemical distance scales with exponent $d_{\min}$, where various estimates of $d_{\min}$ are $d_{\min} \approx 1.130 \pm 0.005$ [21] and $d_{\min} \approx 1.1307 \pm 0.0004$ [22,23]. (This figure is courtesy of S. Schwarzer.)

fractal object with a dimension somewhat larger than unity (see, e.g., Refs. [21–23] and references cited therein). Notice also in Fig. 4 the singly connected bonds (indicated in red), which have a dimension of 0.75 [11,12].

The quantity of interest here is the conditional probability, $P(\ell|r)$, that two sites taken from the same cluster, separated by Euclidean distance r, are a chemical distance ℓ apart. For example, in oil recovery the first passage time from the injection well to a production well a distance r away is related to $P(\ell|r)$. There has been an extensive theoretical and computer work done on studying the scaling of $P(\ell|r)$ [1,24], and the complete scaling form of $P(\ell|r)$ – including finite-size effects, and off-critical behavior is recently becoming clear [25].

The breakthrough time – the time needed for the first bit of oil to come out is another important quantity that is only just now beginning to be studied [26]. This quantity appears to be connected to a subset of the incipient infinite cluster, the "backbone", which is defined to be the subset carrying oil from one point to another. Associated with this backbone is a unique fractal dimension d_{backbone} whose value has recently been calculated very accurately [22].

Thus by considering a practical problem, one is led to consider new features of percolation clusters and new fractal dimensions to characterize these features – the above example being the concept of the "minimum path" and "backbone" and their fractal dimensions. We have invoked an increasing number of fractal dimensions. We started out with only two, corresponding to the fractal dimension of the cluster and the fractal dimension of the singly connected red bonds. Each new application brings in a new dimension: d_{walk} for diffusion, d_{spectral} for localized fracton vibrations, and now d_{min} and d_{backbone} for the oil field problem.

We go to one last example and pass an electrical current through our system. We assume that each bond in the percolation cluster is a unit resistor, pass a current through the cluster, and measure the current as a function of p (a procedure pioneered 15 years ago in Ref. [27]). If p is less than the threshold, no current flows through a big system. Above the threshold, a current begins to flow, and an exponent t describes the way in which this current increases with $p - p_c$. The converse problem is that each bond in a cluster is a superconductor and the non-cluster bonds are unit resistors [28]. If $p = 0$ there are only normal bonds. We start to add superconducting bonds until we reach the threshold at which there is a connected path of superconducting links, and the current goes to infinity with an exponent s. These two exponents, s and t, are also related by fractal dimensions.

Related to this electrical problem is the problem of fluid flow. The model used in the electrical problem is also a good model for the problem of fluid flow. To some degree, the equations for fluid flow are very similar to those for electrical current. In the limit of low Reynolds number, they are the same; in the limit of high Reynolds number, they are not. What happens when we actually solve the Navier–Stokes equation as a function of Reynolds number? This has been done by Andrade and his collaborators using methods of computational fluid dynamics [29]. They find that when the Reynolds number is low, the flow "finds" all the connected paths on the backbone from one end to the other. When the Reynolds number is high, it does not easily discover the best paths to take, and they smash into the walls (very much like a crowd in a Tokyo subway station), creating zones of recirculation. Corresponding to this qualitative picture is a difference in behavior of the basic distribution function. Instead of a log-normal distribution there is a power-law function for the relevant distribution [29].

The concept of fractals and the use of percolation models to describe disordered media represent today important ingredients to analyse and predict properties of anomalously diffusive systems of transport. For example, a recent study concerning a system in which there occurs both diffusion and reaction in a percolation geometry suggested that the fractal geometry of the porous media can have a strong influence on the reactive effectiveness of porous catalysts [30].

4.2. Connectivity in water

Connectivity is also a fundamental feature of liquid water, since liquid water is characterized by a high degree of hydrogen bonding. Each water molecular is bound together with other water molecules, resulting in clusters of strongly bonded molecules in the water network [31]. Remarkably, these off-lattice (continuum) clusters formed by strongly interacting water molecules have the identical critical properties as on-lattice random percolation clusters [32]. These clusters may "condense" at low enough temperatures [33] giving rise to a second critical point in the volume–temperature–pressure phase diagram at a temperature of about 220 K and at about 1 kbar of pressure [34] – approximately the pressure at the bottom of the Mariana Trench. This second critical point – if it exists – would provide a parsimonious explanation for all of water's unusual properties, such as the familiar volume minimum at 4°C, below which the volume starts to rise [35].

4.3. Connectivity and human behavior

Connectivity is a fundamental feature of some human behavior. Makse et al. have studied experimental data on how people behave when they are "linked together" to form cities [36,37]. They find that the development units (people, capital, and resources) that form a city have features of gradient percolation of Sapoval and collaborators, i.e., the units have a larger probability of occurring near the center [38]. Gradient percolation refers the fact that the fraction of occupied sites is not homogeneous, but depends on where you are in the system. In the case of cities we let this be a monotonic function that decreases exponentially from the center.

The development units also interact with each other; they are not completely independent. To model this, we combine gradient percolation with Coniglio's correlated percolation [39], thereby marrying Sapoval to Coniglio and creating a Coniglio–Sapoval percolation model. Thus the probability that a site will be occupied by a development unit is an exponentially decreasing function of the distance from the center, and the development units are placed with a correlation parameter, i.e., they are not added at random, but each location is affected by the neighborhood's current occupancy [36,37].

Does this correlated gradient percolation model fit the data? Data in this area are not trivial to obtain. We have two sets: Berlin for the years 1875, 1920 and 1945, and London for 1981. The correlation is readily apparent, since if there were no correlation, the exponent describing the area distribution function $N(A)$ would be 2.45 and the data in fact have an exponent 2.06 characteristic of correlated percolation [36,37].

5. Summary

In summary, percolation is the study of connected objects. We can approach percolation as a worthwhile occupation in itself and discover the parameters that quantify percolation clusters, such as d_{fractal}, and d_{red}. When we study the theory of percolation, we find these parameters are related to scaling powers: $y_H = d_{\text{fractal}}$, $y_T = d_{\text{red}}$. When we study the *applications* of percolation we begin to discover new parameters are needed, such as d_{walk}, d_{spectral}, d_{backbone}, and d_{min}. When we study applications, we enrich our knowledge of percolation in ways we would not if we kept the topic in splendid isolation. Applications provide grist for the mill – real stimulation for the mind – and increase the probability of making new discoveries.

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