LETTER TO THE EDITOR

A real-space renormalization group for site and bond percolation†

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Abstract. We develop a real-space renormalization group which renormalizes probabilities directly in the percolation problem. An exact transformation is given in one dimension, and a cluster approach is presented for other lattices. Our results are in excellent agreement with series calculations for the critical percolation concentration p_c (both site and bond), and in good agreement for the correlation length exponent v_p. Additionally, in one dimension we include a field-like variable and calculate the remaining exponents.

The percolation problem (Shante and Kirkpatrick 1971) has been receiving renewed attention lately, and with its similarity to thermal phase transitions, the renormalization group (RG) comes readily to mind. See, for example, Wilson and Kogut (1974) and Niemeyer and Van Leeuwen (1974). Several workers have in fact attempted various RG approaches, and these have met with fair success. Harris *et al* (1975) and Dasgupta (1976) used the fact that the bond percolation problem has a direct mapping onto the s-state Ashkin–Teller–Potts model when $s \to 1$ (Kasteleyn and Fortuin 1969). They then use this model to do RG transformations both in real space and in ϵ-expansion from $d = 6$ dimensions. Others have transformed the bond probabilities directly. In particular, Young and Stinchcombe (1975) and Stinchcombe and Watson (1976) use decimation, while Kirkpatrick (1977) uses Migdal recursion relations.

The previous work concerns only the bond percolation problem. In this Letter we present a cluster approach which works on the probabilities directly, and applies to site as well as bond percolation. First we treat the site problem. We start by choosing a lattice which we partition into cells that both cover the lattice and maintain its original symmetry (figure 1). These cells will play the role of renormalized sites. Given that sites in the original lattice are independently occupied with probability p, we must choose a cell occupation probability $p' = \mathscr{R}(p)$ in such a way that $\mathscr{R}(p)$ contains the essential physics of percolation. To this end we note that percolation involves the formation of an infinite connected network—that is, one that actually 'gets across' the entire lattice. Below percolation only finite clusters are present. *Thus it is sensible to define a cell as occupied if and only if it contains a set of sites such that the cell 'percolates.'* This will define $\mathscr{R}(p)$ for us.

Since our transformation rescales the lattice spacing by a factor of b, we expect the

mean size of a cluster, ξ_p, to decrease as $\dagger$

$$\xi'_p = b^{-1}\xi_p. \tag{1}$$

Thus (see figure 2) if we start off at some $p < p_c$, then our transformation should give us $p' < p$, with iteration taking us to a stable fixed point $p^* = 0$. Likewise if $p > p_c$, we
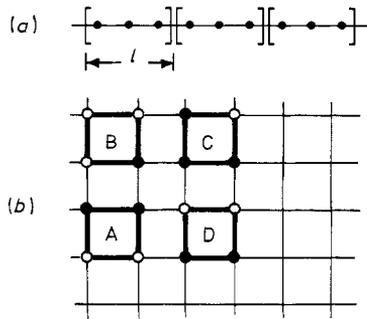
Figure 1. Rescaling a lattice by forming cells out of groups of sites: (*a*) The linear chain lattice is covered with cells of *l* sites, with $l = 3$ pictured. (*b*) Four-site cells on the square lattice are shown, with dots representing occupied sites. Cells C and D, for example, are both occupied (each cell can be traversed), but we cannot get from one cell to the other. This problem is overcome in the cluster approach treated in the text. Cells A, B, and C are connected on the site level, though only the next-nearest neighbour cells, A and C, are occupied. This forces us to introduce further-neighbour percolation probabilities.

expect $p' > p$ in order that ξ_p decrease, and we should flow toward a stable fixed point $p^* = 1$. If, however, we start off at $p = p_c$, ξ_p is infinite, and equation (1) is satisfied. Therefore, we expect $\mathscr{R}$ to have an *unstable* fixed point $p^* = p_c$. Further, as $p \to p_c$, ξ_p diverges as

$$\xi_p \sim |p - p_c|^{-\nu_p}, \tag{2}$$

where ν_p is the critical exponent for the 'correlation length'. We may calculate ν_p by linearizing about the fixed point and looking for the eigenvalue λ_p of the linearized transformation $\mathscr{R}^L$:

$$\mathscr{R}^L(p - p_c) = \lambda_p(p - p_c), \tag{3}$$

where here

$$\lambda_p = \mathrm{d}\mathscr{R}(p)/\mathrm{d}p|_{p=p^*}. \tag{4}$$

From equation (3) we get

$$\xi'_p = \lambda_p^{-\nu_p}\xi_p. \tag{5}$$

Comparing this with equation (1) we immediately find for the 'temperature' scaling power

$$y_p = \ln \lambda_p/\ln b = \nu_p^{-1}. \tag{6}$$

We demonstrate our approach first in one case where the transformation is exact.

$\dagger$ For $p > p_c$, ξ_p refers to the mean size of the *finite* clusters.
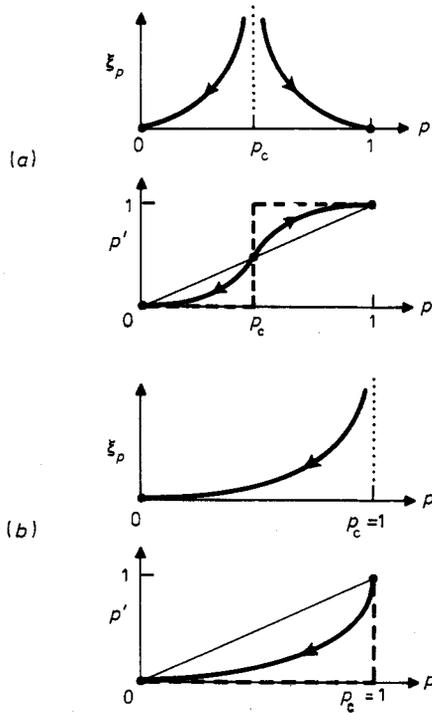
Figure 2. (*a*) The mean cluster size ξ_p diverges at p_c, and thus ought to be an unstable fixed point under an RG transformation. At $p = 0$ and $p = 1$ we have stable fixed points. We also sketch a typical transformation $p' = \mathscr{R}(p)$ (full line) and its limiting form as the cell size goes to infinity (broken line). The intersections with the line $p' = p$ locate the fixed points. (*b*) We show the special case $d = 1$. The transformation is $p' = p^l$ (full line). Due to the simple form of $\mathscr{R}(p)$ we may take the limit $l \to \infty$ (broken line), which should be compared with (*a*).

For the $d = 1$ linear chain lattice (figure 1(*a*)) we define an l-site cell as occupied if we can get across it, and this requires that all the sites be present. Thus

$$p' = \mathscr{R}(p) = p^l. \tag{7}$$

Setting $p' = p$, we find fixed points at $p^* = 0, 1$. The unstable fixed point gives p_c, and so (cf equations (4) and (6)) $p_c = 1$ and $\nu_p = 1$. These are in fact the exact results. When we plot $p' = \mathscr{R}(p)$ (figure 2(*b*)) for some finite l, and then take the limit $l \to \infty$, we get precisely the features we expect as the cell size approaches infinity. In particular, this shows that our rule of requiring a cell to 'percolate' makes sense, since by the definition of percolation, if $p < p_c$ we cannot 'get across' the (infinite) cell, and so $p' = 0$; if $p > p_c$ we can get across with probability unity, and so $p' = 1$ (note the broken curves in figure 2).

For $d = 2$ the simplest cell is the three-site cell on the triangular lattice. Applying our rule, the cell is occupied if all three sites are occupied or if any two sites are occupied and one is vacant, since in both cases we can 'get across'. However, with one or no sites the cell is vacant. Thus,

$$p' = \mathscr{R}(p) = p^3 + 3p^2(1 - p), \tag{8}$$

with fixed points at $p^* = 0, 1, \frac{1}{2}$. This RG therefore predicts $p_c = \frac{1}{2}$ for the triangular

lattice, which is in agreement with the exact results known for this lattice (Shante and Kirkpatrick 1971). Using equations (4) and (6) we calculate v_p,

$$v_p = \ln \sqrt{3}/\ln (3/2) = 1\cdot354\ldots, \tag{9}$$

which is in excellent agreement with the series result of Dunn *et al* (1975)†, $v_p = 1\cdot34 \pm 0\cdot02$.

When $d > 1$, our transformation involves an approximation. This can be seen by considering a group of neighbouring cells (cf figure 1(*b*)). There are some site configurations in which two nearest-neighbour cells (e.g. C and D) are occupied, though no path from one to the other exists, while other site configurations have two *next-nearest* neighbour cells joined via a 'vacant' intermediate cell.

To improve our approximation for $d > 1$, we use a cluster approach where we choose a group of cells for which the connectivity problem is treated exactly. The single-cell renormalization described earlier is then just a zeroth-order approximation, accounting for only the internal ('getting across') degrees of freedom. When clusters contain more than one cell we introduce 'interactions' (getting between cells). We now indicate how these interactions may be included. We will consider the square lattice with cells of four sites (figure 1(*b*)) because here the single-cell approximation is considerably worse. It leads to the recursion relation

$$\mathscr{R}(p) = p^4 + 4p^3(1 - p) + 4p^2(1 - p)^2, \tag{10}$$

since, as before, we must be able to traverse a cluster for it to be considered full. Equation (10) gives $p^* \approx 0\cdot38$ and $v_p \approx 1\cdot6$, whereas numerically (Shante and Kirkpatrick 1971) $p_c \approx 0\cdot59$. The next larger cluster contains two adjacent cells. For both cells to be present (probability $(p')^2$) we require that each cell is separately occupied, *and* that you can get from one cell to the other. (For example, none of the two cell configurations shown in figure 1(*b*), are counted.) One finds

$$\mathscr{R}(p) = [p^8 + 8p^7(1 - p) + 20p^6(1 - p)^2 + 20p^5(1 - p)^3 + 7p^4(1 - p)^4]^{1/2} \tag{11}$$

for which $p^* \approx 0\cdot57$ and $v_p \approx 1\cdot78$.

We have noticed that going to larger clusters of cells generally improves p_c, unless the cluster has a node-like structure where a single site is crucial for getting between cells (as in the case of clusters of three-site cells on the triangular lattice). The value of the exponent v_p however, seems sensitive to the symmetry of the cluster. For example our cluster of two cells which leads to equation (11) no longer has the full 90° rotational symmetry of the square lattice, and v_p gets worse.

At present we are generalizing our procedure both by continuing to larger clusters, and by including various kinds of further-neighbour percolation 'interactions'. Such 'interactions' are generated upon iteration for clusters such as that in figure 1(*b*). In particular, a next-nearest neighbour interaction included in the simple four-site cell improves the exponent to $v_p \approx 1\cdot40$.

For bond percolation it is not as obvious how to choose a cell that covers the lattice with bonds, and rescales to another *bond*. We choose (figure 3(*a*)) a simple eight-bond cell on the square lattice. If the cell can be traversed horizontally, then the heavy (renormalized) horizontal bond is present, and similarly for the vertical direction. We find

$$\mathscr{R}(p) = p^5 + 5p^4(1 - p) + 8p^3(1 - p)^2 + 2p^2(1 - p)^3, \tag{12}$$

† Although this result is for bond percolation on the square lattice, it is consistent with and more precise than other estimates of $v_p(d = 2)$.

with $p^* = 0.5$ (exact) and $v_p \approx 1.43$. The closeness of this value of v_p with those calculated earlier for the site problem supports 'site-bond' universality. The generalization of this simple cell to the cubic lattice gives $p^* \approx 0.22$ and $v_p \approx 1.04$. Numerical work predicts
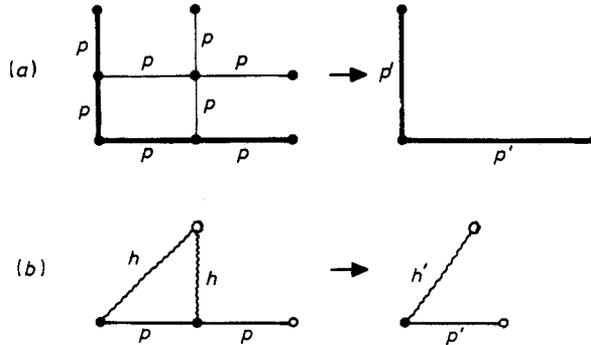
Figure 3. (*a*) A single cell on the square lattice used for bond percolation. The heavy lines represent the renormalized vertical and horizontal bonds. These are independently present if the cell may be traversed vertically or horizontally, respectively. (*b*) A cell for $d = 1$ bond percolation. Each bond in the original lattice (solid lines) is present with probability p, while additional bonds (wavy lines) connect the sites in this cell to the 'ghost site' with probability h.

(Shante and Kirkpatrick 1971 and Kirkpatrick 1976) $p_c \approx 0.247$ and $v_p = 0.86 \pm 0.05$.

For the bond problem, in addition to investigating the generalizations discussed earlier for the site problem, we are also studying the inclusion of a 'ghost spin', or magnetic-field-like variable (Kasteleyn and Fortuin 1969), to make possible the calculation of other exponents. We have done this exactly for $d = 1$. We choose a cell made of two bonds and two sites (see figure 3(*b*)). Note that one end site (open circle) belongs to another cell. Each site is joined to the 'ghost site' with probability h. Then, the probability that we can traverse the cell is

$$p' = p^2 + p(1 - p)h^2, \tag{13a}$$

where the second term arises from a path through the 'ghost site'. By rescaling the probability that we can get to the 'ghost site' from where we enter the cell (the open end in figure 3(*b*)) we arrive at

$$h' = h[p(1 - h) + 1]/[p + (1 - p)h^2]. \tag{13b}$$

From these coupled equations we find an unstable fixed point at $p^* = 1$, $h^* = 0$ with a field eigenvalue of $\lambda_h = 2$. It follows that

$$\eta_p = d + 2 - 2 \ln \lambda_h / \ln b = 1, \tag{14a}$$

and

$$\gamma_p = (2 - \eta_p)v_p = 1, \tag{14b}$$

which are the exact results in one dimension.

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