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Stochastic processes with power-law stability and a crossover in power-law correlations

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Abstract

Motivated by the goal of finding a more accurate description of the empirically observed dynamics of financial fluctuations, we propose a stochastic process that yields three statistical properties: (i) short-range autocorrelations in the index changes, (ii) long-range correlations in the absolute values of the index changes, with a crossover between two power-law regimes at approximately one week, and (iii) power-law stability in the tails of the probability distributions of the index changes. We find that this stochastic process can surprisingly well reproduce statistical properties observed in the high-frequency data of the S&P 500 stock index.
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The recent activity at the interface of statistical physics and economics [1–8] is partly due to the finding that financial data exhibit power-law spatial and temporal scaling behaviors, which are commonly encountered in many different natural phenomena [9]. One common features of those systems is that the power-law spatial or temporal scaling behavior extends over several orders of magnitude. Here, we investigate the possibility that power-law scaling in distributions and correlations may have the same dynamical origin. To exemplify that hypothesis, we study an extensively studied financial time series, the S&P 500 stock index s_t, [1] which has been found to possess the following

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[1] Here Δt denotes the sampling time interval, and we set $\Delta t = 10$ min throughout this paper. We use the S&P 500 data, sampled at 10-min intervals, covering the period 1 January 1984 through 31 December 1995. The length of a trading day is roughly 400 min, and the length of a trading week is 5 days, which corresponds to roughly 2000 min.

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three intriguing statistical features:

(i) The price *changes* defined as $\tilde{r}_t \equiv \log s_{t+\Delta t} - \log s_t$ [1] are short-range correlated [10,11].
(ii) $|\tilde{r}_t|$ are long-range correlated [10,11], and the correlations in $|\tilde{r}_t|$ can be approximated by two piece-wise power laws [11].
(iii) The tails of the probability distributions of the price changes exhibit a stable power-law functional form over a wide range of time scales, called *power-law stability* [12].

In order to develop some understanding of the dynamical origin and of the interrelation of these three statistical features, we propose a stochastic process r_t that is capable of reproducing—qualitatively and quantitatively—the statistical features (i)–(iii) observed in the empirical data $\tilde{r}_t$. Specifically, we define r_t by the following set of coupled equations:

$$r_t = c r_{t-\Delta t} + x_t \ , \tag{1}$$

$$x_t = v_t e_t \ , \tag{2}$$

$$v_t = \sum_{n=1}^{\infty} a_n |x_{t-n\Delta t}| \ . \tag{3}$$

Here, e_t denotes an independent and identically distributed (i.i.d.) random variable with truncated Lévy [13] probability distribution $P(e_t)$,[2] and the *weights* a_n [3] are defined by

$$a_n \sim \begin{cases} n^{-1-\delta_1} & [n < n_\times] \ , \\ n^{-1-\delta_2} & [n \geqslant n_\times] \ , \end{cases} \tag{4}$$

where c, δ_j, and $n_\times$ are four free parameters.[4] The parameter c, which models the short-range correlations, as well as the scaling parameters δ_j and the crossover parameter $n_\times$ can be easily obtained from the data. Note that the values of x_t are not correlated with each other and independent of v_t because e_t are i.i.d. random variables. In contrast, the absolute values of x_t are correlated with each other through the choice of v_t.

The long-range correlations in $|x_t|$ are accomplished through Eqs. (2) and (3), and the specific functional form of the correlations depends on the choice of the weights a_n. If the weights are chosen to decay as a geometric series in n, then the correlations

[2] We find that the choice of the probability distribution $P(e_t)$ is *not* relevant for the correlation analysis. We find that once the parameters responsible for the correlations in r_t and $|r_t|$ are fixed, the probability distribution for the data can be better approximated by r_t if $P(e_t)$ is chosen to be a truncated Lévy distribution [13] rather than a Gaussian distribution.

[3] Precisely, the weights a_n are defined as $D_j \delta_j \Gamma(n-\delta_j)/(\Gamma(1-\delta_j)n!)$, where the two constants D_j are set to meet normalization and continuity in the weights a_n. Due to the asymptotic behavior of the Gamma function Γ, the weights a_n can be approximated by $n^{-1-\delta_j}$.

[4] Our choice of a_n is inspired by Ref. [10], which can be understood as a special case of x_t for $\delta_1 = \delta_2$ and $D_1 = D_2 = 1$.

in $|x_t|$ versus time scale τ decay exponentially in τ [14]. If the weights are chosen to decay as a power-law series in n with a single exponent for the whole range of n, then the correlations in $|x_t|$ decay with a power-law in τ with a unique scaling exponent [10]. (That case refers to Eq. (4) when, e.g., $\delta_1 = \delta_2$ or $n_\times = 1$ or $n_\times \to \infty$ [10].) In order to obtain a *crossover* in the power-law correlations of $|x_t|$, we choose the weights to decay by two piece-wise power laws $n^{-1-\delta_j}$ in the two regimes $n < n_\times$ and $n \geqslant n_\times$, with a crossover at $n_\times$.

We find that Eq. (1) generates short-range correlations in r_t [result (i)]. We will show that Eqs. (2)–(4) generate the piece-wise power-law correlations in $|r_t|$ [result (ii)], i.e., we will show that the piece-wise power-law form of the weights a_n as a function of n is responsible for creating the piece-wise power-law correlations in $|x_t|$ and $|r_t|$. We will demonstrate that Eqs. (2)–(4) are capable of generating power-law stability in the distribution of the sum of m consecutive stochastic variables r_t [result (iii)]. We will also show that with the choice of $c = 0.23$, $\delta_1 = 0.21$, $\delta_2 = 0.43$, and $n_\times = 100$, [5] the stochastic process r_t can reproduce—qualitatively and quantitatively—the statistical features of results (i)–(iii) observed in the empirical data $\tilde{r}_t$. [1]

In studying the effect of the choice of parameter values on the correlation properties of $|x_t|$ and $|r_t|$, we will focus on the three parameters δ_1, δ_2, and $n_\times$, because the parameter c affects only the short-range correlation properties of r_t. As we cannot derive closed-form expressions for the autocorrelation functions of $|x_t|$ or $|r_t|$ (except for $|x_t|$ when $\delta_1 = \delta_2$), [6] we study the temporal correlations in $|x_t|$ and $|r_t|$ by numerical simulations and compare them to the correlations observed in the empirical data $|\tilde{r}_t|$.

In order to reliably compute correlations from the empirical and simulated time series, we employ a method called *detrended fluctuation analysis* (DFA) [15], which yields results in cases where traditional correlation analyses fail due to the presence of nonstationarities [11]. The idea of the DFA is to construct a random walk

$$z_t(\tau) \equiv \sum_{n=1}^{m} y_{t+n\Delta t} \tag{5}$$

based on the studied time series y_t (where y_t may stand for $|\tilde{r}_t|$, $|r_t|$, or $|x_t|$, with $\tau \equiv m\Delta t$, where m is a positive integer) and to compute the mean standard deviation $F(\tau)$ of the *detrended* fluctuations of $z_t(\tau)$ as a function of τ.

Fig. 1 shows plots of the mean standard deviation $F(\tau)$ of the detrended fluctuations of $|\tilde{r}_t|$, $|r_t|$, and $|x_t|$ versus τ. We find that for the empirical data $F(\tau)$ is composed of two power-law regimes [11],

$$F_j(\tau) \sim \tau^{\beta_j}\,, \tag{6}$$

[5] In order to determine the values of the parameters D_1 and D_2, we compute numerically $S_1 \equiv \sum_{n=1}^{n_\times} a_n$ and $S_2 \equiv \sum_{n=n_\times+1}^{\infty} a_n$. As a_n must be continuous at $n_\times$ and normalized to 1, we calculate D_1 and D_2 from the following two equations: $D_1 a_{n=n_\times} = D_2 a_{n=n_\times+1}$ and $S_1 D_1 + S_2 D_2 = 1$.

[6] For the special case of $\delta \equiv \delta_1 = \delta_2 < \frac{1}{2}$ the autocorrelation function of $|x_t|$, $C(\tau) \equiv (\langle |x_t||x_{t+\tau}|\rangle - \langle |x_t|\rangle^2)/(\langle |x_t|^2\rangle - \langle |x_t|\rangle^2)$, converges for asymptotically large τ to the power-law form $C(\tau) \sim \tau^{-1+2\delta}$ [10]. Ref. [15] shows that a power-law functional form of $C(\tau) \sim \tau^{-\gamma}$ corresponds to a power-law functional form of $F(\tau) \sim \tau^{\beta}$, where the exponents γ and β are related by $\beta = 1 - \gamma/2$ [15]. Hence, we obtain that δ and β are related by $\delta = \beta - 0.5$.

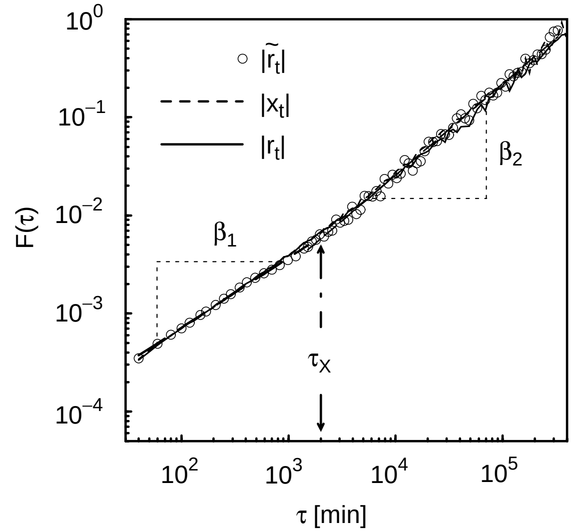

Fig. 1. Log–log plot of the mean standard deviation $F(\tau)$ of the detrended fluctuations of $|\tilde{r}_t|$ for the high-frequency data of the S&P 500 stock index (○). We find that for $\tau < \tau_\times$, $F(\tau)$ follows a power law, $F(\tau) \sim \tau^{\beta_1}$, with exponent $\beta_1 = 0.71$, while for $\tau > \tau_\times$, $F(\tau)$ also follows a power law, $F(\tau) \sim \tau^{\beta_2}$, with a different exponent $\beta_2 = 0.93$. We find that the crossover time scale $\tau_\times$ is approximately 2×10^3 min ≈ 1 week.[1] We also find that the stochastic processes $|x_t|$ and $|r_t|$, with $\delta_1 = 0.21$, $\delta_2 = 0.43$ [18], and $n_\times = 100$ yield a function $F(\tau)$ that is almost identical to the function $F(\tau)$ obtained for the empirical data $|\tilde{r}_t|$.

with a crossover time $\tau_\times \approx 2 \times 10^3$ min and exponents $\beta_1 = 0.71$ for $\tau < \tau_\times$ and $\beta_2 = 0.93$ for $\tau \geqslant \tau_\times$.[7] Fig. 1 also shows that the stochastic processes x_t and r_t possess almost identical correlations in $|x_t|$ and $|r_t|$ and, more importantly, Fig. 1 shows that both stochastic processes x_t and r_t can reproduce the piece-wise power-law long-range correlations $F(\tau)$ observed in the empirical data.[8]

We next investigate how the correlation behavior of $|r_t|$ depends on the choice of $n_\times$. Fig. 2 shows $F(\tau)$ of $|r_t|$ for $n_\times = 1$, $n_\times = 100$, and $n_\times \to \infty$. We find that the empirical crossover time $\tau_\times$ is proportional to $n_\times$ and can be approximated by $\tau_\times \approx 2 n_\times \Delta t$.[1] Hence, we choose $n_\times = 100$ in the simulations leading to Fig. 1, which reproduces the empirically observed crossover at $\tau_\times \approx 2 \times 10^3$ min. In the limit $n_\times \to \infty$ the crossover vanishes, and we recover the exponent $\beta_1 = 0.71$, which is in agreement with analytic results that can be derived for the stochastic process x_t [10] with unique exponent $\delta_1 = 0.21$.[6] Likewise, for $n_\times = 1$ the crossover vanishes, and we recover the exponent $\beta_2 = 0.93$, which is again in agreement with analytic results for the stochastic process x_t [10] with unique exponent $\delta_2 = 0.43$.[6]

[7] Here, we analyze the distributions and correlations for the *same* time series of empirical data. The DFA exponent β_1 and the crossover time scale $\tau_\times$ are slightly larger than those found in Ref. [11], where the "normalized" time series of $|\tilde{r}_t|$ is analyzed, from which the "intraday pattern" is excluded. In contrast, the authors of Ref. [12] investigate the distributions for the *same* time series as in this paper.

[8] Note that the variance growth $F(\tau)$ for both the empirical data of the S&P 500 stock index $|\tilde{r}_t|$ and the model time series $|\tilde{r}_t|$ can be equally well approximated by a stretched exponential. We focus on the approximation of the variance growth $F(\tau)$ by two piece-wise power laws, because this is the approximation proposed in the original publication [11].

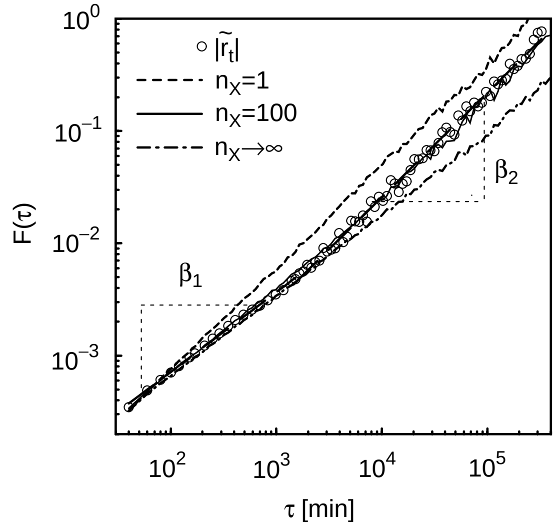

Fig. 2. Log–log plot of $F(\tau)$ of $|\tilde{r}_t|$ (○) and $|r_t|$ (lines) for different choices of $n_\times$. Generally, we find that the observed crossover time $\tau_\times$ increases monotonically with $n_\times$ and is typically larger than $n_\times \Delta t$. We also show $F(\tau)$ for the two limit cases $n_\times = 1$ and $n_\times \to \infty$, and we find—in agreement with theoretical predictions for these two limiting cases—that $\delta_2 = 0.43$ yields $\beta_2 = 0.93$ in case of $n_\times = 1$, and that $\delta_1 = 0.21$ yields $\beta_1 = 0.71$ in case of $n_\times \to \infty$.

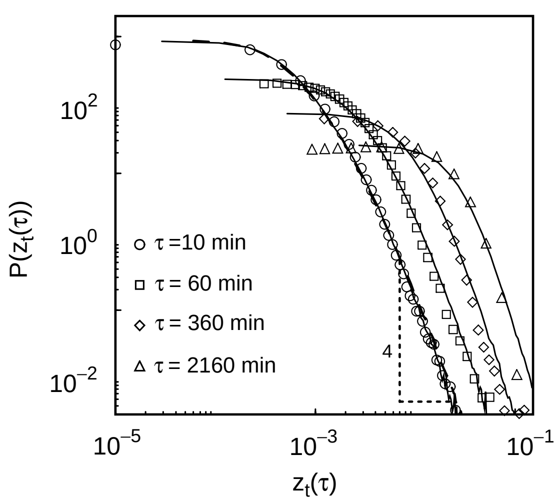

Fig. 3. For small time scales $\tau \equiv m\Delta t$,[1] the power-law tails of the probability distribution $P(\tilde{z}_t(\tau))$ of the *integrated* index changes, $\tilde{z}_t(\tau) \equiv \sum_{n=1}^{m} \tilde{r}_{t+n\Delta t}$, remain practically stable with exponent $\zeta = 4$. We find that the stochastic process r_t of Eqs. (1)–(3) can reproduce—qualitatively and quantitatively—the probability distribution $P(\tilde{z}_t(\tau))$, in addition to reproducing the correlation behavior of $\tilde{r}_t$ and $|\tilde{r}_t|$. Moreover, we find that $P(z_t(\tau))$ exhibits power-law stability in the tails even for time scales longer than those that can currently be measured from empirical data.

We study the probability distribution $P(r_t)$ and compare it to the probability distribution $P(\tilde{r}_t)$ of the empirical data $\tilde{r}_t$ (Fig. 3). In order to perform this comparison over a wide range of time scales, we define by $z_t(\tau) \equiv \sum_{n=1}^{m} r_{t+n\Delta t}$ the random walk of r_t, starting at time t and ending at time $t + \tau$, with $\tau \equiv m\Delta t$. Consistent with recent work

[12], we find that for the empirical data the central part of $P(\tilde{z}_t(\tau))$ exhibits—over a wide range of time scales τ—a scaling behavior similar to that of the Lévy distribution [16], and that the tails can be well approximated by a power law,

$$P(\tilde{z}_t(\tau)) \sim \tilde{z}_t^{-(1+\zeta)} \tag{7}$$

with exponent $1+\zeta=4$, which is well beyond the range $(0<\zeta<2)$ expected for the Lévy distribution.

We find that—caused by Eqs. (2) and (3)—the tails of the probability distributions $P(z_t(\tau))$ can be approximated by power laws, independently of the choice of $P(e_t)$ in Eq. (2).[9] Moreover, we find that the tails of $P(z_t(\tau))$ exhibit power-law stability even *beyond* the time range that is currently measurable from experimental data. This finding is intriguing as it indicates the possibility that also in the empirical data of the S&P 500 stock index the observed power-law stability of the probability distributions might extend beyond the currently observable time scale.

In summary, we proposed a stochastic process r_t that is capable of reproducing three statistical features observed in the fluctuations $\tilde{r}_t$ of the S&P 500 stock index, namely (i) short-range correlations in $\tilde{r}_t$, (ii) two regimes of long-range power-law correlations in $|\tilde{r}_t|$ with a crossover at $\tau_\times \approx 2\times 10^3$ min, and (iii) power-law stability of the tails of the probability distribution $P(\tilde{z}_t(\tau)) \sim \tilde{z}_t^{-(1+\zeta)}$, which is seemingly independent of τ, and which holds even for time scales τ that go beyond the empirically measurable range of all currently available data. As power-law correlations with crossover have been found in many diverse systems or processes, such as in DNA sequences [15], time series of heart beat fluctuations [17], or small-world networks [18], we studied—in a wider sense—to which extent the presence of correlations in a physical variable r_t may contribute to the form of its probability distribution $P(r_t)$, and which class of stochastic processes could possibly be responsible for the emergence of stable power-law tails in $P(r_t)$ [6,7,19–21].

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