Quantifying fluctuations in market liquidity: Analysis of the bid-ask spread

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Quantifying the statistical features of the bid-ask spread offers the possibility of understanding some aspects of market liquidity. Using quote data for the 116 most frequently traded stocks on the New York Stock Exchange over the two-year period 1994–1995, we analyze the fluctuations of the average bid-ask spread \( S \) over a time interval \( \Delta t \). We find that \( S \) is characterized by a distribution that decays as a power law \( P(S > s) \sim s^{-\zeta} \), with an exponent \( \zeta \approx 3 \) for all 116 stocks analyzed. Our analysis of the autocorrelation function of \( S \) shows long-range power-law correlations, \( \langle S(t)S(t+\tau) \rangle \sim \tau^{-\delta} \), similar to those previously found for the volatility. We next examine the relationship between the bid-ask spread and the volume \( Q \), and find that \( S \sim \ln Q \); we find that a similar logarithmic relationship holds between the transaction-level bid-ask spread and the trade size. We then study the relationship between \( S \) and other indicators of market liquidity such as the frequency of trades \( N \) and the frequency of quote updates \( U \), and find \( S \sim N \) and \( S \sim \ln U \). Lastly, we show that the bid-ask spread and the volatility are also related logarithmically.

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I. INTRODUCTION

The primary function of a market is to provide a venue where buyers and sellers can transact. The more buyers and sellers at any time, the more efficient the market is in matching buyers and sellers, so a desirable feature of a competitive market is liquidity, i.e., the ability to transact quickly with buyers and sellers at any time, the more efficient the market is in matching buyers and sellers. The more buyers and sellers can transact to the prevalent market demand. The market maker sells which is related to the volatility, and order processing costs, e.g., costs incurred in setting up, order-driven markets are generally cheaper to buy and sell according to the prevalent market demand. The market maker sells at the “ask” (offer) price \( A \) and buys at a lower “bid” price \( B \); the difference \( S = A - B \) is the bid-ask spread.

The ability to buy at a low price and sell at a high price is the main compensation to market makers for the risk they incur while providing liquidity. Therefore, the spread must cover costs incurred by the market maker [1–7] such as: (i) order processing costs, e.g., costs incurred in setting up, fixed exchange fees, etc., (ii) risk of holding inventory, which is related to the volatility, and (iii) adverse information costs, i.e., the risk of trading with a counterparty with superior information. Since the first component is a fixed cost, the interesting dynamics of liquidity is reflected in (ii) and (iii). Analyzing the statistical features of the bid-ask spread thus also provides a way to understand information flow in the market.

The prevalent bid-ask spread reflects the underlying liquidity for a particular stock. Quantifying the fluctuations of the bid-ask spread thus offers a way of understanding the dynamics of market liquidity. In this paper, we show that the fluctuations of the average bid-ask spread over a fixed time interval display power-law distributions and long-range correlations. We further explore the relationship between the bid-ask spread and other indicators of liquidity such as the frequency of trade occurrence \( N \) and the frequency of quote updates \( U \). We find \( S \sim N \) and \( S \sim \ln U \). We find a similar logarithmic relation between the bid-ask spread and the share volume, both over a fixed time interval and on a transaction level. Lastly, we report logarithmic relationships between bid-ask spread, order flow, and two different measures of volatility.

Our analysis focuses on stocks that are listed on the NYSE. The NYSE is a hybrid market in which both the specialist and limit-order traders play a role in price formation. The hybrid market system ensures that specialists incorporate the best bid and ask in the order book while posting their quotes. The NYSE hybrid market contrasts with a purely order-driven market, such as the Tokyo Stock Exchange, where orders are submitted before prices are determined, or a “quote-driven” system such as used in NASDAQ, where different competing market makers are required to provide bid-ask quotes continuously.

In an order-driven market, orders are submitted to a centralized location (electronic or physical), where they are matched, executed, or deleted. Here, the bid price represents the largest sell limit order price and the ask price represents the smallest buy limit order price. Their difference defines the spread. Order-driven markets are generally cheaper to trade since they have smaller bid-ask spreads, in part because fixed costs such as (i) discussed above, are not present.

We analyze the trades and quotes (TAQ) database for the two-year period January 1994 to December 1995. The TAQ database, which has been published by the NYSE since 1993, covers all trades and quotes for all stocks listed at three major U.S. stock markets (NYSE, AMEX, and NASDAQ). Our analysis focuses on a subset of these stocks that are traded on the NYSE.

This paper is organized as follows. Sections II and III present our results on the distribution and time correlations of the bid-ask spread, respectively. Section IV presents our results on the relationship between the bid-ask spread, the
share volume, and the number of trades, both over fixed time
intervals and at the transaction level. Sections V and VI
describe the relationship of the bid-ask spread to the order flow
and to the frequency of quote updates. Section VII explores
the relationship between the bid-ask spread and the volatility.

II. DISTRIBUTION OF FLUCTUATIONS IN THE BID-ASK
SPREAD

From the TAQ database, we select the 116 most fre-
quently traded stocks listed in the NYSE, and form for each
stock a list of all trades and quotes. For each trade \( i \), the
database provides the trade price \( P_i \) and the trade size \( q_i \). We
use the procedure of Ref. [8] to identify for each trade \( i \), the
bid price \( B_i \) and the ask price \( A_i \).

We first compute a time series of the average spread \( S \)
over fixed time intervals \( \Delta t \), where \( s_i = A_i - B_i \) [9] and \( N = N_{\Delta t}(t) \) denotes the number of trades in \( \Delta t \) [10]. In the
following analysis, we show results for \( \Delta t = 15 \text{ min} \). We find
similar results for \( \Delta t = 30 \) and 60 min.

Denote by \( S_j(t) \) the time series of the bid-ask spread for
stock \( j \). Figure 1(a) shows that the \( S_j(t) \) for a typical stock
displays large fluctuations. We compute the cumulative dis-
tribution function \( P(S_j > x) \) for each of the \( j = 1, \ldots , 116 \)
stocks. We find that each distribution is consistent with a
power law

\[
P(S_j > x) \sim x^{-\xi_j}. \tag{2}
\]

Our estimates of the individual exponents \( \xi_j \) are similar
across all 116 stocks in our sample. Although the functional
forms of individual distributions are similar, their widths
(standard deviations) vary. We obtain a good “collapse” of
the distributions by normalizing each time series \( S_j(t) \) by
transforming it to zero mean and unit standard deviation.
Based on the hypothesis that the functional forms of the
distributions \( P(S_j > x) \) are the same for all stocks, we com-
pute the cumulative distribution function \( P(S > x) \) using the
normalized spreads for all stocks \( j = 1, \ldots , 116 \). Figure 1(b)
shows that \( P(S > x) \) decays as

\[
P(S > x) \sim x^{-\xi_S}, \tag{3}
\]

and we find the mean value \( \xi_S = 3.0 \pm 0.1 \).

We note that \( \xi_S \) is similar in value to the exponent found
for \( \xi_G \) describing the distribution \( P(G > x) \) of price change \( G \)
[11,12], and the tail exponent describing the distribution of
volatility [13]. Our analysis of the relationship between
return \( G \) and \( S \) shows an approximate power-law dependence
\( |G| \sim S^\alpha \) with \( \alpha \approx 0.7 \). For most stocks this dependence holds
only up to a threshold, after which there is a drop off. Since
the relationship is weaker than linear (\( \alpha < 1 \) followed by a
drop off), our results do not seem to support the simplistic
hypothesis that the power-law tails of \( |G| \) with \( \xi_G \approx 3 \) can be
explained by \( \xi_S = 3 \).

III. TIME CORRELATIONS IN THE BID-ASK
SPREAD

We next consider temporal correlations in the bid-ask
spread. Figure 2(a) shows the autocorrelation function
\( \langle S(t)S(t+\tau) \rangle \) for a typical stock, where \( S(t) \) is transformed to
zero mean and unit variance. We find that \( \langle S(t)S(t+\tau) \rangle \) de-
cays slowly and displays pronounced peaks at multiples of 1
day (390 min). The peaks originate from the \( U \)-shaped intrada-
y day pattern in the bid-ask spread [14], similar to the previ-
ously reported intradaily patterns in volatility [13,15–17].

To test the presence of long-range correlations, we first
remove the intraday pattern from \( S(t) \) using the procedure
outlined in Ref. [13]. To accurately quantify the long-range
 persistence of the bid-ask spread correlation function, we use
the method of detrended fluctuation analysis (DFA). We
therefore calculate the detrended fluctuation function \( F(\tau) \),
deﬁned as the root-mean-square ﬂuctuation around a poly-
nomial ﬁt to the integrated time series of \( S \) (for details see Ref.
[13]). In the analysis presented in this paper we use linear
detrending. We find that the detrended fluctuation function
\( F(\tau) \) for \( S \) scales as
The mean value of $n_s = 0.73 \pm 0.01$ for all 116 stocks. The correlation function therefore decays as $F(t) = \tau^{n_s}$, with the mean value of $n_s = 0.73 \pm 0.01$ for all 116 stocks. The autocorrelation function $\langle S(t)S(t+\tau) \rangle$ displays peaks at multiples of one day for Exxon Corp. The detrended fluctuation function $F(\tau)$ for the same stock displays long-range power-law correlations that extend over almost 3 orders of magnitude. A histogram of slopes are obtained by fitting $F(\tau) = \tau^{n_s}$ for all 116 stocks. We find a mean value of the exponent $n_s = 0.73 \pm 0.01$. The error bar denotes the standard error of the mean of the distribution of exponents, which, under i.i.d. assumptions, is estimated as the ratio of the standard deviation of the distribution to the square root of the number of points. In reality, the i.i.d. assumptions do not hold, so the error bar thus obtained is likely understated.

FIG. 2. (a) The autocorrelation function $\langle S(t)S(t+\tau) \rangle$ displays peaks at multiples of one day for Exxon Corp. (b) The detrended fluctuation function $F(\tau)$ for the same stock displays long-range power-law correlations that extend over almost 3 orders of magnitude. (c) A histogram of slopes are obtained by fitting $F(\tau) = \tau^{n_s}$ for all 116 stocks. We find a mean value of the exponent $n_s = 0.73 \pm 0.01$. The error bar denotes the standard error of the mean of the distribution of exponents, which, under i.i.d. assumptions, is estimated as the ratio of the standard deviation of the distribution to the square root of the number of points. In reality, the i.i.d. assumptions do not hold, so the error bar thus obtained is likely understated.

FIG. 3. (a) Equal-time conditional expectation $\langle S \rangle_Q$ of the spread for a given value of $Q$ averaged over all 116 stocks over a time interval $\Delta t = 15$ min. Here $S$ is normalized to have a zero mean and unit variance, and $Q = Q_\Delta(t)$ is normalized by its first centered moment. The solid line shows a logarithmic fit to the data extending over almost two orders of magnitude. (b) Conditional expectation $\langle S \rangle_N$. As before, $S$ is normalized to have a zero mean and unit variance, and we normalize $N = N_\Delta(t)$ by its standard deviation. The solid line shows a logarithmic fit to the data. Note that for both (a) and (b) the ordinate takes negative values because of our normalization of the spread to zero mean. We have tested that these relationships are robust under other normalization schemes such as scaling by the first moment.

$F(\tau) \sim \tau^{n_s}$, with the mean value of $n_s = 0.73 \pm 0.01$ for all 116 stocks [Figs. 2(b) and 2(c)]. The correlation function therefore decays as

$\langle S(t)S(t+\tau) \rangle \sim \tau^{\mu_s}$, with $\mu_s = 2 - 2 \nu_s = 0.54 \pm 0.02$.

The power-law distributions and long-range correlations that we find in $S$ are similar to those found in the volatility (measured, e.g., by $|G|$) [13,15–18]. The similarity in statistical properties of spread and volatility is qualitatively consistent with the notion that spreads reflect the market maker’s risk of holding inventory, which is, in turn, an increasing function of volatility [19].
IV. BID-ASK SPREAD, SHARE VOLUME, AND TRADING ACTIVITY

A. Fixed time interval analysis

We next examine the relationship between the bid-ask spread $S$ and the share volume $Q$ displayed on its intraday patterns and are large near the open and close of the market. Figure 3(a) shows that the increase of spread with volume is consistent, over 2 orders of magnitude, with the logarithmic relationship $S \propto \ln Q$. One may expect that the logarithmic relation that we find, particularly at the transaction level (below), reflects the distribution of the order book.

For both $Q$ and $N$, we test and confirm that the logarithmic relationships hold individually for each stock.

Recent empirical studies [17] show that the long-range correlations in volatility $\Delta t=15$ min. The solid line shows a logarithmic fit to the data. Here, $\Omega$ is normalized by its first moment after setting to zero mean. (b) The conditional expectation of the spread conditioned on the time interval between trades. The solid line shows a logarithmic fit to the data. Here $\delta t$ is normalized by its standard deviation.

FIG. 4. (a) Conditional expectation of the transaction-level spread conditioned on trade size averaged over all 116 stocks. The solid line is a logarithmic fit to the data. Here $s$ has been normalized to have a zero mean and unit variance, and $q$ is normalized by its centered first moment. (b) The expectation of the spread conditioned on the time interval between trades. The solid line shows a logarithmic fit to the data. Here $\delta t$ is normalized by its standard deviation.

FIG. 5. (a) Conditional expectation of the spread for a given value of $\Omega$ averaged for all 116 stocks over a time interval $\Delta t=15$ min. The solid line shows a logarithmic fit to the data. Here, $\Omega$ is normalized by its first moment after setting to zero mean. (b) The conditional expectation of the spread conditioned on the time interval between trades. The solid line shows a logarithmic fit to the data. Here, $\delta t$ is normalized by its standard deviation.
B. Transaction level analysis

We next analyze the relationship between spread and volume at the trade level. To test the time dependence between spreads and volume, we first analyze the correlation function $k_q$ is $i + k_l$. The correlation function has its largest value at $k = 0$; for $k$, 0, correlations are almost zero while for $k > 0$ we find correlations that decay to zero quickly. Beyond $k = 4$ trades, we find no statistically significant correlation.

We next analyze the conditional expectation $k_s l q$ of the transaction-level bid-ask spread conditioned on the trade size. We find similar to Eq. 6d, the logarithmic relationship $S$, $\ln Q$ can therefore be understood because $S = 1/N \sum_{i=1}^{N} s_i$, and $\sum_{i=1}^{N} s_i = \sum_{i=1}^{N} \ln q_i$ can be expressed to leading order in terms of $\ln(\sum_{i=1}^{N} q_i) = \ln Q$, and consequently $S \sim \ln Q$.

We next examine the relationship between $s_i$ and the time interval $\delta t$ between trades. The average intertrade time interval $\langle \delta t \rangle$ can be thought of as a reciprocal of $N_{\Delta t}$ to a first approximation. We find that as $\delta t$ increases, the bid-ask spread decreases, and the functional relationship is $\langle s \rangle$, $-\ln \delta t$, where the brackets denote an average over all transactions $i$ conditioned on $\delta t$.

V. SPREADS AND ORDER FLOW

Similar logarithmic functional forms also describe the relation between the bid-offer spread and order flow. During periods of large demand or supply, we expect $S$ to be large, since a market maker increases the spread to compensate for the additional risk.

Denote $a_i = -1$ if the trade is seller initiated and $a_i = 1$ if the trade is buyer initiated. The volume imbalance can then be defined as [24]
\[
\Omega_{\Delta t}(t) = \Omega = \sum_{i=1}^{N} q_i a_i, \tag{10}
\]
and the number imbalance
\[
\Phi = \sum_{i=1}^{N} a_i, \tag{11}
\]
which quantify, respectively, the difference between the trading volume and number [25] of buyer-initiated and seller-initiated trades in a time interval \(\Delta t\). We next compute the equal-time conditional expectations \(\langle S \rangle_{\Omega} \) [Fig. 5(a)] and \(\langle S \rangle_{\Phi} \) [Fig. 5(b)] of the spread for a given value of volume imbalance and number imbalance, respectively. We find
\[
\langle S \rangle_{\Omega} \sim \ln(|\Omega|), \tag{12}
\]
and
\[
\langle S \rangle_{\Phi} \sim \ln(|\Phi|). \tag{13}
\]

The logarithmic relationship \(\langle S \rangle_{\Omega} \sim \ln(|\Omega|)\) is not surprising, since \(|\Omega| \sim Q\) for large volumes, and we have seen above that \(S \sim \ln Q\).

**VI. FREQUENCY OF QUOTE UPDATES**

Our analysis thus far has focused on the properties of the bid-offer spread. A closely related indicator of liquidity is the quote-update frequency \(U = U_\Delta(t)\), i.e., the number of times a new bid or offer is posted in the market in a time interval \(\Delta t\). Note that the prevalent bid or offer can change either because of incoming market orders, limit-order cancellations, or by the specialist posting an improved quote over the prevalent best limit-order book bid or ask. We therefore analyze the statistics of \(U\) to understand the behavior of liquidity in terms of quote updates.

Figure 6(a) shows that the distribution \(P(U)\) decays almost exponentially, unlike the frequency of trades \(N\), which has power-law fluctuations [17]. Performing power-law fits gives a very large value of exponent, consistent with an approximately exponential behavior.

We next consider temporal correlations in the quote-update frequency. Figure 6(b) shows the autocorrelation function \(U(t)U(t+\tau)\) for a typical stock where \(U(t)\) is transformed to zero mean and unit variance. As before, we find that \(U(t)U(t+\tau)\) decays slowly and displays pronounced peaks at multiples of 1 day (390 min), similar to the intraday pattern that we find in \(S\). To accurately quantify these correlations, we use the DFA method and find that the detrended fluctuation function for \(U\) scales as
\[
F_U(\tau) \sim \tau^{\nu_U}, \tag{14}
\]
with the mean value \(\nu_U=0.78\pm0.03\) for all 116 stocks [Fig. 6(c)]. Here, we have first excluded the effects of the intraday pattern, and have performed linear detrending for computing \(F_U(\tau)\).

The correlation function \(U(t)U(t+\tau)\) correspondingly decays as
\[
\langle U(t)U(t+\tau) \rangle \sim \tau^{\mu_U}, \tag{15}
\]
with \(\mu_U=2-2\nu_U=0.44\pm0.06\). Somewhat related results are obtained in Ref. [26].

As we have found previously, spreads depend logarithmically on the number of trades. Similarly, we find a logarithmic relationship between spreads and the frequency of quote revisions. Figure 7(d) shows that the conditional expectation
\[
\langle S \rangle_U \sim \ln U. \tag{16}
\]

**VII. RELATION BETWEEN SPREADS AND VOLATILITY**

Finally, we study the dependence of the bid-ask spread on the volatility of price movements. As a short-term estimate of volatility, we consider two measures: (a) the magnitude \(|G|=|G|_{\Delta t}\) of the price changes and (b) the magnitude \(|M|=|M|_{\Delta t}\) of the midquote change over the time interval
\( \Delta t \). Figure 6(a) shows that the equal-time conditional expectation \( \langle S_{|G} \rangle \) of the spread for a given value of \( |G| \) increases logarithmically,

\[
\langle S_{|G} \rangle \sim \ln(|G|). \tag{17}
\]

We find similar behavior for the spread when conditioned on the absolute changes in the midquote price \( |M| \) [Fig. 6(b)],

\[
\langle S_{|M} \rangle \sim \ln(|M|). \tag{18}
\]

It is interesting to contrast our finding of a logarithmic dependence of the bid-ask spread on the volatility with the somewhat related results of Ref. [27], where an almost linear relationship is reported between the time-averaged spread and the time-averaged volatility for LSE stocks. While the linear relationship of Ref. [27] is a time-averaged property that holds between the mean spread and the mean volatility for a particular stock, our finding of Eq. (18) reflects more on the dynamics of the joint evolution of \( S(t) \) and \( |M(t)| \).

**VIII. DISCUSSION AND SUMMARY**

The relationships that we uncover for the bid-ask spread are interesting from the perspective of recent work [22,23,26,28–30]. Reference [26] analyzes the Island electronic communication network (ECN) order book, which is one of many electronic platforms that comprise NASDAQ. They report the long-memory behavior of the rates of order placement and cancellation, that is related to our finding of long memory in \( S \). Studying double-auction limit-order markets within a model where order arrivals and cancellations follow a Poisson process, Ref. [29] finds an exponential tail for the distribution of spread, and a weak approximately linear relationship between spreads and order size. It is possible that their inherent assumptions about thin-tailed distribution of order sizes or the i.i.d. nature of order flow gives rise to the disparity with our empirical finding of \( S \sim \ln Q \).

Reference [30] studies the evolution of spreads and volatility following large price moves in the NYSE and NASDAQ. They find that for the NYSE, both the volatility and the bid-ask spread decay as a power law following a large price move. For NASDAQ, however, they find that the bid-ask spread reacts in a much milder way than for the NYSE (a 20% increase compared to a 600% increase for the NYSE [30]). On the other hand, Ref. [23] finds that their analysis of the Island ECN order book for NASDAQ stocks seems to give qualitatively similar results to their analysis of the Paris Bourse. In light of these findings, it is interesting to see if the relationships that we uncover for the spread and its relation to volume in the NYSE hold for quote-driven markets as well.

In sum, we have analyzed the statistical properties of the bid-ask spread for the most frequently traded 116 stocks in the NYSE. We have found that the bid-ask spread \( S \) over a fixed time interval \( \Delta t \) displays power-law distributions and long-range temporal correlations. Our finding that \( \langle S \rangle \sim \ln N \) suggests that the long-range correlations in \( S \) arises from those of \( N \). We have explored the relationship between the bid-ask spread and the transaction volume and find a logarithmic relationship both over fixed time intervals and at the trade-by-trade level. Lastly, we have found logarithmic relationships between spreads, order flow, and volatility. Our results add to the existing literature on the relationships between spreads and volatility, and uncover interesting logarithmic relationships that may offer a guide to modeling the microstructural dynamics of spreads, returns, volume, and volatility.

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presented in Ref. [3]. Consider the price $P(t) = P^* + \epsilon(t) S(t)/2$, where $P^*$ denotes a “fundamental” value around which prices move and $\epsilon(t)$ denotes $i.i.d.$ variables $\epsilon \in \{-1,1\}$, which implies that the variance of price changes is $\text{Var}\Delta P(t) = s^2/2$, making volatility linearly dependent on $S$, where $\Delta P(t) = P(t+1) - P(t)$.


[21] Fitting a power-law function gives generally worse quality fits. The exponents that we obtain thus are quite small $\sim 0.1-0.2$, which is consistent with a logarithmic relation. We find similar results for other logarithmic relationships reported herein.


[25] We took the difference between the number of buyer-initiated and seller-initiated trades into consideration because price changes, when conditioned on the number of trades, do not show a significant dependence on volume per trade [C. Jones, G. Kaul, and M. Lipson, Rev. Financ. Stud. 7, 631 (1994)].


[27] G. Zumbach, eprint cond-mat/0407769, Quantitative Finance (to be published).

