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Scale-invariant truncated Lévy process

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Abstract. – We develop a scale-invariant truncated Lévy (STL) process to describe physical systems characterized by correlated stochastic variables. The STL process exhibits Lévy stability for the distribution, and hence shows scaling properties as commonly observed in empirical data; it has the advantage that all moments are finite and so accounts for the empirical scaling of the moments. To test the potential utility of the STL process, we analyze financial data.

In recent years, the Lévy process [1] has been proposed to describe the statistical properties of a variety of complex phenomena [2–14], due to scaling behavior in distributions similar to that observed in empirical data. However, the application of the Lévy process to empirical data is limited because, opposite to commonly encountered data, it is characterized by no correlations in the moments.

Lévy walks [7] have been proposed to account for the finite moments observed for empirical data. Another way to retain the finite variance is by means of truncated Lévy (TL) flights [15] defined to have a Lévy distribution in the central regime, truncated by a function decaying faster than a Lévy distribution in the tails. However, the TL process with either abrupt [15] or smooth [16] truncation has limitations when applied to empirical data. i) The TL process is introduced for independent and identically distributed (i.i.d.) stochastic variables, while variables describing many physical systems are long-range correlated [17–20], and so are not i.i.d. ii) The distributions for a variety of complex systems, however, are often characterized by regions of scale-invariant behavior, while the TL process tends to the Gaussian distribution and hence does not exhibit scale invariance.

Here we introduce a stochastic process which we call the scale-invariant truncated Lévy (STL) process. The STL process might be regarded as a generalization of the truncated Lévy process —due to scaling transformations of the Lévy type, the stochastic variables exhibit scale-invariant behavior in the distributions, and due to truncation even in the moments. We also propose a dynamical mechanism to account for the regimes of different scaling behavior.

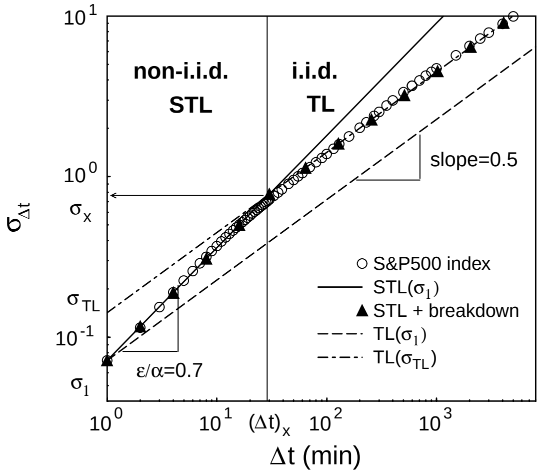

Fig. 1

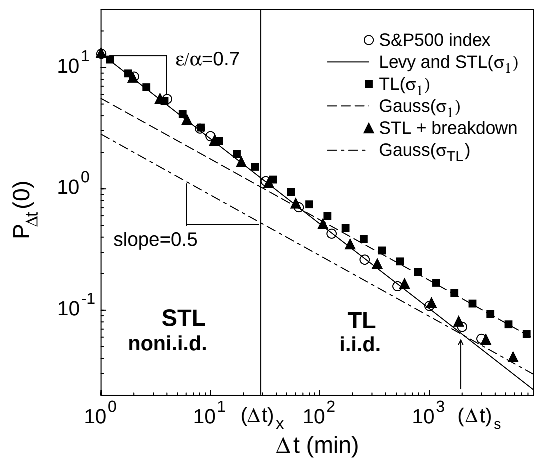

Fig. 2

Fig. 1 – $S\&P500$ index changes show two scaling regimes for the standard deviation σ. The regime at small time scales Δt, where $\sigma_{\Delta t} = \sigma_1 (\Delta t)^{0.7}$, and σ_1 is calculated for the data with $\Delta t = 1$, we model by the STL process (eqs. (1) and (2)) with $\alpha = 1.43$ (fig. 3). From eq. (4) we obtain $\epsilon = 1$. The index changes for the entire range of time scales with the crossover at $\Delta t_\times = 30$, we model by the STL process with a breakdown (eqs. (1), (6), and (7)). The breakdown in the STL is equivalent to a transition to a TL process at large time scales $\Delta t > (\Delta t)_\times$, where $\lambda_{\Delta t} \equiv \lambda_\times = \text{const}$ and $A_{\Delta t} \equiv \Delta t A_\times$, with σ_{TL} larger than σ_1. This is the reason for the delay (at time scale $(\Delta t)_s \approx 10^3$) in the transition from Lévy to Gaussian behavior observed for $\mathcal{P}_{\Delta t}(0)$ in fig. 2. Note that the TL process with a standard deviation σ_1 would exhibit for $\mathcal{P}_{\Delta t}(0)$ a transition from Lévy to Gaussian at smaller time scales (fig. 2).

Fig. 2 – Maximum of the distribution $\mathcal{P}_{\Delta t}(0)$ of the $S\&P500$ index changes R_t follows the power law scaling for more than 3 decades in Δt. We show the STL process with $\alpha = 1.43$ and $A_1 = 0.0027$ obtained by fitting $\mathcal{P}_{\Delta t=1}(R_t)$ (fig. 3). The maximum of the distribution, as expected from eq. (3), follows the distribution of the Lévy process with the same α and A_1 for all time scales. The TL process (with $\sigma_1 = 0.07$, identical to the empirical value) exhibits a transition at short time scales to the Gaussian process (with the same value of σ_1), in disagreement with the data. The STL process with a breakdown at $(\Delta t)_\times$ with distribution of eqs. (1), (6), and (7), however, is in agreement with the data and explains the delayed transition (at $(\Delta t)_s \approx 10^3$) to the Gaussian observed in the data.

To exemplify the motivation for the STL process, we analyze the changes of the $S\&P500$ stock index, denoted by R_t, over varying time scales Δt sampled for the 12-year period Jan. '84–Dec. '95. In particular, we focus on the scaling behavior of several statistical characteristics: i) In fig. 1 the standard deviation of index changes as a function of time scale Δt shows two different scaling regimes with a crossover at $(\Delta t)_\times \approx 30$ min [21]. The regime at small time scales is characterized by exponent 0.7, indicating the presence of positive correlations in the index changes. The second regime has exponent 0.5, indicating absence of correlations. Therefore, for the whole range of time scales, the stochastic process underlying index changes cannot be described by an i.i.d. stochastic process, such as the Lévy or the TL process. ii) In fig. 2 the maximum of the distribution $\mathcal{P}_{\Delta t}(0)$, as a measure of behavior of the distribution in the whole central part, however, follows the distribution expected for a Lévy process for more than three decades [14]. iii) The scaling exponent of $\mathcal{P}_{\Delta t}(0)$ is identical to the exponent of 0.7 for the standard deviation in the first regime. However, the crossover in the scaling of the standard deviation is not followed by a change in the slope of $\mathcal{P}_{\Delta t}(0)$.

The first regime at small time scales in fig. 1 characterized by correlations in index changes, we describe by the STL process specified by the distribution

$$\tilde{\mathcal{P}}(z) \equiv \frac{1}{2\pi} \int \phi(k) e^{ikz} \mathrm{d}k, \tag{1}$$

where $\phi(k) \equiv \exp[-\int_{-\infty}^{\infty} \mathrm{d}z(1 - e^{-ikz}) f(z)]$ is the characteristic function [16, 22] and $f(z) \equiv Ae^{-\lambda|z|^\beta}|z|^{-1-\alpha}$. The parameters α $(0 < \alpha < 2)$ and A $(A > 0)$ are of the Lévy process [1], where α governs the scaling behavior of the distribution. The parameter λ makes $f(z)$ decrease faster than $z^{1+\alpha}$ expected for the Lévy process, and so ensures a smooth truncation of the Lévy distribution, and makes the moments finite. The parameter β can take any positive value but, for the sake of simplicity, we set $\beta = 1$ in order to obtain an analytic form for $\phi(k)$ (eq. (1)) [16].

Despite the truncation, we maintain scale invariance of the Lévy type for $\mathcal{P}(z)$, over the entire range including the tails, by defining the scaling transformations [23]

$$A_{\Delta t} \equiv (\Delta t)^{\epsilon} A_1, \qquad \lambda_{\Delta t} \equiv (\Delta t)^{-\epsilon\beta/\alpha} \lambda_1, \tag{2}$$

where Δt is the time scale and ϵ can take any positive value. Under these transformations, the distribution $\tilde{\mathcal{P}}(z) \equiv \tilde{\mathcal{P}}_{\Delta t}(z)$ scales as the Lévy stable distribution:

$$z \equiv (\Delta t)^{\epsilon/\alpha} z_1, \qquad \tilde{\mathcal{P}}_{\Delta t}(z) \equiv \frac{\tilde{\mathcal{P}}_1(z_1)}{(\Delta t)^{\epsilon/\alpha}}. \tag{3}$$

These scaling transformations relate the variable z_1 for the time scale $\Delta t = 1$ with the linear combination of those stochastic variables, denoted by z, at any given Δt. With the transformations of eqs. (2) and (3), we obtain a process with controlled dynamical properties —$\tilde{\mathcal{P}}_{\Delta t}(z)$ for any value of Δt can be calculated from the distribution at any chosen Δt (*e.g.*, $\Delta t = 1$). Note that for $\lambda = 0$, and $\epsilon = 1$, the distribution (1) reduces to the distribution expected for the Lévy process [24]. For that reason, with an appropriate choice for λ, for small values of z, $\mathcal{P}(z)$ has a Lévy profile in the central part. Also, for $\epsilon = 1$ and $\lambda = \mathrm{const}$ in eq. (2), the scale invariance in distributions of eq. (3) is lost and that case corresponds to the TL process [16].

Although the distribution $\tilde{\mathcal{P}}_{\Delta t}(z)$ exhibits scaling properties identical to the Lévy stable distribution, the Lévy and the STL process defined by eqs. (1) and (2) are different. While the Lévy process is defined for i.i.d. variables, the STL process is characterized by correlated stochastic variables. To demonstrate this, we consider the scaling of the second moment σ^2 over varying time scales Δt [25]:

$$\sigma^2_{\Delta t} = \frac{2A\ \Gamma((2-\alpha)/\beta)\ \lambda^{(\alpha-2)/\beta}}{\beta} = (\Delta t)^{2\epsilon/\alpha} \sigma_1^2, \tag{4}$$

where the last term is obtained after inserting the scaling transformations of eq. (2) in the second term, and σ_1 is the standard deviation for $\Delta t = 1$. Clearly, the presence of correlations is indicated by the scaling exponent ϵ/α chosen to be different than 0.5, expected for uncorrelated process. The scaling exponent 0.5 is obtained for the i.i.d.

In addition, the STL process exhibits scaling not only for the second moment but also for all higher moments:

$$\langle |z|^n \rangle \equiv \int \mathrm{d}z\ |z|^n\ \tilde{\mathcal{P}}_{\Delta t}(z) = \Delta t^{\epsilon n/\alpha} \langle |z_1|^n \rangle\,. \tag{5}$$

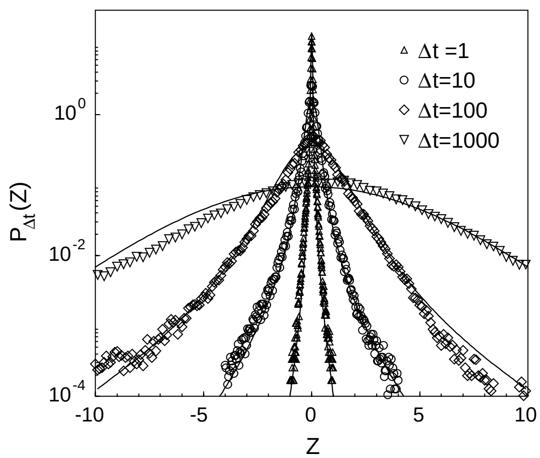

Fig. 3 – Distributions $\mathcal{P}_{\Delta t}(R_t)$ of the $S\&P500$ index changes R_t for different time scales Δt. By solid lines, we show the distributions $\tilde{\mathcal{P}}_{\Delta t}(z)$ of eqs. (1), (6), and (7) for the same time scales. We fit only $\mathcal{P}_{\Delta t=1}(R_t)$, when $\Delta t = 1$. We set $\alpha = 1.43$ (eq. (1)), since the central part of $\mathcal{P}_{\Delta t=1}(R_t)$ is nice fit by the Lévy distribution with given α. The rest of parameters (eqs. (6) and (7)) $A_1 = 0.0027$ and $\lambda_1 = 2.6$ are found to give the standard deviation (eq. (4)) for $\Delta t = 1$ as calculated for the data, and to fit better $\mathcal{P}_{\Delta t=1}(R_t)$ in the entire range. For any $\Delta t > 1$, $\tilde{\mathcal{P}}_{\Delta t}(z)$ is calculated from $\tilde{\mathcal{P}}_{\Delta t=1}(z)$.

Hence, the STL is a process for which the distribution $\tilde{\mathcal{P}}_{\Delta t}(z)$, the second moment σ^2, and all higher moments $\langle |z|^n \rangle$ scale with the same scaling exponent ϵ/α, due to the scaling transformations we introduce in eq. (2). The STL process yields the same scaling exponent for the maximum of the distribution and the standard deviation as found for the data in figs. 1 and 2.

To account for the crossover behavior shown in fig. 1 in the scaling of the standard deviation of index changes at time scale $(\Delta t)_\times$, we introduce a new stochastic process with two different regimes of time scales —the STL process with a breakdown— with the distribution of eq. (1) and the scaling transformations

$$\lambda_{\Delta t} = \left\{ \begin{array}{rr} (\Delta t)^{-\epsilon_1 \beta/\alpha} \lambda_1, & 1 \leq \Delta t \leq (\Delta t)_\times \\ \lambda_\times, & \Delta t > (\Delta t)_\times \end{array} \right\}, \tag{6}$$

$$A_{\Delta t} = \left\{ \begin{array}{rr} (\Delta t)^{\epsilon_1} A_1, & 1 \leq \Delta t \leq (\Delta t)_\times \\ \Delta t \ A_\times, & \Delta t > (\Delta t)_\times \end{array} \right\}. \tag{7}$$

At the crossover time scale $(\Delta t)_\times$ the STL process with breakdown exhibits a transition from a non-i.i.d. STL process, where standard deviation scales as $\sigma_{\Delta t} \propto (\Delta t)^{\epsilon_1/\alpha}$, to an i.i.d. TL process for which $\sigma_{\Delta t} \propto (\Delta t)^{1/2}$. Here α, A_1 and λ_1 are free parameters, chosen to fit the distribution $\mathcal{P}_{\Delta t}(R_t)$ for the $S\&P500$ data at time scale $\Delta t = 1$. The parameter $\alpha = 1.43$ is set to fit the central region of $\mathcal{P}_{\Delta t=1}(R_t)$ that is approximately the Lévy distribution, while A_1 and λ_1 are set to reproduce $\mathcal{P}_{\Delta t=1}(R_t)$ over the entire range (fig. 3). The first regime in eqs. (6) and (7) at small time scales $\Delta t < (\Delta t)_\times$, which we call the STL regime, accounts for the empirical regime in fig. 1 with correlations in index changes, where $\sigma_{\Delta t} \propto (\Delta t)^{0.7}$. Equating this scaling exponent with that from eq. (4), $\epsilon_1/\alpha = 0.7$, we obtain $\epsilon_1 = 1$. The second regime at large time scales, or the TL regime, accounts for the uncorrelated regime in index changes in fig. 1, where $\sigma_{\Delta t} \propto (\Delta t)^{0.5}$. Continuity of the distributions and the moments

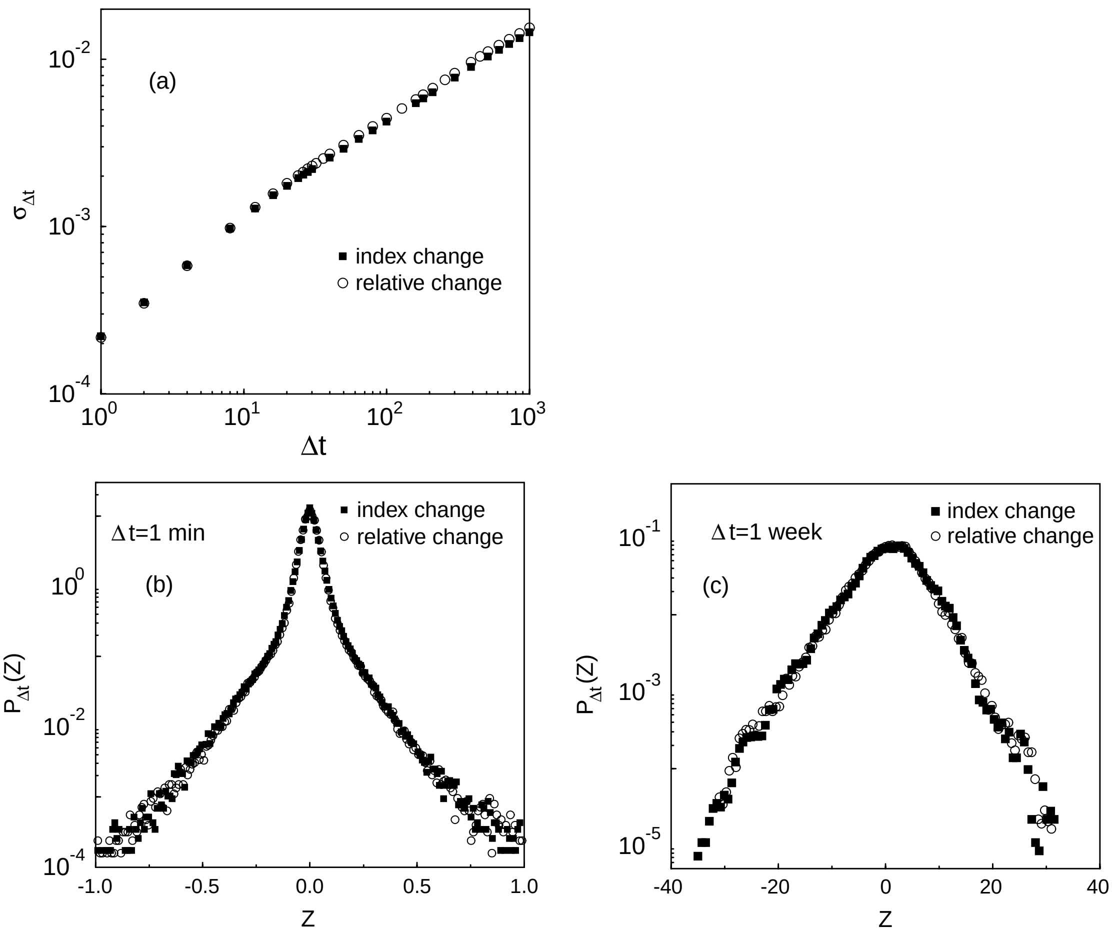

Fig. 4 – (a) For the $S\&P500$ index (Jan. '84–Dec. '95), we show how the standard deviation of index changes and relative index changes completely overlap, after rescaling by the average value of the index for that period. The standard deviation is calculated for a long period of time (12 years). In (b) and (c) we show how the distributions of the same stochastic variables collapse after rescaling by the same average value of the index. The collapse is obtained again for both high-frequency (1 min) and low-frequency (1 week) data. We also subdivide the 12 year period in two equal subintervals, and find the same collapse.

at the crossover time scale $(\Delta t)_\times$ is ensured by continuity in the values of A and λ: from eqs. (6) and (7) we find $A_\times \equiv (\Delta t)_\times^{\epsilon_1 - 1} A_1$ and $\lambda_\times \equiv (\Delta t)_\times^{-\beta\epsilon_1/\alpha}\lambda_1$, where $\beta = 1$.

We find that the maximum of the distribution $\tilde{\mathcal{P}}_{\Delta t}(0)$ of eqs. (1), (6), and (7) for the STL process with a scaling breakdown is in good agreement with $\mathcal{P}_{\Delta t}(0)$ for the data for more than three decades (fig. 2). $\tilde{\mathcal{P}}_{\Delta t}(0)$ gradually scales from a value expected for a Lévy process, $L(0) \propto (\Delta t)^{-1/\alpha}$ to a value expected for a Gaussian process, $G(0) = 1/(\sqrt{(2\pi)}\sigma_{\rm TL})(\Delta t)^{-1/2}$, where $\sigma_{\rm TL}$ is the standard deviation for the limiting Gaussian distribution. Since $\sigma_{\rm TL}$ is larger than σ_1 calculated for the data with $\Delta t = 1$, in fig. 2 we see that the transition from the Lévy to the Gaussian regime is delayed compared with the transition from the Lévy to a limiting Gaussian process if the whole range of time scales is characterized by i.i.d. TL stochastic variables with standard deviation σ_1. The reason for the delay is that for the first STL regime in eqs. (6) and (7), $\sigma_{\Delta t}$ increases with exponent 0.7, that is much faster than 0.5

expected for the second i.i.d. TL regime (see fig. 1). At the crossover time scale $(\Delta t)_\times$, the standard deviation reaches the value $\sigma_\times = (\Delta t)_\times^{0.7}\sigma_1$. From the continuity of the standard deviation for different regime of time scales at $(\Delta t)_\times$, we calculate $\sigma_\times = (\Delta t)_\times^{0.5}\sigma_{\mathrm{TL}}$, where σ_{TL} is the standard deviation for the limiting Gaussian distribution. The time scale $(\Delta t)_s$ of the transition between the Lévy and the Gaussian regime can be calculated by equating the maximum of the probability $\mathcal{P}_{\Delta t}(0)$ for the Lévy and Gaussian distributions [26]. We find that $(\Delta t)_s = \mathcal{B}(\Delta t)_\times$, where $\mathcal{B} \approx 70$ (fig. 2). Such a relation is interesting, since it explicitly connects the crossover from the Lévy to the Gaussian regime in scaling of distribution with the crossover from non-i.i.d. to i.i.d. process in scaling of standard deviation.

Finally, in fig. 3 for different time scales Δt, we compare the empirical distributions $\tilde{\mathcal{P}}_{\Delta t}(R_t)$ of the $S\&P500$ index changes with the distributions $\tilde{\mathcal{P}}_{\Delta t}(z)$ of eqs. (1), (6), and (7). Good agreement between the data and the theoretical results is observed both for the central part and for the tails. At small time scales, the scale-invariant behavior of $\tilde{\mathcal{P}}_{\Delta t}(z)$ is maintained in the entire range due to the scaling transformations of the STL process (eq. (2)). The crossover to an i.i.d. TL process at large time scales ensures a smooth transition to a Gaussian-like profile. We find that the proposed mechanism of a STL process with a breakdown provides a reliable control of the dynamical properties of the distribution.

Up to now, we have considered the changes of the $S\&P500$ index as the stochastic variables analyzed. The choice of stochastic variable depends on the type of the stochastic process: *e.g.*, for an *additive* process one considers increments, while for *multiplicative* processes the appropriate choice are relative increments. In finance, it is traditionally assumed that economic indicators arise from a multiplicative process, and correspondingly the preferred quantity to analyze are the relative changes commonly defined as the difference in the natural logarithm of the index. The additive and multiplicative processes are related for high-frequency data (small Δt) and short period of analysis, so the use of index changes or relative index changes leads to similar results. We find that even for low-frequency data (large Δt) and for long period of analysis (up to 12 years), the results for the distribution and the standard deviation remain similar for both the index changes and the relative index changes (fig. 4).

We have proposed a stochastic process that even in the presence of correlations among the stochastic variables exhibits a Lévy stability for the distribution. The STL process is characterized by identical scaling exponent for both the moments and the PDF. The STL process provides a unified dynamical picture to describe different statistical properties, and can be generalized for situations when the moments and the PDF exhibit different scaling behavior. The STL process can be utilized —as we show in the case for financial data— not only for processes with a single scaling regime but also for systems with different regimes of scaling behavior. Recently, the crossover behavior in scaling of moments has been found for DNA sequences [27].

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[24] Note that the distribution for the STL process characterized by given Lévy index α can scale with any scaling exponent ϵ/α in contrast to the distribution of Lévy stable process which scales with the scaling exponent $1/\alpha$. The parameter ϵ controls the dynamics of the process —distributions characterized with the same α can exhibit different scaling behavior for different values of ϵ.
[25] The first term is calculated as the second derivative of $\phi(k)$ (eq. (1)) at small values of k [22].
[26] We obtain the following analytic expression: $\mathcal{B} = [\sqrt{2\pi}\sigma_1 \mathcal{L}_1(0)]^{2\alpha/(2-\alpha)}$, where $\mathcal{L}_1$ is the Lévy PDF at $\Delta t = 1$ [16].
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