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Systems with correlations in the variance: Generating power law tails in probability distributions

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Abstract. – We study how the presence of correlations in physical variables contributes to the form of probability distributions. We investigate a process with correlations in the variance generated by i) a Gaussian or ii) a truncated Lévy distribution. For both i) and ii), we find that due to the correlations in the variance, the process "dynamically" generates power law tails in the distributions, whose exponents can be controlled through the way the correlations in the variance are introduced. For ii), we find that the process can extend a truncated distribution *beyond the truncation cutoff*, which leads to a crossover between a Lévy stable power law and the present "dynamically generated" power law. We show that the process can explain the crossover behavior recently observed in the $S\&P500$ stock index.

Many natural phenomena are described by distributions with scale-invariant behavior in the central part and power law tails. To explain such a behavior the Lévy process [1] has been employed in finance [2], fluid dynamics [3], polymers [4], city growth [5], geophysical [6] and biological [7] systems. An intense activity has been developed in order to understand the origin of these ubiquitous power law distributions [8]. The Lévy process, however, is characterized by a distribution with infinite moments and in applications this might be a problem, *e.g.*, analysis of autocorrelations in time series requires a finite second moment. To address this problem, probability distributions of the Lévy type with both abrupt [9] and exponential cutoffs [10] have been proposed. A second problem is that the Lévy process has been introduced for independent and identically distributed stochastic variables, while for some systems there is a clear evidence of correlations in the variance (*e.g.*, for many important market indices [11]). Moreover, a crossover between a Lévy stable power law and a power law with an exponent out of the Lévy regime, was recently found in the analysis of price changes [12]. We investigate how a stochastic process with no correlations in the variables but rather in their variance can be introduced to account for the empirical observations of a Lévy stable form of the probability distribution in the central part and a crossover to a power law behavior different than the Lévy in the far tails.

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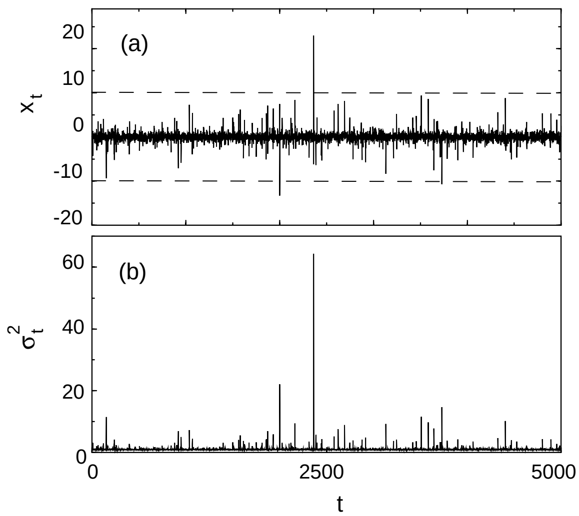

Fig. 1 – The dependence on time of (a) a realization of an ARCH process x_t of eq. (1) and (b) the variance σ_t^2 of eq. (2). The ARCH process x_t is defined by $a = 0.88$ and $b = 0.12$. For these a and b values, $\sigma_x^2 = 1$ (eq. (3)). $P(w_t)$ is the TL PDF of eq. (5) with $\alpha = 1.5$, $\gamma = 0.27$, and the cutoff length $\ell = 10$ (horizontal dashed line). Large values of x_t are followed by large values of σ_t^2. Due to the dynamical features of the ARCH process, values of x_t can exceed the truncation cutoff ℓ of the TL distribution, and the regime beyond the cutoff length becomes populated (see fig. 3).

In finance, a stochastic process with autoregressive conditional heteroskedasticity (ARCH) [13] is often used to explain systems characterized by correlations in the variance. The ARCH process is a discrete time process x_t whose variance σ_t^2 depends conditionally on the past values of x_t. The ARCH process is specified by the form of the probability density function (PDF) $P(w_t)$ of the process w_t which generates the random variable x_t.

Here we ask to what extent the presence of correlations in the variance σ_t^2 contributes to the form of the probability distribution $P(z_n)$, where $z_n \equiv \sum_{i=1}^{i=n} x_{t+i-1}$ and x_t is the ARCH process (in finance z_n is called temporal aggregation of the ARCH process x_t). We perform numerical simulations and investigate the form of $P(z_n)$, when the PDF $P(w_t)$ of the process w_t which generates x_t is either of two cases, i) a Gaussian, or ii) a truncated Lévy distribution. For case i) and $n = 1$ (*i.e.* $z_1 \equiv x_t$), the ARCH process generates $P(x_t)$ which, to a good approximation, can be fit at the tails by a single power law [14]. For case ii) and $n = 1$, we show that the interplay between the Lévy form of the distribution in the central part and the dynamics of the ARCH process can give rise to a crossover between two power law regimes in the tails of $P(x_t)$. For both i) and ii), at large n, we find clear convergence of $P(z_n)$ to a Gaussian [15].

Suppose x_t is generated by an independent and identically distributed (i.i.d.) stochastic process w_t, drawn from a PDF $P(w_t)$ with zero mean ($\langle w_t \rangle = 0$) and unit variance ($\langle w_t^2 \rangle = 1$),

$$x_t = \sigma_t w_t. \tag{1}$$

Then x_t follows an ARCH process if the variance σ_t^2 evolves in time as

$$\sigma_t^2 = a + b\, x_{t-1}^2, \tag{2}$$

where a and b are two non-negative constants [13]. For $b = 0$, the ARCH process x_t reduces to the i.i.d. stochastic process w_t —no correlations in all moments. For $b \neq 0$, the stochastic

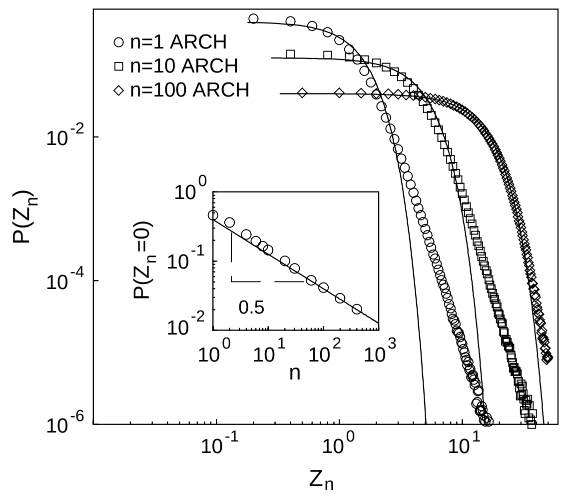

Fig. 2 – Log-log plot of $P(z_n)$ for the temporal aggregation z_n of the ARCH process x_t calculated for different time scales $n = 1$ (○), 10 (□), and 100 (◇). The ARCH process x_t with $a = 0.49$ and $b = 0.51$ has the expected variance $\sigma_x^2 = 1$ (eq. (3)) and kurtosis $\kappa_x = 10$ (eq. (4)). w_t in eq. (1) is chosen from a Gaussian distribution ($\kappa_w = 3$). Due to the correlations in the variance σ_t^2, the temporal aggregation of the ARCH process generates power law tails in $P(z_n)$ for small time scales n. By solid lines, for the same time scales ($n = 1, 10$, and 100), we denote the Gaussian process with the same variance as the ARCH process. In the inset, we show a log-log plot of the probability of return to the origin $P(z_n = 0)$ for the same ARCH (○) and for the Gaussian process (solid line); the line has slope 0.5 —the value predicted for the Gaussian— which suggests that $P(z_n)$ gradually tends towards the Gaussian with increasing n.

process x_t is uncorrelated —$\langle x_t x_{t'} \rangle \sim \delta_{tt'}$— but has correlations in the variance σ_t^2. Indeed, the ARCH process is characterized by exponentially decaying correlations in the variance $\langle \sigma_t^2 \sigma_{t'}^2 \rangle \sim \exp[-(t'-t)/\tau]$ with decay constant $\tau = |\log(b)|^{-1}$.

In fig. 1 we see that σ_t^2 is large when x_{t-1}^2 is large. The ARCH process is specified by the functional form of the PDF $P(w_t)$ which controls the stochastic variable w_t and also by the value of the parameter b (eqs. (1) and (2)). In the ARCH process $P(w_t)$ is traditionally the Gaussian, but other choices are possible [16]. Furthermore, the correlations in the variance σ_t^2 can be controlled by changing the parameter b —larger values of b relate to stronger correlations.

The expected variance of x_t is a constant: $\sigma_x^2 \equiv \langle x_t^2 \rangle = \langle \sigma_t^2 \rangle$ [13]. From this relation and eqs. (1) and (2) there follows:

$$\sigma_x^2 = \frac{a}{1-b} \tag{3}$$

and

$$\kappa_x \equiv \frac{\langle x^4 \rangle}{(\langle x^2 \rangle)^2} = \kappa_w + \frac{\kappa_w(\kappa_w - 1)b^2}{1 - \kappa_w b^2}, \tag{4}$$

where κ_w is the kurtosis of $P(w_t)$, a common measure of the degree of "fatness" of the distribution related to the fourth moment [11, 13]. From eq. (3), σ_x^2 is finite if $b < 1$, while finiteness of kurtosis κ_x (and the fourth moment of x_t) requires $\kappa_w b^2 < 1$. No matter what functional form for $P(w_t)$ we choose, the ARCH process generates PDFs with slower decaying tails compared to $P(w_t)$ — $\kappa_x > \kappa_w$ from eq. (4). Such a slow decay in the tails of $P(z_n)$ for the ARCH process may come both from the choice of the $P(w_t)$ and the presence of correlations in the variance, *i.e.* κ_w and b, respectively.

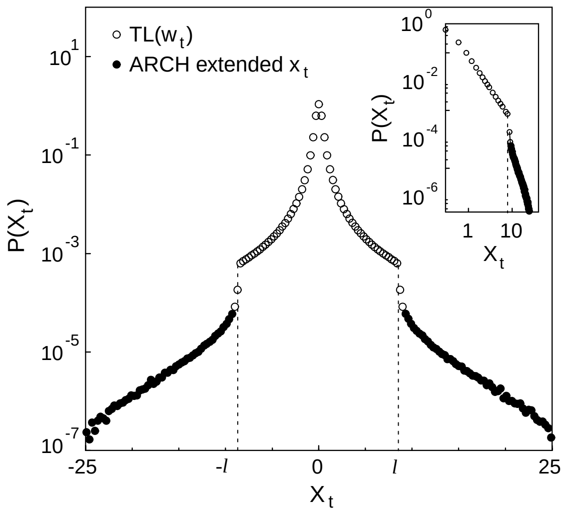

Fig. 3 – Linear-log plot of the PDFs of (○) the truncated Lévy process w_t without correlations and (●) the ARCH process x_t ($n = 1$), where w_t of eq. (1) has the TL distribution of eq. (5) with $\alpha = 1.2$, $\gamma = 0.21$, and cutoff length $\ell = 9$ (vertical dotted lines), implying $\kappa_w = 24.2$. Parameters of the ARCH process are $a = 0.9$ and $b = 0.1$ giving $\kappa_x = 31.6$ (eq. (4)). While the central part (○) of the PDF remains practically the same for both w_t and x_t processes, the ARCH process x_t extends the range of the TL to $x_t > \ell$ due to the correlations in the variance σ_t^2 (eq. (2)). This ARCH-extended regime (●) is rarely populated and therefore decays faster. In the inset, we show the log-log plot of the PDF of the ARCH process x_t. The process leads to extension of the range of the original TL distribution, with a crossover between two different power law regimes: the original Lévy stable power law and the dynamically generated power law (appearing beyond the truncation cutoff ℓ). The exponent of the power law in the ARCH-extended regime is defined by correlations in the variance σ_t^2 through the parameter b (eq. (2)).

First, we consider the temporal aggregation z_n of an ARCH process for a particular choice of $P(w_t)$ given by a Gaussian distribution (see fig. 2). Due to the correlations in the variance (eq. (2)), the ARCH process, for small n, generates tails which can be approximated by a power law. Given that the parameter b controls the correlations in the variance σ_t^2, we find that the slope of this power law is directly linked to these correlations —the stronger the correlations in σ_t^2, the smaller the slope of the power law tail. For large n, the PDF of the temporal aggregation z_n of the ARCH process resembles the form of the Gaussian distribution, *i.e.* $P(z_n)$ tends to the Gaussian, with a variance of $n\sigma_x^2$.

Second, we analyze the ARCH process where $P(w_t)$ (eq. (1)) is the truncated Lévy (TL) distribution [9], defined by

$$\mathcal{T}_{\alpha,\gamma,l}(w) \equiv \left\{ \begin{array}{ll} \mathcal{N}\mathcal{L}_{\alpha,\gamma}(w), & |w| \leq \ell \\ 0, & |w| > \ell \end{array} \right\}. \tag{5}$$

Here, $\mathcal{L}_{\alpha,\gamma}(w)$ is the symmetrical Lévy stable distribution [1,2], α is the scale index ($0 < \alpha < 2$), γ is the scale factor ($\gamma > 0$), $\mathcal{N}$ is the normalizing constant and ℓ is the cutoff length. Note that the TL distribution is characterized by power law tails from zero up to the cutoff length ℓ. In performing numerical simulations [17], we employ an algorithm of ref. [18].

In fig. 3, we show the PDF of an ARCH process x_t ($n = 1$) where $P(w_t)$ is the TL distribution. We find two regimes where the values of x_t are smaller and larger than the cutoff length ℓ of $P(w_t)$. The first regime ($x_t < \ell$) is characterized by the Lévy power law due

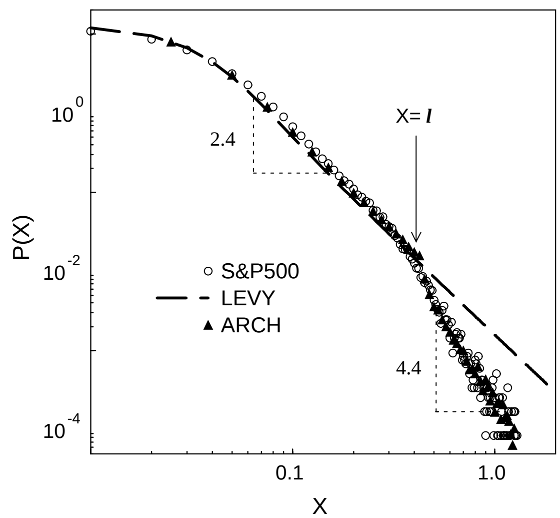

Fig. 4 – PDF of the 1 min price changes for the $S\&P500$ stock index over the 12-year period Jan. '84-Dec. '95. The central part of the empirical PDF is well fit with the Lévy distribution ($\alpha = 1.4$) up to the cutoff length ℓ (arrow), followed by crossover to a second "dynamically generated by ARCH" regime ($x_t > \ell$), which can be approximated by a power law with slope 4.4 ($\alpha = 3.4$). We show a realization of the ARCH process, where $P(w_t)$ is the TL distribution of eq. (5) with $\alpha = 1.4$, $\gamma = 0.275$, length $\ell = 8$, and the process is characterized by $b = 0.4$, where a is chosen to give the empirical standard deviation $\sigma = 0.07$. Note that the bump at the truncation cutoff ℓ in the PDF of the ARCH process in fig. 3 is more pronounced compared to the bump in the PDF of the $S\&P500$ data and the ARCH process in this figure. This is due to the fact that the correlations in the variance σ_t^2 introduced for the particular realization of the ARCH process in fig. 3 are weaker ($b = 0.1$) compared to the correlations in the $S\&P500$ data ($b = 0.4$).

to the choice of the $P(w_t)$ (eq. (5)). Due to the correlations in the variance σ_t^2 (eq. (2)), the ARCH process extends the range of the PDF to $x_t > \ell$, generating a new power law different from the Lévy power law in the first regime. This second regime, where $x_t > \ell$, is rarely populated —events in that regime occur only when two large values of x_t follow each other. Since such events have small probability, the PDF of the ARCH process x_t in that regime decreases faster compared with the first regime where the fluctuations of x_t are smaller than the cutoff length ℓ.

The existence of two regimes in the PDF, characterized by two different power laws, is empirically observed in high-frequency data on price changes [12]. In addition the central part of the empirical PDF in financial data is well described by a Lévy distribution and exhibits the same scale-invariant behavior [2]. Such behavior of the empirical PDF can be mimicked by the ARCH process with a TL distribution, since the ARCH process generates power laws in the far tails while preserving the form of the PDF in the central region (figs. 2 and 3). Empirical data also show a bump in the PDF of the price changes calculated for a delay of 1 min. A similar bump is observed in fig. 3 for the ARCH process x_t around the cutoff length ℓ. We find that this bump is a signal for changing the power law exponent in the probability distribution. In fig. 4 we show the PDF of the ARCH process in good agreement with the PDF of 1 min price changes for the $S\&P500$ index. The central part ($x_t < \ell$) of the PDF characterized by Lévy-stable power law with slope 2.4 ($\alpha = 1.4$) is followed by a crossover to a second "dynamically generated" by ARCH regime ($x_t > \ell$) with a different behavior, approximately power law with slope 4.4 ($\alpha = 3.4$). As in the case of the ARCH process

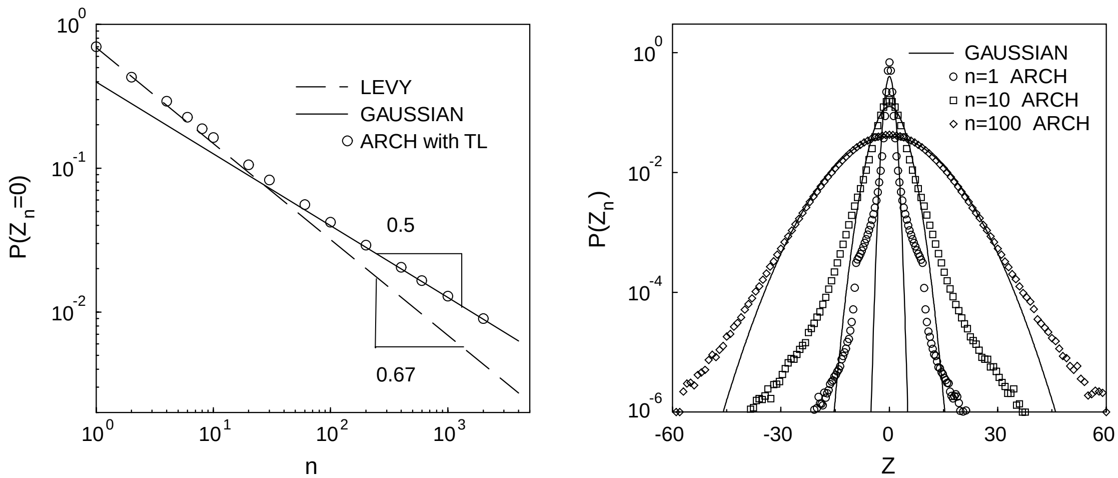

Fig. 5 – Left panel: log-log plot of the probability of return $P(z_n = 0)$, where z_n is the temporal aggregation of the ARCH process with $a = 0.88$, $b = 0.12$, $\sigma_x^2 = 1$ (eq. (3)) and kurtosis $\kappa_x = 31.4$ (eq. (4)). $P(w_t)$ is the truncated Lévy distribution (eq. 5) (with $\sigma_w^2 = 1$ and $\kappa_w = 21.9$ (eq. 4)) obtained by setting $\alpha = 1.5$, $\gamma = 0.27$, and $\ell = 10$. Also shown is the Lévy stable process with $\alpha = 1.5$ (slope equals $1/\alpha$) and $\gamma = 0.27$, and the Gaussian process with variance $\sigma^2 = 1$ equal to the expected variance σ_x^2 of the ARCH process. For small values of n, z_n follows the Lévy process and then converges to the Gaussian. Right panel: linear-log plot of $P(z_n)$ calculated for different time scales n. For the same time scales, we also show the Gaussian process to which z_n gradually converges. Parameters for both Gaussian and ARCH process are defined as for the left panel.

with Gaussian PDF, we can control the exponent of the power law in the second dynamically generated by ARCH regime by tuning b, while the parameter a is chosen to give the empirical standard deviation.

Next we study the asymptotic behavior (at large time scales n) of the PDF of the temporal aggregation z_n, where $P(w_t)$ is the TL distribution. In fig. 5 we show $P(z_n = 0)$ for the ARCH process. For small n, $P(z_n = 0)$ approximately follows the Lévy distribution $\mathcal{L}_{\alpha,\gamma}(z_n = 0) = \Gamma(1/\alpha)/[\pi\alpha(n\,\gamma)^{1/\alpha}]$ with the same α and γ as $P(w_t)$. For $n > 30$, $P(z_n = 0)$ tends to a Gaussian distribution $G(z_n = 0) = 1/[\sqrt{2\pi}\sigma_x n^{1/2}]$ with variance σ_x defined by eq. (3) and equal to the variance of $P(x_t)$ [19]. Thus, despite the presence of correlations in the variance σ_t^2, the temporal aggregation z_n of the ARCH process behaves like an i.i.d. process, characterized by a fast transition from Lévy to Gaussian process ("ultra-fast" TL flight [9]). Since the correlations in the variance are of short range, for sufficiently large time scales n, $P(z_n)$ approaches a Gaussian form, a behavior normally associated with uncorrelated stochastic variables (fig. 5).

In summary, we find that power law tails in distributions can be dynamically generated by introducing correlations in the variance of stochastic variables, even when the initial distribution of these variables is the Gaussian. We also find that when the initial distribution is a truncated Lévy distribution, the process of introducing correlations in the variance can extend the range of the PDF beyond the truncation cutoff. This extension of the range of the original probability distribution is characterized by a crossover between two different power law regimes: the original Lévy stable power law (within the limits of the truncation cutoff) and the "dynamically generated" power law (beyond the truncation cutoff). This behavior appears to explain the empirically observed crossover in the PDF of price changes for the $S\&P500$, and carries information about the relevant parameters of the underlying stochastic process. Our findings can help understand to what extent the presence of correlations in physical variables contributes to the form of probability distributions, and what class of stochastic processes could be responsible for the emergence of power law behavior.

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[17] To obtain a random variable with $\gamma \neq 1$, we note that the PDF of eq. (5) rescales under the transformations $\tilde{w} \equiv w/\lambda^{1/\alpha}$, $\tilde{\mathcal{T}}_{\alpha,\tilde{\gamma},\tilde{l}}(\tilde{w}) \equiv \mathcal{T}_{\alpha,\gamma,l}(w)\lambda^{1/\alpha}$, where $\tilde{\gamma} \equiv \gamma/\lambda$, $\tilde{l} \equiv l/\lambda^{1/\alpha}$, and λ is the scaling parameter. If any algorithm corresponding to random variable w with $\gamma = 1$ is provided, the scaling transformation allows us to obtain any other variable $\tilde{w}$ with $\tilde{\gamma} \neq 1$.

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