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High-frequency trading model for a complex trading hierarchy

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Financial markets exhibit a complex hierarchy among different processes, e.g. a trading time marks the initiation of a trade, and a trade triggers a price change. High-frequency trading data arrive at random times. By combining stochastic and agent-based approaches, we develop a model for trading time, trading volume, and price changes. We generate intertrade time (time between successive trades) Δt_i, and the number of shares traded $q(\Delta t_i)$ as two independent but power-law autocorrelated processes, where Δt_i is subordinated to $q(\Delta t_i)$, and Δt_i is more strongly correlated than $q(\Delta t_i)$. These two power-law autocorrelated processes are responsible for the emergence of strong power-law correlations in (a) the total number of shares traded $N(\Delta T)$ and (b) the share volume $Q_{\Delta T}$ calculated as the sum of the number of shares q_i traded in a fixed time interval ΔT. We find that even though $q(\Delta t_i)$ is weakly power-law correlated, due to strong power-law correlations in Δt_i, the (integrated) share volume $Q(\Delta T) \equiv \sum_{i=1}^{\Delta T} q(\Delta t_i)$ exhibits strong long-range power-law correlations. We propose that intertrade times and bid–ask price changes share the same volatility mechanism, yielding the power-law autocorrelations in absolute values of price change and power-law tails in the distribution of price changes. The model generates the log-linear functional relationship between the average bid–ask spread $\langle S \rangle_{\Delta T}$ and the number of trade occurrences $N_{\Delta T}$, and between $\langle S \rangle_{\Delta T}$ and $Q_{\Delta T}$. We find that both results agree with empirical findings.

Keywords: Subordinated processes; High-frequency data; Fractionally integrated processes

1. Introduction

The study of price dynamics is the study of price changes (Clark 1973, Schwert 1989, Farmer *et al*. 2004, Gillemot *et al*. 2006). Empirical evidence indicates that extremely complex trading activities affect price changes. In one of the first attempts to model this activity, Clark (1973) uses a discrete stochastic process t_i to represent times at which trading occurs. Upon this stochastic process, a new stochastic process $X(t_i)$ is defined representing, for example, a stock price at time t_i. The process t_i is said to be subordinated to $X(t_i)$. Clearly, how quickly prices respond to trades occurring at t_i determines market liquidity, and liquidity is related to the ease with which securities are bought and sold without substantial price changes. To emphasize the importance of subordinate stochastic processes, Riccaboni *et al*. (2008) recently proposed a subordinate stochastic process for the model of proportional growth.

There are two main approaches to model price dynamics: the stochastic approach (Simon 1955, Clark 1973, Fu *et al*. 2005) and an agent-based approach (Maslov 2000, Maslov and Mills 2000, Kullmann and Kertész 2001, Slanina 2001, Scalas *et al*. 2006, Scalas 2006). These two approaches we can understand, for example, by comparing modeling long-range correlations in price changes, ΔS_t. In the stochastic approach, one models these correlations by assuming that ΔS_t depends on its previous values $\Delta S_t \equiv \sum_i a_i \Delta S_{t-i}$. The choice for statistical weights a_i determines, first, whether we want long- or short-range dependence in the autocorrelations of ΔS_t, and, second, which functional dependence we want to obtain for the autocorrelation function.

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The agent-based approach models security market microstructure starting from different traders (agents) and defining the trading rules among the agents, which, for example, may finally yield long-range correlations in price changes. Several papers propose models for artificial markets populated with heterogeneous agents endowed with learning and optimization capabilities (Palmer *et al.* 1994, Levy *et al.* 1994, Lux and Marchesi 1999, Raberto *et al.* 2001, Eisler *et al.* 2009, Ponta *et al.* 2011).

Here we combine stochastic and agent-based approaches to create a hybrid price dynamics model to simulate empirical evidence reported on bid–ask spread, stock price autocorrelations, and trading volume. In part, we follow the subordinated stochastic process proposed by Clark (1973). We first define a process for trading times t_i. When trading occurs, a package of stocks (volume) denoted by q_i changes owner. Thus, in terms of the Clark (1973) process, in our model t_i is subordinated to the number of shares traded $q(t_i)$. However, in contrast to Clark (1973), we define both t_i and $q(t_i)$ as long-range correlated processes. When trading occurs it triggers a price change. In our model a co-movement between intertrade time, defined as $\Delta t \equiv t_i - t_{i-1}$, and volatility, defined as the absolute value of the price change, exists because the process controlling Δt also controls the bid–ask spread (the difference between ask and bid). The model generates power-law autocorrelations in absolute returns (Ding *et al.* 1993, Cizeau *et al.* 1997, Liu *et al.* 1997, 1999) and power-law tails in distributions of returns (Lux 1996, Gopikrishnan *et al.* 1999, Plerou *et al.* 1999). It also yields a log-linear functional relationship between the average bid–ask spread $\langle S\rangle_{\Delta T}$ and the number of trades $N_{\Delta T}$, and between $\langle S\rangle_{\Delta T}$ and the share volume traded $Q_{\Delta T}$.

2. Empirical evidence

When ink particles diffuse in water, the collision of each ink particle with numerous water molecules causes it to move in a random walk pattern (Chandrasekhar 1943). The distance covered by the particle after a time ΔT is $X_{\Delta T} = \sum_{i=1}^{N_{\Delta T}} \Delta x_i$ where $X_{\Delta T}$ is Gaussian distributed and short-range correlated, $N_{\Delta T}$ denotes the number of collisions during the interval ΔT, and Δx_i is the change of position of the ink particle after collision. A more complex variation of the classic diffusion problem exists in finance, with intertrade times—which are the time intervals between two consecutive trades in the market. First, intertrade times are not Gaussian uncorrelated, but are power-law correlated variables (Ivanov *et al.* 2004). Second, financial markets are characterized by many complex hierarchies among different processes, and the number of trading times is only one variable among others such as the number of shares traded and the share price. The hierarchy is roughly the following: the trading time marks the initiation of the trade, and then a trade triggers the price to change. This implies that in explaining market activities we must consider not a univariate model, but rather a multivariate model where different time series are subordinated and frequently power-law autocorrelated.

(i) *Empirical evidence in bid–ask spread*: The ability to buy at a low price and sell at a high price is the main compensation to traders for the risk they incur (Cohen *et al.* 1981, Glosten and Milgrom 1985, Schwartz 1988, Admati and Pfleiderer 1988, George *et al.* 1991, Brock and Kleidon 1992, McInish and Wood 1992, Plerou *et al.* 2005, Lillo *et al.* 2005, Wyart *et al.* 2008, Ponzi *et al.* 2009). The trader sells at the 'ask' (offer) price A and buys at a lower 'bid' price B, where the difference is the bid–ask spread. Schwartz (1988) identifies four indicators that determine bid–ask spreads: activity, risk, information, and competition. More specifically,

(a) greater trading activity (shorter trading times) can lead to lower spreads since the higher the level of trading, the greater the chance that buy and sell orders will tend to balance during a trading period;
(b) there is a direct relationship between the level of risk and spreads;
(c) there is a direct relationship between spreads and the amount of information coming to the market—large trades convey more information than small trades; and
(d) There is an inverse relationship between spreads and the level of competition.

Competition varies with volume—the number of traders is more active as volume levels increase. In addition, analysing NYSE stocks, McInish and Wood (1992) show that the mean of (ask − bid)/(ask + bid)/2 for each minute of the trading day shows that spreads are relatively high at minute three, decline at a decreasing rate until minute 293 and then increase at an increasing rate until the close of trading. Thus, the plot of spreads over the trading day exhibits a crude reverse J-shaped pattern. By studying the bid–ask quotations and transactions information during 1988, Chung and Charoenwong (1998) find that spreads are negatively associated with the number of exchange listings, share price, and firm size. Different models are proposed to explain bid–ask spread properties (Cohen *et al.* 1981, Admati and Pfleiderer 1988, George *et al.* 1991, Brock and Kleidon 1992, Maslov 2000, Maslov and Mills 2000, Kullmann and Kertész 2001, Slanina 2001, Scalas *et al.* 2006).

(ii) *Empirical evidence in stock price correlations*: Analysing the daily recorded SP500 financial index, Ding *et al.* (1993) report a power-law long memory in autocorrelations of absolute returns. Podobnik *et al.* (2010a) report power-law cross-correlations of absolute returns between 1340 members of NYSE. Wang *et al.* (2011) report power-law cross-correlations of absolute returns between world-wide indices. By analysing the high-frequency S&P500 index and individual U.S. firms, Liu *et al.* (1999) find a crossover in correlations of absolute returns between two power-law regimes at approximately 1.5 days. Analysis accomplished on the time series of time intervals between consecutive S&P500 stock trades of different US firms revealed the same crossover between power-law regimes, implying a parallel with the crossover in the scaling of absolute price returns (Ivanov *et al.* 2004). Engle (2000) reports a Weibull distribution in IBM intertrade times.

(iii) *Empirical evidence in trading volume*: By analysing a database documenting every transaction for 1000 U.S. stocks for the two-year period 1994–1995, Plerou *et al.* (2000) quantify the relation between trading activity measured by the number of transactions $N_{\Delta t}$ and the price change $G_{\Delta t}$ for a given stock, over a time interval $[t, \Delta t]$. Denoting by $W^2_{\Delta t}$ the variance of the price changes for all transactions in Δt, it was found that the power-law tails of $P(G_{\Delta t})$ are due to $P(W_{\Delta t})$ and the long-range correlations in $|G_{\Delta t}|$ are due to $N_{\Delta t}$. For the 1000 stocks analysed, the cumulative distribution of $N_{\Delta t}$ displays power-law behavior with a mean value 3.40 ± 0.2, close to the exponent of the cubic law found in the tails of $P(G_{\Delta t})$ (Gopikrishnan *et al.* 1999). For the number of shares traded $Q_{\Delta t}$, the distribution $P(Q_{\Delta t})$ displays a power-law decay $P(Q_{\Delta t}) \propto (Q_{\Delta t})^{-1-\alpha}$, where $\alpha = 1.7 \pm 0.1$ (Gopikrishnan *et al.* 2000). Also, the long-range correlations in $Q_{\Delta t}$ are largely due to those of $N_{\Delta t}$. The results are consistent with the interpretation that the large equal-time correlations between $Q_{\Delta t}$ and the absolute value of price change $|G_{\Delta t}|$ are largely due to $N_{\Delta t}$. However, expressing $Q_{\Delta t}$ as the sum of the number of shares traded for all transactions, $Q_{\Delta t} = \sum q_i$, Gopikrishnan *et al.* (2000) and Plerou *et al.* (2001) report only weak correlations in q_i. Recently, based on the detrending cross-correlations analysis of Podobnik and Stanley (2008) Podobnik *et al.* (2009a) and Horvatic *et al.* (2011), Podobnik *et al.* (2009b) report long-range cross-correlations between volatility and the absolute values of volume changes. They also report the existence of a cubic law in trading volume changes, supporting the intriguing possibility that the cubic law in price changes has its origin in trading activities.

3. Model

Our goal is to construct a common framework for modeling trading time, trading volume, and price changes. To test our model, we select Exxon, a stock typical of the U.S. market and, according to the Trades and Quotes database (NYSE, New York, 1993), one of the most traded U.S. companies during the four-year period January 1993–December 1996. Our model is comprised of three stages: (i) we stochastically generate the duration or intertrade times (the interval between two trading times) Δt_i; (ii) at each Δt_i we stochastically generate the number of shares traded $q(\Delta t_i)$; and (iii) we propose a mechanism that explains how both Δt_i and $q(\Delta t_i)$ affect price change.

(i) We first define trading at times indexed by a set of numbers $t_1, t_2, t_3, \ldots$. These numbers are a realization of a discrete stochastic process with positive increments (since $t_i \geq 0$), implying that $t_1 < t_2 < t_3 \cdots$. In order to reproduce long-range power-law correlations in Δt_i as found for the three-year period January 1993–December 1996 (Ivanov *et al.* 2004), we model Δt_i using a fractionally integrated autoregressive conditional duration (FIACD) (Engle and Russell 1998, Jasiak 1998),

$$\Delta t_i = \psi_i(\rho_1)\epsilon_i, \tag{1}$$

where ϵ_i is independent and identically distributed (i.i.d.) with an exponential probability distribution ($a_1 \exp(-a_1\epsilon)$) (i.e. with one free parameter a_1) that is an approximation of the Weibull distribution found for U.S. firms (Engle 2000, Ivanov *et al.* 2004), $\psi_i(\rho_1)$ is the expectation of duration i (Jasiak 1998), and Δt_i at each moment t_i depends only on its previous values. The time series $\{\Delta t_i\}$ of equation (1) in figure 1(a) is generated using the fractional parameter $\rho_1 = 0.4$, which is used to reproduce the power-law scaling in Δt (figure 1(b)). To quantify the power-law memory, we use detrended fluctuation analysis (DFA) (Peng *et al.* 1994). The fractional parameter $\rho_1 = 0.4$ corresponds to the DFA exponent $\alpha = 0.9$ found for the Exxon company for the three-year period (Ivanov *et al.* 2004). The free parameter a_1 of the (i.i.d.) exponential $[a_1 \exp(-a_1\,\epsilon)]$ in equation (1) can be estimated from the average intertrade times. When a trade occurs at t_i, a number of shares $q(\Delta t_i)$ changes ownership.

(ii) We next model a process for the time series $q(\Delta t_i)$ for the same three-year period. For $q(\Delta t_i)$ of Exxon company trades, we obtain the DFA exponent $\alpha = 0.62$, which implies the presence of weak but long-range power-law correlations. We assume that $q(\Delta t_i)$ depends not on previous Δt values, but on previous q values. Motivated by Clark's subordinated process (Clark 1973), we assume that Δt_i is subordinate not to share price as in Clark (1973), but to $q(\Delta t_i)$, and model $q(\Delta t_i)$ using a fractionally integrated moving average process (FIARCH) (Ding *et al.* 1993),

$$q(\Delta t_i) = \sigma_i(\rho_2)\epsilon'_i. \tag{2}$$

Here, $\sigma_i = \sum_{n=1}^{\infty} a_n(\rho_2) q(\Delta t_{i-n})$ and $a_n \Gamma(n-\rho_2)/[\Gamma(-\rho_2)\Gamma(1+n)]$ are statistical weights where Γ denotes the Gamma function, $\rho_2 \in (0, 0.5)$ is a single free parameter (Ding *et al.* 1993), and ϵ'_i is i.i.d., for simplicity taken from an exponential distribution $a_2 \exp(-a_2\epsilon')$, the parameter of which

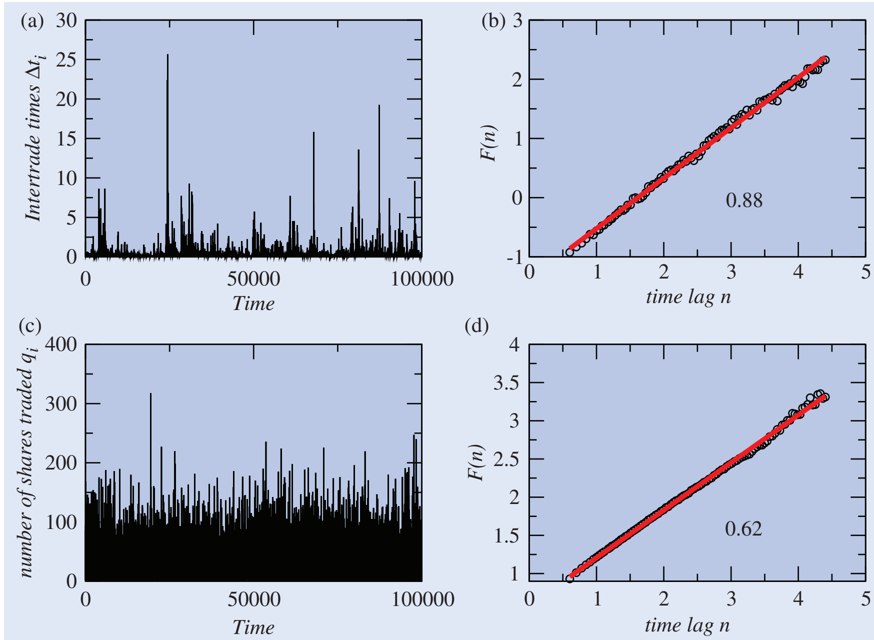

Figure 1. Model assumptions. Modeling power-law correlations in intratrading time and number of shares traded in the Exxon corporation using the stochastic process of equation (1) with fractional parameter $\rho_1 = 0.4$, and the stochastic process of equation (2) with fractional parameter $\rho_1 = 0.12$. (a) Intertrade time Δt of equation (1). (b) For Δt, detrended fluctuation function $F(n)$ versus time lag n yields strong long-range autocorrelations in Δt. (c) Number of shares traded $q(\Delta t_i)$ of equation (2). (d) For $q(\Delta t_i)$, we find weak long-range power-law autocorrelations.

(a_2) can be estimated to give the average number of shares traded. Since there is a simple relation between the DFA exponent α and the FIARCH parameter ρ_2 ($\alpha = 0.5 + \rho_2$), from $\alpha = 0.62$ calculated for power-law correlations in q_i we obtain $\rho_2 = 0.12$.

Thus we model Δt_i and $q(\Delta t_i)$ as two mutually independent but individually autocorrelated power-law processes in which the correlations in Δt_i are much stronger than those in $q(\Delta t_i)$. There are four parameters at this stage: ρ_1 and ρ_2, responsible for power-law scaling in intertrade times Δt_i, the number of shares $q(\Delta t_i)$, and two parameters, a_1 and a_2, corresponding to the distributions of i.i.d. variables in equations (1) and (2).

Figure 2(a) and (b) show that, in equations (1) and (2), these two power-law scalings are responsible for the strong power-law correlations in the sum of the number of shares q_i traded (the trading volume) in a fixed time interval ΔT (where $\Delta T \gg \langle \Delta t \rangle$),

$$Q(\Delta T) = \sum_{i=1}^{N_{\Delta T}} q_i(\Delta t_i), \tag{3}$$

where $N_{\Delta T}$ is the total number of trades within a time interval $\Delta T = \sum_{i=1}^{N_{\Delta T}} \Delta t$. Thus, even though the time series of the individual number of shares traded $q(\Delta t_i)$ is weakly power-law correlated, because of strong power-law correlations in the intertrade time Δt_i the integrated trading volume $Q(\Delta T)$ exhibits strong long-range power-law correlations, which were found empirically by Gopikrishnan *et al.* (2000). Figure 2(c) and (d) show that equations (1) and (2) also generate long-range power-law autocorrelations in the total number of shares traded in the fixed time interval ΔT, where $\Delta T \gg \langle \Delta t \rangle$.

Trading strategies play a key role in price dynamics, and the literature on this topic is huge. Diamond and Verrecchia (1987) model trading activities by assuming that, at the beginning of a trading day, traders are greeted with news that is either good or bad, and that long durations are likely to be associated with news that is bad. Bagehot (1971) assumes that informed traders possess non-public information that allows them to better estimate a future security price than uninformed traders. Easley and O'Hara (1987) assume that informed traders trade only when they have information and thus variations in trading rates are associated with the changing number of informed traders. In the model proposed by Bak *et al.* (1997), buyers and sellers are represented by particles subject to a reaction–diffusion process. According to the Maslov and Mills (2000) model, traders can either buy or sell stock at the market price or place a limit order to automatically buy or sell a particular amount of stock. In this case, traders are allowed to trade only one unit of stock ($q_i = 1$) in each transaction. A mean-field variant of the Maslov and Mills (2000) model proposed by Slanina is found to exhibit a power-law tail with exponent 2 (Slanina 2001). Other models with non-trivial agent strategies have also been proposed (Takayasu *et al.* 1992, Caldarelli *et al.* 1997,

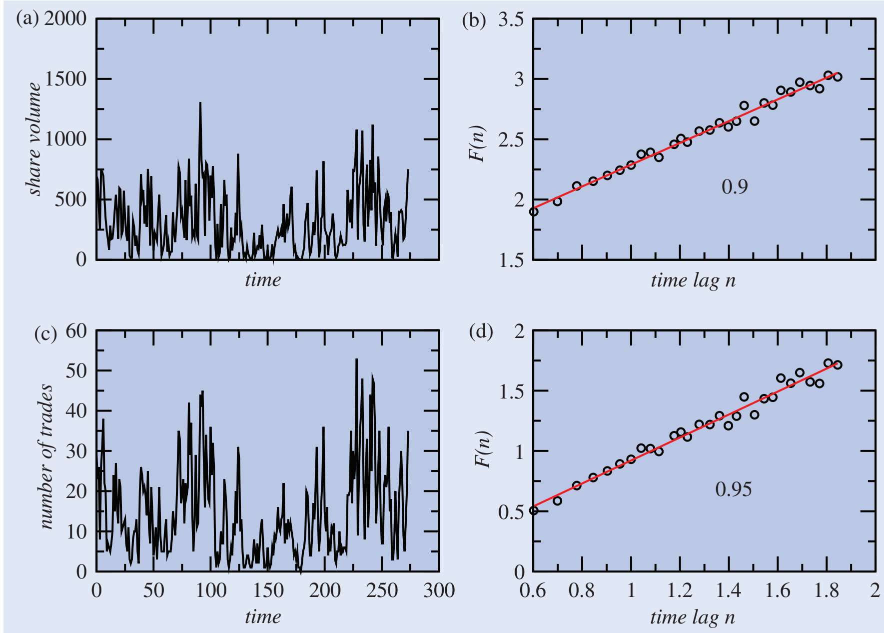

Figure 2. Model outcomes. Modeling power-law correlations in share volume $Q(\Delta T)$ and number of trades (transactions) $N_{\Delta T}$ within ΔT using the stochastic process of equation (1) with fractional parameter $\rho_1 = 0.4$, and the stochastic process of equation (2) with fractional parameter $\rho_2 = 0.12$ (a) Share volume $Q(\Delta t)$. (b) For $Q(\Delta t)$, detrended fluctuation function $F(n)$ versus time lag n yields strong long-range power-law autocorrelations in Δt. (c) Number of trades $N_{\Delta T}$. For $N_{\Delta T}$, detrended fluctuation function $F(n)$ versus time lag n yields strong long-range power-law autocorrelations.

Challet and Zhang 1997, Cont and Bouchaud 2000).

We now simplify the trading process, but at a level that can still provide us with the scaling properties found in price and trading dynamics. In this model, bid and ask prices are stochastically generated at each time coordinate. We do this in response to the Gopikrishnan *et al*. (2000) finding that the correlations in the absolute values of price changes are largely due to correlations in trading volume. Engle and Russell (1998) report evidence of co-movements between intertrade time and volatility—the absolute value of price changes. Similarly, Ivanov *et al*. (2004) quantify this co-movement finding as an analogy in the power-law scaling between the absolute value of price changes and the time intervals between consecutive stock trades. Finally, for the 116 stocks analysed, Plerou *et al*. (2005) report that the average bid–ask spread S is characterized by a cumulative distribution that decays as a cubic power law. These results clearly suggest a common origin for price change dynamics and trading time dynamics. In our model, at each trading time a single trader trades stock while other traders put either bid or ask prices. We therefore suggest the following process for generating the trader's (agents') ask and bid price changes, respectively:

$$\Delta S^a = \psi_i(\rho_1)\epsilon_i'', \tag{4}$$

$$\Delta S^b = -\psi_i(\rho_1)\epsilon_i'', \tag{5}$$

where $\psi_i(d)$—the volatility process shown in equation (1)—is responsible for the long memory in intertrade times, and ϵ_i'' is from an exponential function. Thus in our model intertrade times and bid–ask price changes share the same volatility mechanism. In our simulations we keep the number of bid and ask traders equal and constant. Clearly this is an approximation, since the number of bid and ask traders changes over time and at certain times, e.g. during market crashes, substantially increases.

(iii) To illustrate how trading influences price changes, consider a simple example with only two ask traders. Suppose trader A puts an ask order with 3000 shares and requires that its price be at least \$100 per share. Trader B puts an ask order with 6000 shares and requires that the price exceed \$110 per share. Trader C decides to buy the cheapest 6000 shares. Clearly, trader C can buy 3000 shares from trader A at \$100 per share and 3000 shares from trader B at \$110 per share. We assume that for the trader who trades shares, the probability of a bid offer is equal to the probability of an ask offer, and this assumption ensures that there will be no serial correlations (figure 3(b)). Based on this trading decision, using the stochastic process of equation (1) to generate intertrade time Δt_i, the stochastic process of equation (2) to generate the number of shares traded at Δt_i, $q_i(\Delta t_i)$, and the choice for bid and ask price changes in equations (4) and (5), we generate a price time series (see figure 3(a)). Using the detrended

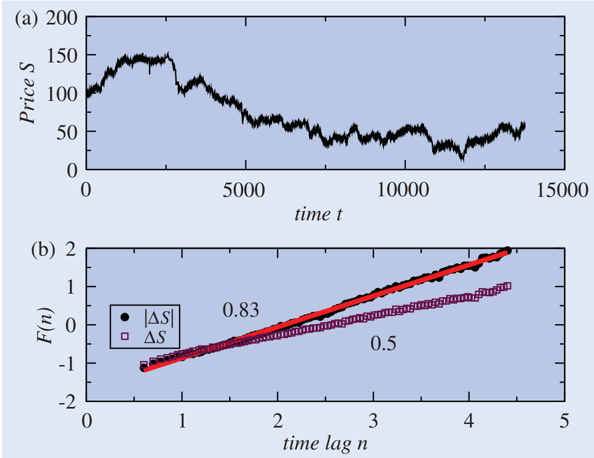

Figure 3. Model outcomes. Modeling power-law autocorrelations in absolute values of price change using the stochastic process of equation (1) with fractional parameter $\rho_1 = 0.4$, and the stochastic process of equation (2) with fractional parameter $\rho_2 = 0.12$ as in figures 1 and 2. (a) Time series of price for 100,000 time steps with average intertrade time $\langle \Delta t \rangle = 0.137$. (b) Detrended fluctuation function $F(n)$ versus time lag n yields strong long-range power-law autocorrelations. We also show that there are no correlations in price changes.

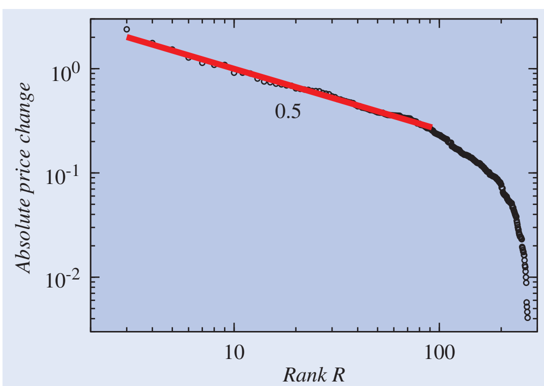

Figure 4. Model outcomes. Zipf plot with power-law tails in absolute values of price changes using the stochastic process of equation (1) with fractional parameter $\rho_1 = 0.4$, and the stochastic process of equation (2) with fractional parameter $\rho_2 = 0.12$ as in figures 1 and 2.

fluctuation function $F(n)$, in figure 3(b) we show that the absolute values of price changes exhibit strong power-law autocorrelations.

For power-law distributed variables with cumulative distribution $P(s > x) \sim x^{-\zeta'}$, the Zipf plot of size s vs. rank R usually exhibits a power-law scaling regime with a scaling exponent ζ for a large range of R (Stanley *et al.* 1995, Podobnik *et al.* 2010b, 2011), $\zeta = 1/\zeta'$. Using the Zipf ranking approach, in figure 4 we show that the tails of the distribution of absolute values of price change exhibit a power law. The Zipf exponent corresponds to the scaling exponent $\zeta = 2$. Using different parameters and different i.i.d. distributions in equations (1), (2), (4), and (5), it is clear that we can eventually obtain an exponent corresponding to a cubic law (Gopikrishnan *et al.* 1999).

Using quote data for the 116 most frequently traded stocks on the New York Stock Exchange over the two-year period 1994–1995, Plerou *et al.* (2005) analyse the relationship between the bid–ask spread and other indicators of liquidity such as the number of trades occurring $N_{\Delta T}$, and the share volume traded $Q_{\Delta T}$. They found $S \propto \ln N_{\Delta T}$ and $S \propto \ln Q_{\Delta T}$. They also examined the relationship between the spread expectation conditioned by the time interval between trades. They found that, as Δt increases, the bid–ask spread decreases, and the functional relationship is approximately $\langle s \rangle_{\Delta t} \propto -\ln \Delta t$. In order to reproduce the last finding and to keep the rest of the findings, we modify the bid–ask process of equations (4) and (5), which gives the proportional and not the reciprocal dependence between the spread and the intertrade time interval. Then we generate the trader's (agents') ask price changes,

$$\Delta S^a = (\psi_i(\rho_1))^{-\gamma} \epsilon''_i, \qquad (6)$$

$$\Delta S^b = -(\psi_i(\rho_1))^{-\gamma} \epsilon''_i, \qquad (7)$$

where $\gamma > 0$ and ϵ'' are explained in equations (4) and (5). In figure 5(a) and (b) for $\gamma = 0.25$ we show the log-linear functional relationship between the average of the spread $\langle S \rangle_{\Delta T}$ and the number of trades occurring $N_{\Delta T}$, and between $\langle S \rangle_{\Delta T}$ and the total share volume traded $Q_{\Delta T}$, and both agree with empirical findings. Since the average intertrade time interval Δt can be thought of as a reciprocal of $N_{\Delta T}$, the model accurately gives the the reciprocal dependence between the spread and the intertrade time interval.

We have proposed a stochastic process that may offer a guide to modeling the microstructural dynamics of spreads, returns, volume $q(\Delta t_i)$, and volatility. It gives the statistical properties of the intertrade time interval Δt_i, the bid–ask spread, and the volatility, all in good agreement with empirical findings. We model Δt_i and $q(\Delta t_i)$ as two mutually independent but individually autocorrelated power-law processes in which the correlations in Δt_i are much stronger than those in $q(\Delta t_i)$. There are three exponentially distributed i.i.d. processes in equations (1) and (2) and and (6) and (7), where the parameters a_1 and a_2 defined in equations (1) and (2) can be estimated to fit the average intertrade times and the average number of shares traded, respectively. The fractional parameters ρ_1 and ρ_2 in equations (1) and (2) can be estimated to fit the scaling in the autocorrelations of Δt and $q(\Delta t)$, respectively. The parameter γ in equations (6) and (7) controls the power-law exponent and the strength of the autocorrelations in absolute values of price changes. The larger γ, the smaller the exponent for the power-law tails. We believe that subordinated processes with long-range correlations have a broad range of potential applications.

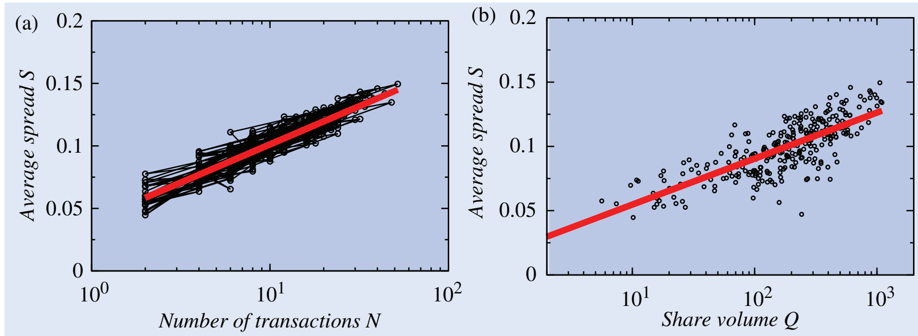

Figure 5. Model outcomes for the stochastic process of equation (1) with fractional parameter $\rho_1 = 0.4$, and the stochastic process of equation (2) with fractional parameter $\rho_2 = 0.12$ as in figures 1 and 2 and equations (6) and (7) with $\gamma = 0.25$. (a) Average spread S versus share volume. (b) Average spread versus number of transactions for a given ΔT. Both exhibit log-linear functional dependence in agreement with empirical findings.

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