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ARCH–GARCH approaches to modeling high-frequency financial data

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Abstract

We model the power-law stability in distribution of returns for S&P500 index by the GARCH process which we use to account for the long memory in the variance correlations. Precisely, we analyze the distributions corresponding to temporal aggregation of the GARCH process, i.e., the sum of n GARCH variables. The stability in the power-law tails is controlled by the GARCH parameters. We model the crossover behavior in magnitude correlations of returns by the so-called two-FIARCH process. Besides detrended fluctuation analysis, we employ the method proposed by Geweke and Porter-Hudak to estimate the fractional parameter in magnitude correlations.

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Much work [1–7] have been devoted to determine precisely the functional form of financial distributions since Mandelbrot [3] and Fama [4] have suggested stable Lévy

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distribution [8] to describe fat tails in the distributions of returns, defined as the first difference of the logarithm of cotton (Mandelbrot) and common stock prices (Fama). For high-frequency S&P500 index recorded each minute, Gopikrishnan et al. [9] have shown that the distribution of 1 min returns for S&P500 index is well described by the crossover behavior between a power-law regime of a Lévy type, found before by Mantegna and Stanley [5], and a power-law regime with an exponent $1+\alpha$ well beyond the Lévy range ($0<\alpha<2$) [10]. Due to large number of data points, it has been shown that the far tails of the distributions of returns appear to exhibit stability for long, but finite time scales (Fig. 1a) [9], while the probability distribution in the central part is well described by stable Lévy distribution (Fig. 1b). The crossover behavior between two power-law regimes has been also found in magnitude correlations [11] (Fig. 2).

To describe power-law stability in the distribution of returns for different time scales, we employ the generalized autoregressive conditional heteroskedastic (GARCH) process: $x_t = \sigma_t \varepsilon_t$, where ε_t is an i.i.d. process and $\sigma_t^2 = a + b x_{t-1}^2 + c\sigma_{t-1}^2$ [12]. The GARCH process is developed to take account of variance correlations typically found in financial data [12] (Fig. 2). Error distribution function $P(\varepsilon_t)$ is defined as $\langle \varepsilon_t \rangle = 0$ and $\langle \varepsilon_t^2 \rangle = 1$, while a, b, and c are nonnegative parameters, where $b+c<1$, for stationarity reason. By iterating the conditional variance σ_t^2, it can be rewritten as a constant plus the weighted average of all prior x_t^2.

Generally, regardless of the choice for $P(\varepsilon_t)$, the GARCH process generates the power-law tails in the distribution of x_t [13–16]. But the choice for $P(\varepsilon_t)$ becomes important when the GARCH process is applied to fit the central region of empirical distribution, since even GARCH process generates the power-law tails in $P(x_t)$, the

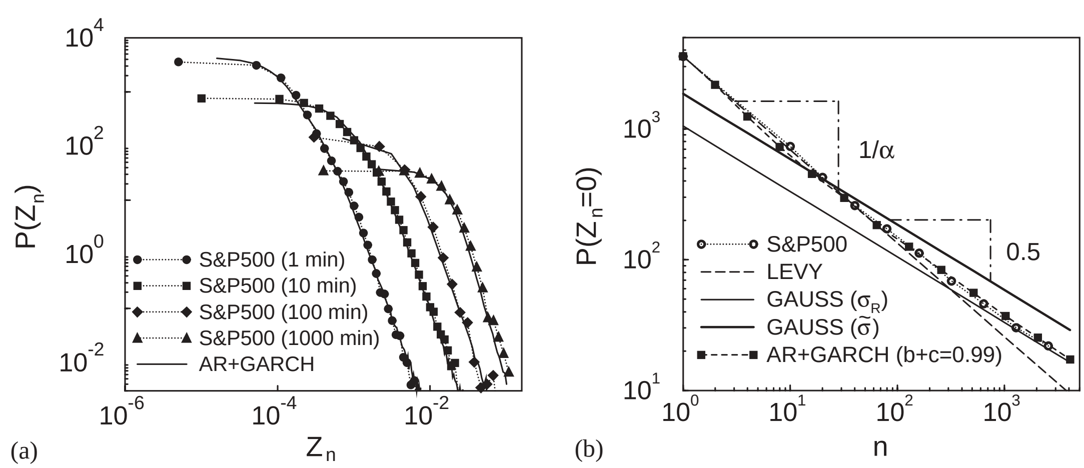

Fig. 1. (a) For finite time scales the power-law tails of the probability distributions of index returns remain practically stable with exponent equal to 4. The probability distributions of temporal aggregations of variables of Eq. (1) exhibit the same behavior. (b) For the same time scales, the probability distribution of index returns at origin follows the Lévy distribution with $\alpha = 1.4$. We present the corresponding Gaussian distributions for cases with and without serial correlations. We also show the probability distribution at origin of the temporal aggregation of variable of Eq. (1), $P(z_n = 0)$.

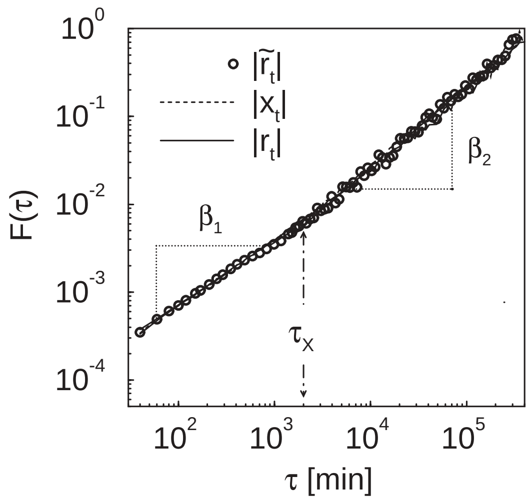

Fig. 2. Log–log plot of the mean standard deviation $F(\tau)$ of the detrended fluctuations of absolute S&P500 returns and absolute values of variables of the two-FIARCH process.

very central region of $P(x_t)$ typically resembles the functional form of error distribution, $P(\varepsilon_t)$ [14,15].

To this end, since the 1 min distribution of returns for the S&P500 index in Fig. 1 is characterized by the crossover behavior from one power-law regime of the Lévy type [8], describing the central region of the distribution, to another power-law regime with an exponent out of the Lévy range, we model the empirical crossover by the GARCH process with $P(\varepsilon_t)$ given by truncated Lévy (TL) distribution [2]. Since the central regime of the 1 min distribution $P(R_t)$ in log–log plot is of slope $1+\hat{\alpha}\approx 2.4$, for the error distribution $P(\varepsilon_t)$ we choose the Lévy exponent $\alpha = 1.4$ [2,15]. With a set of parameters given in Ref. [15], we obtain the power-law tails in $P(x_t)$ with the slope of the tails equal to 4.

Next, to account for the effect of the short-range correlations in 1 min returns, in addition to the GARCH process, we employ an autoregressive (AR) process [2]. To probe for large n the dynamic stability of the distributions, we study a temporal aggregation of the GARCH process, the process z_n that is a sum of n AR + GARCH variables

$$r_t = \phi_0 + \phi_1 r_{t-1} + \sigma_x \varepsilon_t \ . \tag{1}$$

In Fig. 1a, besides empirical distributions for different time scales Δt, we show the distributions $P(z_n)$. Persistence in the power-law tails of slope 4 in the data we model by the GARCH process with $b+c$ taken to be close to 1 [12,15]. The long-range magnitude correlations built in the GARCH process yield stability in the power-law tails of the distribution for a long range of time scales. The closer $b+c$ to one, the longer power-law stability [15].

In Fig. 1(b) we show five distributions of return at origin, one of which is empirical. For small time scales n, $P(z_n=0)$ approximately follows the Lévy distribution with $\alpha = 1.4$. If there were no serial correlations ($\phi_1 = 0$), $P(z_n = 0)$ would follow the upper Gaussian distribution with the variance scaling as $\sigma_r^2 n$, where

σ_r^2 is the 1 min empirical variance ($\sigma_r = \sigma_x$, where σ_x corresponds to the GARCH process). For the lower limit Gaussian distribution, serial correlations exist only for small time scales, where its variance scales as $\hat{\sigma}^2 n$. The theoretical 1 min variance $\hat{\sigma}^2$ is related to the empirical 10 min variance as $\hat{\sigma}^2 10 = \sigma_{10}^2$ [15]. The parameter ϕ_0 and correlation parameter ϕ_1 are calculated from the data [15]. For given parameters a, b, c of the GARCH process, σ_x is related to σ_r, through the relation $\sigma_x^2 = (1 - \phi_1^2)\sigma_r^2$.

Next we focus on the functional form of correlation pattern in magnitudes of returns for S&P500 index. As shown in Ref. [11], magnitude correlations in returns are specified by a *crossover* between two different power-law regimes (see Fig. 2). To see how crossovers in the magnitude correlations can be modeled, note that Engle et al. (see, Ref. [17]) proposed the fractionally integrated ARCH process (FIARCH) ($x_t = \sigma_t \varepsilon_t$) where long-range power-law correlations in $|x_t|$ are accomplished if weights a_n, defined in volatility $\sigma_t = \sum_{n=1}^{\infty} a_n |x_{t-n\Delta t}|$, are chosen to decay for $n \gg 1$ as a power-law series in n, $\propto n^{-1-\alpha}$, with a unique scaling exponent α.

In order to obtain a *crossover* in the power-law magnitude correlations, in Ref. [16] we propose a stochastic process that we call two-FIARCH process $x_t = \sigma_t e_t$, where $\sigma_t = \sum_{n=1}^{\infty} a_n |x_{t-n\Delta t}|$ while the *weights* a_n are defined as $a_n = n^{-1-\delta_1}$ for $n < n_\times$ and $a_n = n^{-1-\delta_2}$ for $n > n_\times$. The crossover time scale $n_\times$ and parameters δ_1 and δ_2 are found from the empirical data, where those parameters are related to DFA exponents [18], calculated for different regimes in magnitude correlations, by a simple linear relation [16]. In Fig. 2, we see that correlation pattern of the process fits nicely the pattern obtained for the empirical data.

After DFA analysis, we apply a method proposed by Geweke and Porter-Hudak (GPH) [19] for an estimator of the fractional parameter, d. The method is based on a regression of the ordinates of the log spectral density on trigonometric function. Suppose that a sample of z of size T is available. Let $\lambda_{j,T} = 2\pi j/T$ denote the harmonic ordinates and $I(\lambda_{j,T})$ denote the periodogram of these ordinates. One may show that the parameter d can be estimated from the least-squares regression

$$\ln(I(\lambda_{j,T})) = A - d \, \ln(4 \sin^2(\lambda_{j,T})/2) + \varepsilon_j \,, \tag{2}$$

where the Schuster periodogram is defined as follows:

$$I(\lambda_{j,T}) = \frac{1}{T^2}\left[\left(\sum_{t=1}^{T}(|z| - |\bar{z}|)\cos(\lambda_{j,T} t)\right)^2 + \left(\sum_{t=1}^{T}(|z| - |\bar{z}|)\sin(\lambda_{j,T} t)\right)^2\right]. \tag{3}$$

Here $T = 127\,000$ (calculated for 10 min returns) and ε_j is assumed to be i.i.d. with zero mean and variance $\pi^2/6$. The estimated d for absolute returns $|r_t|$ is 0.21 with t-statistics of 93, respectively. Since d parameter exceeds $t = 1.96$ with ∞ *df* at the 5% level of significance, we conclude that d parameter is statistically significant at the 5% level. Note that this value for d is considerably different compared to the one obtained by DFA method ($d = 0.34$).

In the paper, we model both power-law stability in distribution of returns and crossover in magnitude correlations found for S&P500 index. We show that the stability in the power-law tails is controlled by the GARCH parameters. We employ different methods to calculate fractional parameter for magnitude correlations.

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