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Time evolution of stochastic processes with correlations in the variance: stability in power-law tails of distributions

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Abstract

We model the time series of the S&P500 index by a combined process, the AR+GARCH process, where AR denotes the autoregressive process which we use to account for the short-range correlations in the index changes and GARCH denotes the generalized autoregressive conditional heteroskedastic process which takes into account the long-range correlations in the variance. We study the AR+GARCH process with an initial distribution of truncated Lévy form. We find that this process generates a new probability distribution with a crossover from a Lévy stable power law to a power law with an exponent outside the Lévy range, beyond the truncation cutoff. We analyze the sum of n variables of the AR+GARCH process, and find that due to the correlations the AR+GARCH process generates a probability distribution which exhibits stable behavior in the tails for a broad range of values n—a feature which is observed in the probability distribution of the S&P500 index. We find that this power-law stability depends on the characteristic scale in the correlations. We also find that inclusion of short-range correlations through the AR process is needed to obtain convergence to a limiting Gaussian distribution for large n as observed in the data. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Application of statistical physics methods to analyze probability distribution functions (PDFs) of financial data has attracted recent interest. Much work [1–21] has been

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devoted to determine precisely the functional form of these PDFs. For the S&P500 stock index, it has been shown [7,21] that the central profile of the PDF is well described by the Lévy distribution [22]. Recent analyses [21] of high frequency data have shown that the PDF is described by a crossover to a power law, with exponent $1 + \alpha$ well beyond the Lévy range ($0 < \alpha < 2$). (The data cover the period 1 January 1985 to 31 December 1995, and the time interval between successive records of the index is 1 min.) The tails of the PDF appear to exhibit stability for long but finite time scales.

In addition to the form of PDFs, other important, but complementary related quantities, are the absolute value and variance of price changes, which are commonly used as measures of the risk [3]. For the S&P500 index, in contrast to time series of price changes that show only short-range correlations [9,21,23], the time series of absolute values of price changes exhibit long-range correlations [23–27]. It is natural to ask how slow-decaying (long-range) correlations in the variance may be related to the scaling behavior observed in PDFs.

To describe the S&P500 index, we analyze the generalized autoregressive conditional heteroskedastic (GARCH) process [28] to take account of correlations in the variance of price changes. (A process is autoregressive if variable x_t depends on its own lagged values. Heteroskedasticity is related with non-constant variances.) In addition to the GARCH process, we employ the autoregressive (AR) process [29] to account for the effect of the short-range correlations in price changes and develop a combined process, the AR+GARCH process. We show that due to the GARCH process, the AR+GARCH process generates the power-law tails in the PDF with an exponent outside the Lévy range. (Power-law tails in distributions can be obtained in multiplicative processes introduced in Refs. [30] and [31].) The GARCH process itself is constructed out of independent and identically distributed (i.i.d.) stochastic variables specified by arbitrary PDF. With the choice of truncated Lévy PDF [32], we model the crossover behavior in the PDF of the AR+GARCH process as observed in the data [21]. We study a process that is the sum of n AR+GARCH variables to probe for large n the stability of the PDF. For this new process, we find long-range correlations in the variance arising from the GARCH process. We also identify the long-range correlations as the source of the empirically observed stability for a range of time scales in the power-law tails of the PDF.

2. The GARCH process for a truncated Lévy distribution

2.1. The GARCH process

Let us define an independent and identically distributed (i.i.d.) stochastic variable v_t with zero mean ($\langle v_t \rangle = 0$) and unit variance ($\langle v_t^2 \rangle = 1$). The generalized autoregressive conditional heteroskedastic (GARCH) process x_t [28] is a discrete time stochastic process generated by a stochastic variable v_t through

$$x_t = \sigma_t v_t ,$$

$$\sigma_t^2 = a + b x_{t-1}^2 + c\sigma_{t-1}^2 , \tag{1}$$

where σ_t^2, called the conditional variance of the GARCH process, depends on both most recent values of x_t^2 and σ_t^2, while a, b, and c are positive parameters. Thus, the GARCH process has no correlations in x_t, $\langle x_t x_{t'} \rangle \propto \delta_{t,t'}$, but has correlations in the variance. (Applied for financial data, the GARCH process x_t implies omission of short-range correlations in the data.) For $c=0$ the GARCH process reduces to the ARCH process [33]. For $b=c=0$, x_t becomes v_t of Eq. (1), an i.i.d. process. Using Eq. (1), the expected variance of the GARCH process x_t, defined by $\langle x_t^2 \rangle = \langle \sigma_t^2 \rangle \equiv \sigma_x^2$, becomes

$$\sigma_x^2 = \frac{a}{1-b-c} . \tag{2}$$

Here $b+c<1$ is a condition for the finiteness of σ_x^2, where σ_x^2, in practice, we calculate from experiment.

The GARCH process was developed in part to take account of the long memory in the variance typically found in financial data [23–27]. By iterating Eq. (1) (i.e., by repeatedly substituting for σ_t on the right side), the conditional variance σ_t^2 can be rewritten as a constant plus the weighted average of all prior x_t^2 (σ_t^2 in Eq. (1) can be expressed as $\sigma_t^2 \propto b\sum_{n=1}^{\infty} c^{n-1} x_{t-n}^2$). The GARCH process is characterized by an exponentially decaying covariance function in the variance, $cov(\sigma_t^2, \sigma_{t-\Delta t}^2)$ [28], with decay rate given by $|\log(b+c)|$. (The covariance function of the variance σ_t^2 is defined as $cov(\sigma_t^2, \sigma_{t-\Delta t}^2) \equiv \langle \sigma_t^2 \sigma_{t-\Delta t}^2 \rangle - \langle \sigma_t^2 \rangle \langle \sigma_{t-\Delta t}^2 \rangle$, where Δt is the time horizon. For the GARCH process $cov(\sigma_t^2, \sigma_{t-\Delta t}^2) \propto \exp(-\Delta t/\tau)$, where decay time τ is the reciprocal value of the decay rate defined as $|\log(b+c)|$.) Even though the covariance function decays exponentially in time, a long-range decay can be mimicked by strongly reducing the decay rate. To achieve this long-range decay, the sum $b+c$ must be chosen close to unity.

2.2. *Truncated Lévy distribution*

To implement the GARCH process of Eq. (1) explicitly, one must specify the form of the initial distribution for v_t, $P(v_t)$. The standard form for $P(v_t)$ in the financial literature is the Gaussian [33,34] or the Student's-*t* distribution [35], but other forms are possible [29]. Generally we show that, regardless of the choice for $P(v_t)$, the GARCH process generates the power-law tails in the PDF. But the choice for $P(v_t)$ becomes important when a process characterized by correlations is applied to fit the central region of empirical distribution. (The TL PDF for the ARCH process, applied to price changes for the S&P500 index, is described in Refs. [36,37]. Price changes of the S&P500 index are defined as $S_{t+\Delta t} - S_t$, where Δt is a time scale.) For the case of the S&P500 index, Ref. [21] shows that the PDF is characterized by the crossover

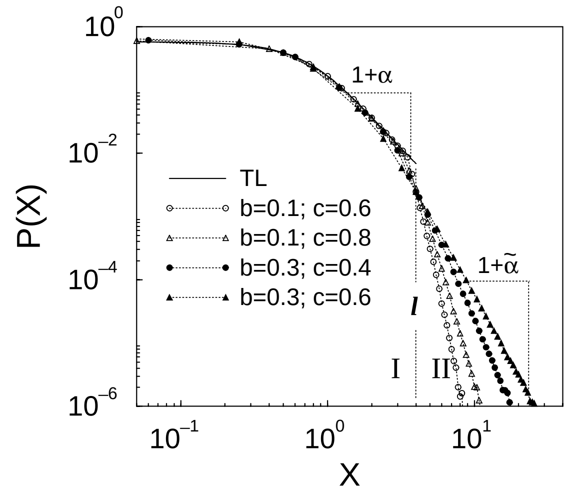


Fig. 1. Two different sets of the GARCH process of Eq. (1) with $P(v_t)$ of the TL PDF [Eq. (3)] where $b+c$ is 0.7 and 0.9, respectively. The TL PDF has $\alpha=1.3$, $\gamma=0.439$, and $\ell=4$ (as required, $\sigma_v^2=1$, numerically calculated). For the GARCH process, we set $\sigma_x=1$ [Eq. (2)]. For all sets, the GARCH process generates the power-law tails. In regime I, the PDF of the GARCH process, $P(x_t)$ typically follows the Lévy power-law tails of the TL PDF, $x^{-(1+\alpha)}$ (up to $x_t \approx \ell$). With increase of $b+c$, difference between $P(x_t)$ and $P(v_t)$ becomes significant, changing the slope of $P(x_t)$. Comparing two realizations with different values of b and c, but the same sum $b+c$, we see that in regime II, the slope of the power-law tails is smaller for b larger. Also, if $b+c$ is fixed, the slope of the power-law tails is smaller if b is larger.

behavior from one power-law regime of the Lévy type, that describes the central region of the PDF, to another power-law regime with an exponent out of the Lévy range. To account for the crossover behavior in the PDF, together with long-range correlations in the variance, we propose the GARCH process with $P(v_t)$ given by truncated Lévy (TL) PDF [32]

$$P(v_t) \equiv \mathscr{T}_{\alpha,\gamma,l}(v) \equiv \left\{ \begin{array}{ll} \mathscr{N}\mathscr{L}_{\alpha,\gamma}(v) & |v| \leqslant \ell \\ 0 & |v| > \ell \end{array} \right\} . \tag{3}$$

Here, $\mathscr{L}_{\alpha,\gamma}(v)$ is the symmetrical Lévy PDF [22], where α $(0<\alpha<2)$, and γ $(\gamma>0)$ are the two parameters of the Lévy distribution. $\mathscr{N}$ is the normalizing constant and ℓ is the cutoff length. We employ an algorithm in Ref. [38] provided for variables v with $\gamma \equiv 1$. Since the probability distribution of Eq. (3) rescales under $v \equiv v/n^{1/\alpha}$, $\mathscr{T}_{\alpha,\tilde{\gamma},\tilde{l}}(\tilde{v}) \equiv \mathscr{T}_{\alpha,\gamma,l}(v)n^{1/\alpha}$, where $\tilde{\gamma} \equiv \gamma/n$ and $\tilde{l} \equiv l/n^{1/\alpha}$, a random variable $\tilde{v}$ with $\tilde{\gamma} \neq 1$ is calculated as $\tilde{v} \equiv v/n^{1/\alpha}$.

Now we analyze how the choice Eq. (3) for $P(v_t)$ affects the form of the PDF $P(x_t)$ of the GARCH process x_t of Eq. (1) (Fig. 1). Due to the form for $P(v_t)$, approximately scaling as $1/(v^{1+\alpha})$ for $|v|<\ell$, and due to the correlations imposed by GARCH process, we find a crossover behavior between two power-law regimes. In regime I, the effects of the correlations in the variance on $P(x_t)$ are generally weak, and the difference between $P(x_t)$ and $P(v_t)$ is very small. For this reason, the parameter α of $P(v_t)$ typically we chose to fit the central region of the empirical PDF. In regime II, the GARCH process generates power-law tails in $P(x_t)$ with an exponent outside the Lévy range. Note that finiteness of the nth moment, defined as $\int x^n P(x_t)$, implies the range of allowed exponent in the power-law tails $x^{-(1+\tilde{\alpha})}$ of $P(x_t)$. The power-law tails are expressed in analogy with stable Lévy power-law tails. For finiteness of the variance ($n=2$), the exponent $1+\tilde{\alpha}$ has to be larger than 3 ($\tilde{\alpha}>2$).

2.3. Asymptotic behavior of the GARCH process for truncated Lévy distribution

To probe asymptotic behavior of PDFs over a varying time scale, we next consider a new stochastic process

$$z_n \equiv \sum_{t=1}^{n} x_t \,, \tag{4}$$

where as before x_t is a GARCH process and n refers to a time scale. In Fig. 2, we show two sets of $P(z_n)$ with $c=0.75$, while b equals 0.24 and 0.15, respectively. We see that, for all n presented, the power-law tails of $P(z_n)$ practically does not change for the set characterized by $b+c=0.99$. The power-law tails of $P(z_n)$ retain stability when $b+c$ is kept close to 1. For the set with $b+c=0.9$, with increase of n, we observe a very slow decrease of the slope in the power-law tails. Hence, the PDF $P(z_n)$ of the stochastic process z_n of Eq. (4) defined by correlations in the variance, where $b+c \approx 1$, preserves its power-law functional form. However, in contrast with the Lévy process [22], the exponent $1+\tilde{\alpha}$ is outside the Lévy range. The same behavior—the stability of the power-law tails of the empirical PDFs with an exponent outside the

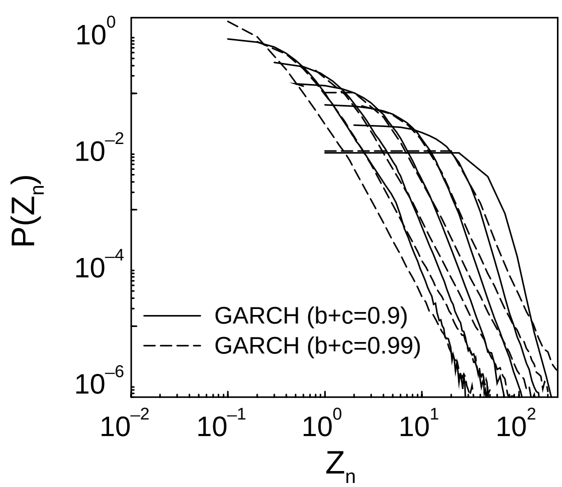


Fig. 2. Test of the dependence of the exponent and the range of scales over which exponent is maintained. Log–log plot of PDFs $P(z_n)$ of the stochastic process z_n of Eq. (4) where x_t is a GARCH process of Eq. (1) where $b=0.24$ and $c=0.75$, together with PDFs $P(z_n)$ where the GARCH process x_t is specified by $b=0.15$ and $c=0.75$. We show n equal to 1, 4, 16, 64, 256, and 1024, from left to right. For given sets, $b+c$ is equal to 0.99 and 0.9, respectively. $P(v_t)$ [Eq. (1)] has the TL PDF of Eq. (3) with the cutoff length $\ell=8$, $\alpha=1.4$, and $\gamma=0.275$. We see that for the set characterized by larger $b+c$, i.e., $b+c=0.99$, the power-law tails remain stable for all values of n analyzed. For the set with $b+c=0.9$, the slope of the tails slowly decreases with increase of n. Thus, the closer $b+c$ to 1, the longer stability of the power-law tails. For both sets, σ_x of Eq. (2) equals 1.

Lévy range for long, but finite time scales—has been recently observed for the S&P500 index [21].

3. Application to the S&P500 index

3.1. Data analysis

For the S&P500 index, Ref. [7] shows that the central region of the PDF of price changes is fit by the Lévy PDF with $\alpha = 1.4$. For a long, but finite time scale, the maximum of the PDF of price changes (probability of return) as a function of time scale Δt, exhibits a power law with a slope $1/\alpha$, inside the Lévy range. Ref. [39] shows that the PDFs of price changes (Fig. 3) and relative price changes,

$$R_t \equiv \log(S_{t+\Delta t}) - \log(S_t) \tag{5}$$

practically overlap after appropriate rescaling. Ref. [21] shows that the tails of the 1 min PDF $P(R_t)$ are power-law distributed with exponent $1 + \tilde{\alpha} = 4$, beyond the Lévy range (Fig. 4a). Further, $P(R_t)$ retains its power-law functional form (Fig. 4a) over a wide range of time scales Δt, after which a slow convergence to Gaussian behavior occurs [21].

The absolute value of R_t exhibits long-range power-law correlations that persist up to several months [40,41]. For a time series X_t, correlations in absolute values and variance of X_t are of the same kind [23]. This feature can be mimicked by exponentially decaying correlations in the variance of the GARCH process with the characteristic decay time $\tau = |\log(b + c)|^{-1}$ where $(b + c)$ is appropriately chosen. We use the sum $(b + c)$ as a measure of the power-law stability of PDFs over varying time scales

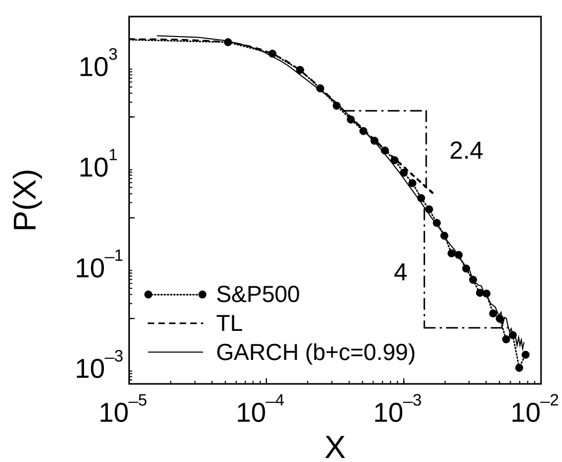
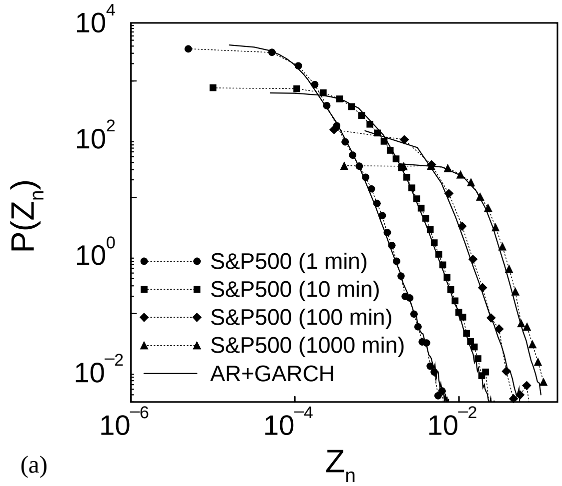
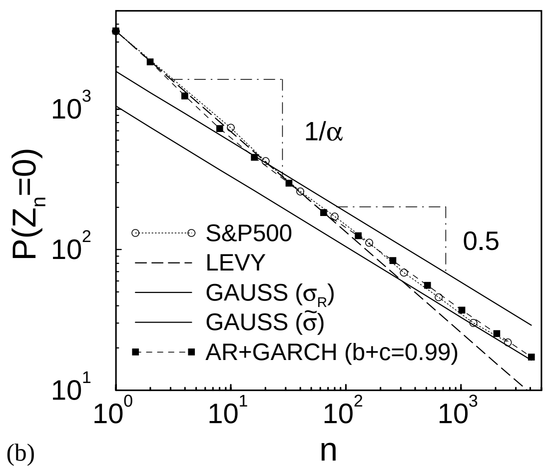


Fig. 3. The tails of the 1-min $P(R_t)$ [Eq. (5)] of the S&P500 index are of slope $1 + \hat{\alpha} \approx 4$ [54]. Near the origin, $P(R_t)$ is characterized by the slope $1 + \alpha = 2.4$. The standard deviation of R_t σ_R equals 0.216×10^{-3}. Also shown is the PDF of the GARCH process x_t of Eq. (1), $P(x_t)$ with $b = 0.06$ and $c = 0.93$ chosen to give the slope of the tails of $P(x_t)$ equal to 4. The choice for the sum $b + c = 0.99$ is explained in Fig. 4. $P(v_t)$ in Eq. (1) has the TL PDF of Eq. (3) with the cutoff length $\ell = 8.0$, and $\gamma = 0.275$ and $\alpha = 1.4$. The choice for $P(v_t)$ specifies that near the origin, the PDF $P(x_t)$ is characterized by the slope $1 + \alpha = 2.4$.



(a)



(b)

Fig. 4. (a) Log–log plot of (symbols) $P(R_t)$ [Eq. (5)] for the S&P500 index for different time scales Δt. We show the PDFs $P(z_n)$ of the sum of n AR+GARCH variables r_t [Eq. (6)]. The parameters of the GARCH process and TL PDF are given in Fig. 3, while $\phi_1 = 0.6$ and $\phi_0 = 0.83\times10^{-6}$ [Eq. (6)]. Persistence in the power-law tails of slope 4 in the data is accomplished through the GARCH process with $b + c = 0.99$. The parameter ϕ_1 in Eq. (6) is set to enable good fit between $P(z_{n=10})$ and $P(R_t)$ for $\Delta t = 10$ min. For given parameter ϕ_1, σ_x of Eq. (2) is calculated from $\sigma_r = 0.216\times10^{-3}$ of Eq. (7) calculated from the data [Eq. (5)] for $\Delta t = 1$ min. (b) Probabilities of return $P(z_n = 0)$ and $P(R_r = 0)$ for small values of n approximately follow the PDF of the Lévy process with $\alpha = 1.4$. In the absence of the AR process, $P(z_n = 0)$ would tend the upper Gaussian distribution with the variance $\sigma_r^2 n$. We also show the limit Gaussian distribution with the variance $\hat{\sigma}^2 n$ to which, for large values of n due to presence of the AR process, $P(z_n = 0)$ finally tends. It holds $\hat{\sigma}10^{1/2} = \sigma_{10}$ where $\sigma_{10} = 0.115\times10^{-2}$ calculated from Eq. (5) for $\Delta t = 10$ min. The time scale shown is half of month long [55].

(Fig. 2)—the closer is $(b + c)$ to 1, the longer is the stability of the power-law tails. Generally, we show that GARCH processes with different choices of "dynamical" parameter $(b + c)$ can generate the same slope of the PDF for a fixed time scale Δt (Fig. 1), but differences in PDFs emerge for larger time scales. (The parameter of the GARCH process, $b + c$ and the Lévy parameter α in Eq. (3) are examples of "dynamical" parameters. They govern the behavior of PDFs for different time scales

Δt. In contrast, the Lévy parameters ℓ and γ in the paper are set to model the 1 min PDF.) Hence, the choice for $b + c$ can be fixed only by fitting empirical distributions corresponding to different time scales.

In Fig. 3, we compare three different PDFs: (a) The 1 min PDF $P(R_t)$ of the S&P500 index, characterized by crossover behavior between two power-law regimes with the standard deviation $\sigma_R = 0.216 \times 10^{-3}$ calculated from the data of Eq. (5) for $\Delta t = 1$ min. (b) The TL PDF (that is $P(v_t)$ of Eq. (1) after rescaling) with the parameter $\alpha = 1.4$ [7,22] chosen to fit $P(R_t)$ in the central regime [32]. (c) The PDF $P(x_t)$ of the GARCH process x_t, with $b = 0.06$ and $c = 0.93$. The choice $(b + c) = 0.99$ is explained below when we analyze PDFs for time scales other than $\Delta t = 1$ min. The specific values for parameters b and c are chosen to generate the slopes of the tails of $P(x_t)$ equal to the slope of 4 found for the 1 min $P(R_t)$ [21]. Due to strong correlations in the variance (through $b + c \approx 1$), the difference between $P(v_t)$ and $P(x_t)$ in the central regime is more pronounced near the crossover (see also Fig. 1).

3.2. Combined AR+GARCH process

Besides long-range correlations in absolute values of price changes R_t of Eq. (5), time series of price changes exhibit short-range correlations. An autoregressive (AR) process [29] is commonly employed to account for such short-range correlations. To encompass empirical evidences found for the data, we define a combined AR+GARCH process as

$$r_t = \phi_0 + \phi_1 r_{t-1} + x_t \,, \tag{6}$$

where x_t is the GARCH process of Eq. (1). The expected variance of r_t is given by

$$\sigma_r^2 = \frac{\sigma_x^2}{1 - \phi_1^2} \tag{7}$$

and σ_x^2 is defined by Eq. (2).

In Fig. 4a, we compare the PDFs of linear combinations of n AR+GARCH variables ($z_n \equiv \sum_{t=1}^{n} r_t$) with the PDFs $P(R_t)$ of the S&P500 index over varying time scales Δt ($\Delta t = 1$ min corresponds to $n = 1$). The slope of the tails in $P(R_t)$ retains stable for a wide range of Δt, and then very slowly decreases for larger Δt. Numerically (see, Fig. 2), we find the choice for $(b+c)$ used in Fig. 3, that ensures the "duration" of the power-law stability as found in the data. We show that different values of parameter ϕ_1, while the GARCH parameters b and c are constant, do not affect $P(z_{n=1})$ of the AR+GARCH process, but change the form of $P(z_n)$ for $n \neq 1$. During the fitting procedure, keeping b and c constant, we find the value of the parameter ϕ_1 that enables good agreement between $P(z_n)$ and $P(R_t)$ for different Δt.

For small values of n in Fig. 4b we see that, $P(R_t = 0)$ and $P(z_n = 0)$ approximately follow the PDF expected for the Lévy process with $\alpha = 1.4$ [9]. The upper straight line corresponds to the Gaussian distribution

$$G(z_n = 0) = 1/[\sqrt{2\pi}\sigma_R n^{1/2}] \,, \tag{8}$$

where σ_R is found for the data of Eq. (5) with $\Delta t = 1$ min. This is the limiting Gaussian distribution to which sums of GARCH variables (without AR process) converges. We show that compared with the case with no AR process, inclusion of the AR process shifts $P(z_n = 0)$ downwards, and so postpones the convergence to the limit Gaussian distribution

$$G(z_n = 0) = 1/[\sqrt{2\pi}\hat{\sigma}n^{1/2}] \tag{9}$$

to which $P(z_n = 0)$ tends for large values of n. The existence of two limit Gaussian distributions is due to the fact that short-range correlations in R_t imply that the time-averaged standard deviation of R_t, $\sigma(\Delta t)$ does not scale with unique scaling exponent for Δt [9,21,39]. For $\Delta t > 10$ min, where short-range correlations in R_t practically vanish [21], we may set $\sigma(\Delta t) = \hat{\sigma}(\Delta t)^{1/2}$, where $\hat{\sigma}(\Delta t)^{1/2} = \sigma_{10}$ and σ_{10} is calculated for the data of Eq. (5) with $\Delta t = 10$ min. Thus, effectively, for small time scales Δt, the AR process combined with the GARCH process enables the approximate scaling of $P(z_n = 0)$ vs. n with the slope $1/\alpha$. The slope $1 + \hat{\alpha} \approx 4$ corresponds to inverse-cubic law ($\hat{\alpha} = 3$) in the tails of cumulative distribution of R_t [21].

4. Summary

In summary, a variety of financial data are characterized by long-range correlations in the variance [23–27], short-range correlations in price changes [9,21,23], Lévy type of scaling in the central region [5,42] and power-law tails in the PDFs with exponent beyond the Lévy range [43]. In this paper, to model time series of the S&P500 index, we derive the AR+GARCH process, where the GARCH process we employ to account for long-range correlations in the variance, and the AR process to take account of short-range correlations in price changes. The GARCH process itself generates power-law tails in the PDFs. We employ the GARCH process with the truncated Lévy distribution to obtain a crossover behavior between a power law of Lévy type and a new power law "dynamically generated" due to correlations in the variance imposed by GARCH process. We analyze the sum of n AR+GARCH variables and show that the long-range correlations in the variance, imposed by the GARCH process alone, are followed by dynamical stability in the power-law tails of the PDF for a long, but finite range of n. In applications, the GARCH process might be very useful since numerous phenomena described by stable power-law statistics are characterized by correlations with long-range decay over wide range of time scales [44–53].

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