The size variance relationship of business firm growth rates

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The relationship between the size and the variance of firm growth rates is known to follow an approximate power-law behavior $\sigma(S) \approx S^{-\beta(S)}$ where S is the firm size and $\beta(S) \approx 0.2$ is an exponent that weakly depends on S. Here, we show how a model of proportional growth, which treats firms as classes composed of various numbers of units of variable size, can explain this size-variance dependence. In general, the model predicts that $\beta(S)$ must exhibit a crossover from $\beta(0) = 0$ to $\beta(\infty) = 1/2$. For a realistic set of parameters, $\beta(S)$ is approximately constant and can vary from 0.14 to 0.2 depending on the average number of units in the firm. We test the model with a unique industry-specific database in which firm sales are given in terms of the sum of the sales of all their products. We find that the model is consistent with the empirically observed size-variance relationship.

preferential attachment | pharmaceutical industry | distributions

Gibrat was probably the first who noticed the skew size distribution of business firms (1). As a simple candidate explanation he postulated the "Law of Proportionate Effect" according to which the expected value of the growth rate of a business firm is proportional to the current size of the firm (2). Several models of proportional growth have subsequently been introduced in economics (3–6). In particular, Simon and colleagues (7, 8) examined a stochastic process for Bose–Einstein statistics similar to the one originally proposed by Yule (9) to explain the distribution of sizes of genera. The Law of Proportionate Effect implies that the variance σ^2 of firm growth rates is independent of size, whereas, according to the Simon model, it is inversely proportional to the size of business firms. The two predictions have not been confirmed empirically and, following Stanley and colleagues (10), several scholars (11, 12) have recently found a nontrivial relationship between the size of the firm S and the variance σ^2 of its growth rate $\sigma \approx S^{-\beta}$ with $\beta \approx 0.2$.

Numerous attempts have been made to explain this puzzling evidence by considering firms as collections of independent units of uneven size (10, 12–18) but existing models do not provide a unifying explanation for the probability density functions of the growth and size of firms as well as the size-variance relationship. Thus, the scaling of the variance of firm growth rates is still an unsolved problem in economics (19, 20). Recent papers (21–25) provide a general framework for the growth and size of business firms based on the number and size distribution of their constituent parts (12–15, 21, 26–29). Specifically, Fu and colleagues (21) present a model of proportional growth in both the number of units and their size, drawing some general implications on the mechanisms which sustain business firm growth. The model in ref. 21 accurately predicts the shape of the distribution of the growth rates (21, 22) and the size distribution of firms (23). In this article, we derive the implications of the model in ref. 21 on the size-variance relationship. The main conclusion is that the size-variance relationship is not a true power law with a single well-defined exponent β but undergoes a slow crossover from $\beta = 0$ for $S \to 0$ to $\beta = 1/2$ for $S \to \infty$. The predictions of the model are tested in both real-world and simulation settings.

The Model

In the model presented in ref. 21 and summarized in the supporting information (SI) *Text*, firms consist of a random number of units of variable size. The number of units K is defined as in the Simon model. The size of the units ξ evolves according to a multiplicative brownian motion (Gibrat process). Thus, both the growth distribution, P_η, and the size distribution, P_ξ, of the units are lognormal.

To derive the size-variance relationship we must compute the conditional probability density of the growth rate $P(g|S, K)$, of a firm with K units and size S. For $K \to \infty$ the conditional probability density function $P(g|S, K)$ develops a tent-shape functional form, because in the center it converges to a Gaussian distribution with the width decreasing in inverse proportion to $\sqrt{K}$, whereas the tails are governed by the behavior of the growth distribution of a single unit that remains to be wide independently of K.

We can also compute the conditional probability $P(S|K)$, which is the convolution of K unit size distributions P_ξ. In case of lognormal P_ξ with a large logarithmic variance V_ξ and mean m_ξ, the convergence of $P(S|K)$ to a Gaussian is very slow (23). Because $P(S, K) = P(S|K)P(K)$, we can find

$$P(g|S) = \sum P(g|S, K)P(S|K)P(K), \qquad [1]$$

where all of the distributions $P(g|S, K)$, $P(S|K)$, $P(K)$ can be found from the parameters of the model. $P(S|K)$ has a sharp maximum near $S = S_K \equiv K_{\mu_\xi}$, where $\mu_\xi = \exp(m_{\xi} + V_\xi/2)$ is the mean of the lognormal distribution of the unit sizes. Conversely, $P(S\,|K)$ as a function of K has a sharp maximum near $K_S = S/\mu_\xi$. For the values of S such that $P(K_S) \gg 0$, $P(g|S) \approx P(g|K_S)$, because $P(S|K)$ serves as a $\delta(K - K_S)$ so that only terms with $K \approx K_S$ make a dominant contribution to the sum of Eq. **1**. Accordingly, one can approximate $P(g|S)$ by $P(g|K_S)$ and $\sigma(S)$ by $\sigma(K_S)$. However, all firms with $S < S_1 = \mu_\xi$ consist essentially of only one unit and thus

$$\sigma(S) = \sqrt{V_\eta} \qquad [2]$$

for $S < \mu_\xi$. For large S, if $P(K_S) > 0$

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$$\sigma(S) = \sigma(K_S) = \sqrt{V/K_S} = \frac{\exp(3V_\xi/4 + m_\xi/2)\sqrt{\exp(V_\eta) - 1}}{\sqrt{S}} \quad [3]$$

where m_η and V_η are the logarithmic mean and variance of the unit growth distributions P_η and $V = \exp(V_\xi)[\exp(V_\eta) - 1]$, as in ref. 21. Thus, one expects to have a crossover from $\beta = 0$ for $S < \mu_\xi$ to $\beta = 1/2$ for $S \gg S^*$, where

$$S^* = \exp(3V_\xi/2 + m_\xi)(\exp(V_\eta) - 1)/V_\eta \quad [4]$$

is the value of S for which Eqs. **2** and **3** give the same value of $\sigma(S)$. Note that for small $V_\eta < 1$, $S^* \approx \exp(3V_\xi/2 + m_\xi)$. The range of crossover extends from S_1 to S^*, with $S^*/S_1 = \exp(V_\xi) \to \infty$ for $V_\xi \to \infty$. Thus, in the double-logarithmic plot of σ vs. S one can find a wide region in which the slope β slowly varies from 0 to 1/2 ($\beta \approx 0.2$) in agreement with many empirical observations.

The crossover to $\beta = 1/2$ will be observed only if $K^* = S^*/\mu_\xi = \exp(V_\xi)$ is such that $P(K^*)$ is significantly larger than zero. For the distribution $P(K)$ with a sharp exponential cutoff $K = K_0$, the crossover will be observed only if $K_0 \gg \exp(V_\xi)$.

Two scenarios are possible for $S > S_0 = K_0\mu_\xi$. In the first, there will be no firms with $S \gg S_0$. In the second, if the distribution of the size of units P_ξ is very broad, large firms can exist just because the size of a unit can be larger than S_0. In this case, exceptionally large firms might consist of one extremely large unit ξ_{max}, whose fluctuations dominate the fluctuations of the entire firm.

One can introduce the effective number of units in a firm $K_e = S/\xi_{max}$, where ξ_{max} is the largest unit of the firm. If $K_e < 2$, we would expect that $\sigma(S)$ will again become equal to its value for small S given by Eq. **2**, which means that under certain conditions $\sigma(S)$ will start to increase for very large firms and eventually becomes the same as for small firms.

Whether such a scenario is possible depends on the complex interplay of V_ξ and $P(K)$. The crossover to $\beta = 1/2$ will be seen only if $P(K > K^*) > P(\xi > S^*)$, which means that such large firms predominantly consist of a large number of units. Taking into account the equation of P_ξ, one can see that $P(\xi > S^*) \sim \exp(-9/8V_\xi)$.

On the one hand, for an exponential $P(K)$, this implies that $\exp(-\exp(V_\xi)/K_0) > \exp(-9/8V_\xi)$ or

$$V_\xi > 8\exp(V_\xi)/(9K_0). \quad [5]$$

This condition is easily violated if $V_\xi \gg \ln K_0$. Thus, for the distributions $P(K)$ with exponential cutoff we will never see the crossover to $\beta = 1/2$ if $V_\xi \gg \ln K_0$.

On the other hand, for a power-law distribution $P(K) \sim K^{-\phi}$, the condition of the crossover becomes $\exp(V_\xi)^{1-\phi} > \exp(-9/8V_\xi)$, or $(\phi - 1)V_\xi < 9/8V_\xi$ which is rigorously satisfied for

$$\phi < 17/8 \quad [6]$$

but even for larger ϕ values this condition is not dramatically violated. Thus, for power-law distributions, we expect a crossover to $\beta = 1/2$ for large S and a significantly large number N of firms: $NP(K^*) > 1$. The sharpness of the crossover mostly depends on V_ξ. For power-law distributions we expect a sharper crossover than for exponential ones because the majority of firms in a power-law distribution have a small number of products K, and hence $\beta = 0$ almost up to S^*, the size at which the crossover is observed. For exponential distributions we expect a slow crossover that is interrupted if V_ξ is comparable to $\ln K_0$. For $S \gg S_1$ this crossover is well represented by the behavior of $\sigma(K_S)$.

We confirm these heuristic arguments by means of computer simulations (see Figs. S1–S4).

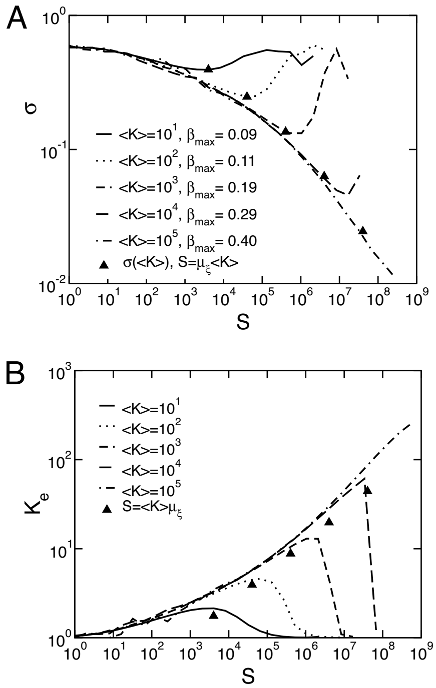

Fig. 1. Simulation results for the size variance relationship and the effective number of units. (*A*) Simulation results for $\sigma(S)$ according to Eq. **1** for exponential $P(K) = \exp(-K/K_0)/K_0$ with $K_0 = 10, 10^2, 10^3, 10^4, 10^5$ and lognormal P_ξ and P_η with $V_\xi = 5.13$, $m_\xi = 3.44$, $V_\eta = 0.36$, $\mu_\eta = 0.016$ computed for the pharmaceutical database. One can see that, for small enough S and for different K_0, $\sigma(S)$ follows a universal curve that can be well approximated with $\sigma(K_S)$, with $K_S = S/\mu_\xi \approx S/405$. For $K_S > K_0$, $\sigma(S)$ departs from the universal behavior and starts to increase. This increase can be explained by the decrease of the effective number of units $K_e(S)$ for the extremely large firms. The maximal negative slope β_{max} increases as K_0 increases in agreement with the predictions of the central limit theorem. (*B*) One can see, that $K_e(S)$ reaches its maximum at approximately $S \approx K\mu_\xi$. The positions of maxima in $K_e(S)$ coincide with the positions of minima in $\sigma(S)$.

Figs. 1 and 2 illustrate the importance of the effective number of units K_e. When K_S becomes larger than K_0, $\sigma(S)$ starts to follow $\sigma(K_e)$. Accordingly, for very large firms $\sigma(S)$ becomes almost the same as for small firms. The maximal negative value

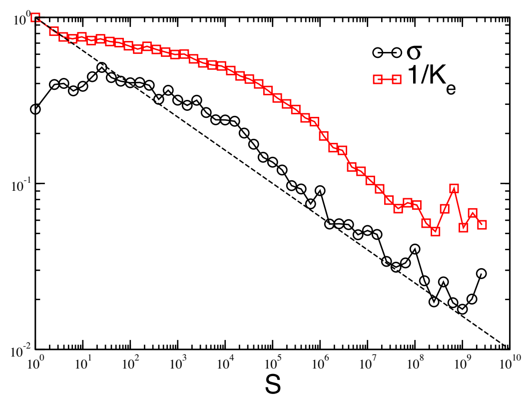

Fig. 2. The standard deviation of firm growth rates (σ) (circles), and the share of the largest products ($1/K_e$) (squares) versus the size of the firms in the pharmaceutical industry (S). As predicted by our model for $S < S_1 = \mu_\xi \approx 3.44$, $\beta \approx 0$. For $S > S_1$ β increases but never reaches 1/2 because of the slow growth of the effective number of products (K_e). The flattening of the upper tail is due to some large companies with unusually large products.

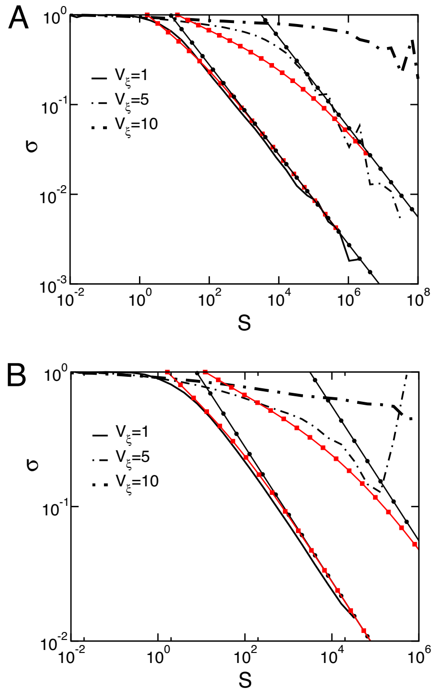

Fig. 3. Size-variance relationship $\sigma(S)$ for various V_ξ with $P(K) \sim K^{-2}$ (*A*) and real $P(K)$ (*B*). A sharp crossover from $\beta = 0$ to $\beta = 1/2$ is seen for the power-law distribution even for large values of V_ξ. In case of real $P(K)$ one can see wide crossover regions in which $\sigma(S)$ can be approximated by a power-law relationship with $0 < \beta < 1/2$. Note that the slope of the graphs (β) decreases with the increase of V_ξ. The graphs of $\beta(K_S)$ and their asymptotes are also shown with squares and circles, respectively.

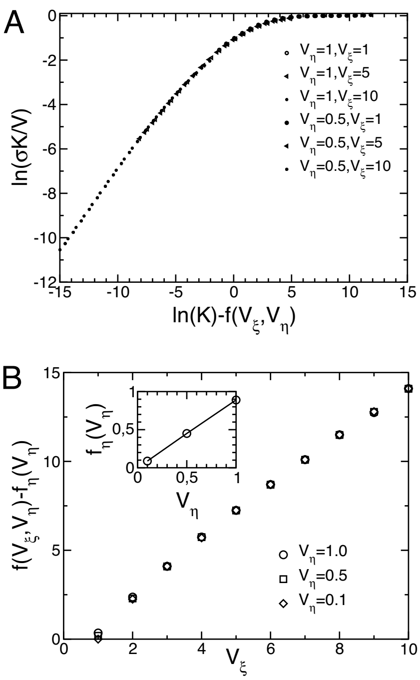

Fig. 4. Simulation results for the variance of firm growth rates with lognormal distributions of the size and growth rates of firm units. (*A*) Simulation results for $\sigma^2(K)$ in case of lognormal P_ξ and P_η and different V_ξ and V_η plotted on a universal scaling plot as a function of scaling variable $z = \ln(K) - f(V_\xi, V_\eta)$. (*B*) The shift function $f(V_\xi, V_\eta)$. The graph shows that $f(V_\xi, V_\eta) \approx f_\xi(V_\xi) + f_\eta(V_\eta)$. Both $f_\xi(V_\xi)$ and $f_\eta(V_\eta)$ (*Inset*) are approximately linear functions.

of the slope $\beta_{\max}$ of the double-logarithmic graphs presented in Fig. 1*A* correspond to the inflection points of these graphs, and can be identified as approximate values of β for different values of K_0. One can see that $\beta_{\max}$ increases as K_0 increases from a small value close to 0 for $K_0 = 10$ to a value close to 1/2 for $K_0 = 10^5$ in agreement with the predictions of the central limit theorem.

To further explore the effect of the $P(K)$ on the size-variance relationship we select $P(K)$ to be a pure power law $P(K) \sim K^{-2}$ (Fig. 3*A*). Moreover, we consider a realistic $P(K)$ where K is the number of products by firms in the pharmaceutical industry (Fig. 3*B*). This distribution can be well approximated by a Yule distribution with $\phi = 2$ and an exponential cutoff for large K. Fig. 3 shows that, for a scale-free power-law distribution $P(K)$, the size-variance relationship depicts a steep crossover from $\sigma = \sqrt{V_\eta}$ given by Eq. **2** for small S to $\sigma = \sqrt{V/K_S}$ given by Eq. **3** for large S, for any value of V_ξ.

As we see, the size-variance relationship of firms $\sigma(S)$ can be well approximated by the behavior of $\sigma(K_S)$ (Fig. 1*A*). It was shown in ref. 24 that, for realistic V_ξ, $\sigma^2(K)$ can be approximated in a wide range of K as $\sigma(K) \sim K^{-\beta}$ with $\beta \approx 0.2$, which eventually crosses over to $K^{-1/2}$ for large K. In other words, one can write $\sigma(K) \sim K^{-\beta(K)}$, where $\beta(K)$, defined as the slope of $\sigma(K)$ on a double-logarithmic plot, increases from a small value dependent on V_ξ at small K to 1/2 for $K \to \infty$. Accordingly, one can expect the same behavior for $\sigma(S)$ for $K_S < K_0$.

Thus, it would be desirable to derive an exact analytical expression for $\sigma(K)$ in case of lognormal and independent P_ξ and P_η. Unfortunately the radius of convergence of the expansion of a logarithmic growth rate in inverse powers of K is equal to zero, and these expansions have only a formal asymptotic meaning for $K \to \infty$. However, these expansions are useful because they demonstrate that μ and σ do not depend on m_η and m_ξ except for the leading term in μ: $m_0 = m_\eta + V_\eta/2$. Not being able to derive close-form expressions for σ (see *SI Text*), we perform extensive computer simulations, where ξ and η are independent random variables taken from lognormal distributions P_ξ and P_η with different V_ξ and V_η. The numerical results (Fig. 4) suggest that

$$\ln \sigma^2(K)K/C \approx F_\sigma[\ln(K) - f(V_\xi, V_\xi)], \qquad [7]$$

where $F_\sigma(z)$ is a universal scaling function describing a crossover from $F_\sigma(z) \to 0$ for $z \to \infty$ to $F_\sigma(z)/z \to 1$ for $z \to -\infty$ and $f(V_\xi,V_\eta) \approx f_\xi(V_\xi)+f_\eta(V_\eta)$ are functions of V_ξ and V_η that have linear asymptotes for $V_\xi \to \infty$ and $V_\eta \to \infty$ (Fig. 4*B*).

Accordingly, we can try to define $\beta(z) = (1 - dF_\sigma/dz)/2$ (Fig. 5*A*). The main curve $\beta(z)$ can be approximated by an inverse linear function of z, when $z \to -\infty$ and by a stretched exponential as it approaches the asymptotic value 1/2 for $z \to +\infty$. The particular analytical shapes for these asymptotes are not known and derived solely from least-square fitting of the numerical data. The scaling for $\beta(z)$ is only approximate with significant deviations from a universal curve for small K. The minimal value for β practically does not depend on V_η and is approximately inverse proportional to a linear function of V_ξ:

$$\beta_{\min} = \frac{1}{pV_\xi + q} \qquad [8]$$

where $P \approx 0.54$ and $q \approx 2.66$ are universal values. (Fig. 5*B*). This finding is significant for our study, because it indicates that near its minimum, $\beta(K)$ has a region of approximate constancy with the value $\beta_{\min}$ between 0.14 and 0.2 for V_ξ between 4 and 8. These values of V_ξ are quite realistic and correspond to the distribution of unit sizes spanning over from roughly 2 to 3 orders of magnitude (68% of all units), which is the case in the majority

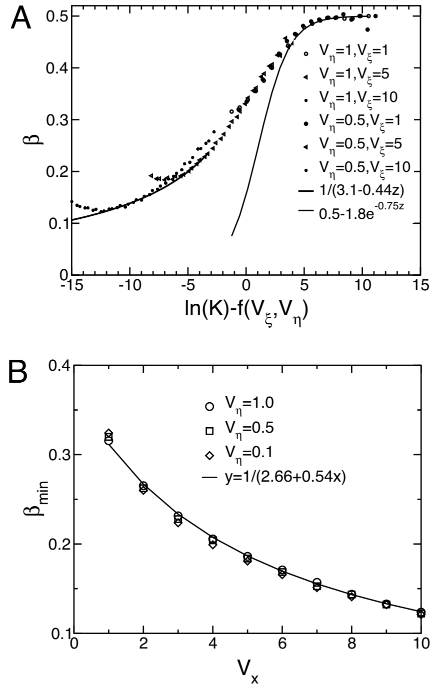

Fig. 5. The effective exponent of the size variance relationship and its dependence on the variance of the growth rates of firm units. (*A*) The effective exponent $\beta(z)$ obtained by differentiation of $\sigma^2(z)$ plotted in Fig. 4*A*. Solid lines indicate least-square fits for the left and right asymptotes. The graph shows significant deviations of $\beta(K, V_\xi, V_\eta)$ from a universal function $\beta(z)$ for small K, where $\beta(K)$ develops minima. (*B*) The dependence of the minimal value of β on V_ξ. One can see that this value practically does not depend on V_η and is inversely proportional to the linear function of V_ξ.

of the economic and ecological systems. Thus our study provides a reasonable explanation for the abundance of value of $\beta \approx 0.2$.

The above analysis shows that $\sigma(S)$ is not a true power-law function, but undergoes a crossover from $\beta = \beta_{\min}(V_\xi)$ for small firms to $\beta = 1/2$ for large firms. However, this crossover is expected only for very broad distributions $P(K)$. If it is very unlikely to find a firm with $K > K_0$, $\sigma(S)$ will start to grow for $S > K_0\mu_\xi$. Empirical data do not show such an increase (Fig. 6), because in reality few giant firms rely on a few extremely large units. These firms are extremely volatile and hence unstable.

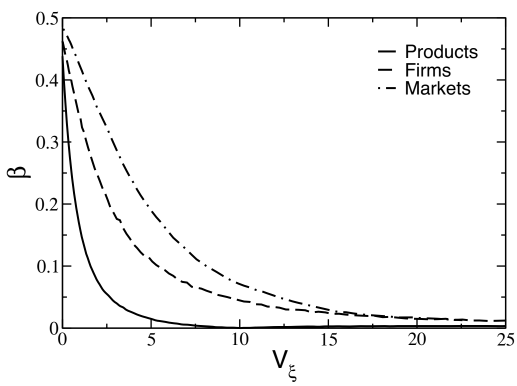

Fig. 6. The scaling of the size-variance relationship as a function of V_ξ. β decays rapidly from 1/2 to 0 for $V_\xi \to \infty$. In the simulation we keep the real $P(K)$ for products, companies, and markets and assign products drawn from a lognormal distribution with the empirically observed mean m_ξ and variance $0 < V_\xi \leq 25$.

Therefore, for real data we do see neither a crossover to $\beta = 1/2$ nor an increase of σ for large companies.

Empirical Evidence

Because the size-variance relationship depends on the partition of firms into their constituent components, to properly test our model one must decompose an industrial system into parts. In this section we analyze a unique database, kindly provided by the European Pharmaceutical Regulation and Innovation Systems (EPRIS) program, which has recorded the sales figures of 916,036 pharmaceutical products commercialized by 7,184 firms worldwide from 1994 to 2004. The database covers the whole size distribution for products and firms and monitors flows of entry and exit at every level of aggregation. Products are classified by companies, markets, and international brand names, with different distributions $P(K)$ with $\langle K \rangle = K_0$ ranging from 5.8 for international products to almost 1,600 for markets (Table 1). If firms have on average K_0 products and $V_\xi \ll \ln K_0$, the scaling variable $z = K_0$ is positive and we expect $\beta \to 1/2$. On the contrary, if $V_\xi \gg \ln K_0$, $z < 0$ and we expect $\beta \to 0$. These considerations work only for a broad distribution of $P(K)$ with mild skewness, such as an exponential distribution. At the opposite extreme, if all companies have the same number of products, the distribution of S is narrowly concentrated near the most probable value $S_0 = \mu_\xi K$ and there is no reason to define $\beta(S)$. Only very rarely $S \gg S_0$, because of a low probability of observing an extremely large product that dominates the fluctuation of a firm. Such a firm is more volatile than other firms of equal size. This would imply negative β. If $P(K)$ is power-law distributed, there is a wide range of values of K, so that there are always firms for which $\ln K \gg V_\xi$ and we can expect a slow crossover from $\beta = 0$ for small firms to $\beta = 1/2$ for large firms. In this case, for a wide range of empirically plausible V_ξ, β is far from 1/2 and statistically different from 0. The estimated value of the size-variance scaling coefficient β goes from 0.123 for products to 0.243 for therapeutic markets with companies in the middle (0.188) (Table 1, Fig. 2).

The model in ref. 21 relies on general assumptions of independence of the growth of products from each other and from the number of products K. However, these assumptions could be violated and other reasons for the scaling of the size-variance relationship such as units interdependence, size and time dependence must be considered (see the *SI Text* for a discussion of candidate explanations). To discriminate among different plausible explanations we run a set of simulations in which we keep the real $P(K)$ and randomly reassign products to firms. In the first simulation we randomly reassign products by keeping the real-world relationship between the size, ξ, and growth, η, of products. In the second simulation we also reassign η. Finally, in the last simulation, we generate elementary units according to a multiplicative brownian motion (Gibrat process) with empirically estimated values of the mean and variance of ξ and η. Table 1 summarizes the results of our simulations.

The first simulation allows us to check for the size dependence and unit interdependence hypotheses by randomly reassigning elementary units to firms and markets. In doing that, we keep the number of the products in each class and the history of the fluctuation of each product sales unchanged. As for the size dependence, our analysis shows that there is indeed strong correlation between the number of products in the company and their average size defined as $\langle \xi(K) \rangle = \langle 1/K \, \Sigma_{i=1}^{K} \xi_i \rangle$, where $\langle \, \rangle$ indicates averaging over all companies with K products. We observe an approximate power-law dependence $\langle \xi(K) \rangle \sim K^\gamma$, where $\gamma = 0.38$. If this would be a true asymptotic power law holding for $K \to \infty$ then the average size of the company of K products would be proportional to $\xi(K)K \sim K^{1+\gamma}$. Accordingly, the average number of products in the company of size S would

Table 1. The size-variance relationship $\sigma(S) \sim S^{-\beta(S)}$: Estimated values of β and simulation results β^* at different levels of aggregations from products to markets

	N	K_0	β_1	β_1^*	β_2^*	β_3^*
Markets	574	1,596.9	0.243	0.213	0.232	0.221
Firms	7,184	127.5	0.188	0.196	0.125	0.127
International products	189,302	5.8	0.151	0.175	0.038	0.020
All products	916,036	–	0.123	0.123	0	0

In simulation 1 (β_1^*) products are randomly reassigned to firms and markets. In simulation 2 (β_2^*) the growth rates of products are reassigned too. In simulation 3 (β_3^*) we reproduce the model in ref. 21 with real $P(K)$ and estimated values of $m_\xi = 7.58$ and $V_\xi = 2.10$.

scale as $K_0(S) \sim S^{1/(1+\gamma)}$ and consequently, due to central limit theorem, $\beta = 1/(2 + 2\gamma)$. In our database, this would mean that the asymptotic value of $\beta = 0.36$. Similar logic was used to explain β in refs. 11 and 15. Another effect of random redistribution of units will be the removal of possible correlations among η_i in a single firm (unit interdependence). Removal of positive correlations would decrease β, whereas removal of negative correlations would increase β. The mean correlation coefficient of the product growth rates at the firm level $\langle\rho(K)\rangle$ also has an approximate power-law dependence $\langle\rho(K)\rangle \sim K^{\zeta}$, where $\zeta = -0.36$. Because larger firms have bigger products and are more diversified than small firms, the size dependence and unit interdependence cancel out and β practically does not change if products are randomly reassigned to firms.

To control the effect of time dependence, we keep the sizes of products ξ_i and their number K_α at year t for each firm α unchanged, so $S_t = \sum_{i=1}^{K_\alpha}\xi_i$ is the same as in the empirical data. However, to compute the sales of a firm in the following year $\tilde{S}_{t+1} = \sum_{i=1}^{K_\alpha}\xi_i'$, we assume that $\xi_i' = \xi_i\eta_i$, where η_i is an annual growth rate of a randomly selected product. The surrogate growth rate $\tilde{g} = \ln\frac{(\tilde{S}_{t+1})}{(S_t)}$ obtained in this way does not display any size-variance relationship at the level of products ($\beta_2^* = 0$). However, we still observe a size-variance relationship at higher levels of aggregation. This test demonstrates that 1/3 of the size-variance relationship depends on the growth process at the level of elementary units which is not a pure Gibrat process. However, asynchronous product life cycles are washed out on aggregation and there is a persistent size-variance relationship that is not due to product autocorrelation.

Finally we reproduced the model in ref. 21 with the empirically observed $P(K)$ and the estimated moments of the lognormal distribution of products ($m_\xi = 7.58$, $V_\xi = 4.41$). We generate N random products according to our model (Gibrat process) with the empirically observed values of V_ξ and m_ξ. As we can see in Table 1, the model in ref. 21 closely reproduce the values of β at any level of aggregation. We conclude that the model in ref. 21 correctly predicts the size-variance relationship and the way it scales under aggregation.

The variance of the size of the constituent units of the firm V_ξ and the distribution of units into firms are both relevant to explain the size-variance relationship of firm growth rates.

Simulations results in Fig. 6 reveal that if elementary units are of the same size ($V_\xi = 0$) the central limit theorem will work properly and $\beta \approx 1/2$. As predicted by our model, by increasing the value of V_ξ we observe at any level of aggregation the crossover of β form 1/2 to 0. The crossover is faster at the level of markets than at the level of products due to the higher average number of units per class K_0. However, in real-world settings the central limit theorem never applies because firms have a small number of components of variable size ($V_\xi > 0$). For empirically plausible values of V_ξ and K_0 $\beta \approx 0.2$.

Discussion

Firms grow over time as the economic system expands and new investment opportunities become available. To capture new business opportunities firms open new plants and launch new products, but the revenues and return to the investments are uncertain. If revenues were independent random variables drawn from a Gaussian distribution with mean m_e and variance V_e one should expect that the standard deviation of the sales growth rate of a firm with K products will be $\sigma(S) \sim S^{-\beta(S)}$ with $\beta = 1/2$ and $S = m_e K$. On the contrary, if the size of business opportunities is given by a multiplicative brownian motion (Gibrat process) and revenues are independent random variables drawn from a lognormal distribution with mean m_ξ and variance V_ξ, the central limit theorem does not work effectively and $\beta(S)$ exhibits a crossover from $\beta = 0$ for $S \to 0$ to $\beta = 1/2$ for $S \to \infty$. For realistic distributions of the number and size of business opportunities, $\beta(S)$ is approximately constant, as it varies in the range from 0.14 to 0.2 depending on the average number of units in the firm K_0 and the variance of the size of business opportunities V_ξ. This implies that a firm of size S is expected to be riskier than the sum of S firms of size 1, even in the case of constant returns to scale and independent business opportunities.

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