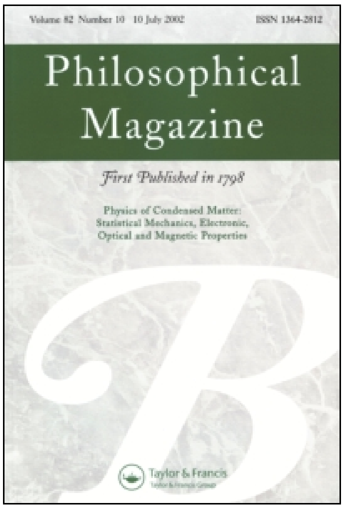

Philosophical Magazine Part B
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713836176

Limit theorems and price changes in financial markets

Rosario N. Mantegna [a]; H. Eugene Stanley [b]
[a] Istituto Nazionale per la Fisica della Materia, Unità di Palermo, and Dipartimento di Energetica ed Applicazioni di Fisica, Università di Palermo, Palermo, Italy [b] Center for Polymer Studies, and Department of Physics, Boston University, Boston, Massachusetts, USA

Online Publication Date: 01 May 1998

To cite this Article Mantegna, Rosario N. and Stanley, H. Eugene(1998)'Limit theorems and price changes in financial markets',Philosophical Magazine Part B,77:5,1353 — 1356

To link to this Article: DOI: 10.1080/13642819808205028

URL: http://dx.doi.org/10.1080/13642819808205028

PHILOSOPHICAL MAGAZINE B, 1998, VOL. 77, NO. 5, 1353–1356

Limit theorems and price changes in financial markets

By ROSARIO N. MANTEGNA

Istituto Nazionale per la Fisica della Materia, Unità di Palermo, and
Dipartimento di Energetica ed Applicazioni di Fisica, Università di Palermo,
Palermo I-90128, Italy

and H. EUGENE STANLEY

Center for Polymer Studies, and Department of Physics, Boston University,
Boston, Massachusetts 02215, USA

ABSTRACT

We discuss the relation between limit theorems in probability theory and price change statistics in financial markets. An analysis of the published empirical results and theoretical models show that the problem of the statistical properties of price (or index) changes is still open. By using the limit theorems of probability theory and the current assumption that stock prices are well described by martingales, we point out that the probability density function (PDF) of price changes is expected to belong to the class of infinitely divisible PDFs.

§1. INTRODUCTION

Financial markets are good examples of 'complex systems'. They can be seen as composed of several subunits interacting in a nonlinear way under almost constant rule. In financial markets, different goods are traded such as stocks, currencies, bonds and derivatives. In this paper we shall consider the price dynamics of a stock traded in a financial market. The most accepted paradigm in finance is that no arbitrage is present in financial markets, that is there is no way to extract money from the market continuously and without risk (Ingersoll 1987). Of course an efficient market (i.e. a financial market where there is no way of extracting money from the market in a continuous way without risk) is an idealized system never encountered in reality. However, a similar conceptual limitation should not worry physicists who are well acquainted with working with idealized systems. To cite a few examples let us consider reversible transformations in thermodynamics, motion without friction, infinite systems at a critical state and so on.

§2. MARTINGALE REPRESENTATION OF A STOCK PRICE IN A FINANCIAL MARKET

Pricing in a financial market in the absence of arbitrage implies that the time series of price is not redundant in information, that is it is a time series of maximal entropy. Maximal-entropy time series cannot be compressed into shorter sequences and are indistinguishable from a random process.

In mathematical finance (Samuelson 1965, 1972), the expected value of the discounted price $P(t_k)$ of an asset at time t_k is assumed to be equal to the last known price (observed at time t_{k-1}) provided that the set of information consisting of the past price changes $\Delta P(t_i)$. In formal terms,

$$E[P(t_k)|P(t_0), \Delta P(t_1), \Delta P(t_2), \ldots, \Delta P(t_{k-1})] = P(t_{k-1}). \tag{1}$$

The rigorous mathematical definition of this stochastic process is the definition of a martingale (Doob 1953). A random walk with zero drift is a special case of a martingale. The converse is not true. Roughly speaking, in a random walk, increments are independent and identically distributed while, in a martingale, increments are independent of previous information but not necessarily identically distributed.

§3. Results of empirical analyses

The concept of a martingale belongs to the theoretical modelling of stock price dynamics. A large amount of empirical analyses have been performed since the 1950s. What are empirical analyses of market data saying?

The most widely accepted conclusion is that the autocorrelation function of price changes is a fast-decaying function in a stock market, with a correlation time of the order of minutes (Lo 1991). However, on very long time scales, correlation between the logarithm of price indices and long-term interest rates are seen on a time scale of several decades (Shiller 1989). In a time interval starting at a few minutes and ending at several decades, the stock price dynamics seem to be well described in terms of a stochastic process with a short time memory.

Concerning the statistical properties of the variations of stock price or of variations in the logarithm of stock price (this last case is more commonly encountered in the economic literature), several different conclusions and models have been proposed. The widely used 'zero-order' approximation in mathematical finance is that the logarithm of a stock price in a financial market is a diffusive process with Gaussian increments chosen from a normal distribution (Cootner 1964, Black and Scholes 1973). However, it is widely recognized that such a log-normal description fails in fully modelling the results of empirical data. An alternative proposal based on a Lévy process was originally proposed by Mandelbrot (1963). Lévy statistics implies non-Gaussian scaling properties of the wings of the distribution and self-similarity in the time evolution of the probability density function (PDF) of logarithmic increments. Other proposals concern mixtures of Gaussian PDFs (Clark 1973), autoregressive conditional heteroskedasticity (ARCH) processes (Engle 1982) and several generalization of the ARCH processes (called GARCH) (Bollerslev *et al.* 1992). Additional proposals also use gamma, student, hyperbolic and stretched-exponential PDFs.

We also performed empirical analyses of market data. A non-Gaussian scaling and its breakdown have been observed in our empirical analysis (Mantegna and Stanley 1995, 1996, 1997). We model the results of our analysis of the Standard & Poor's 500 high-frequency data (Mantegna and Stanley 1995) in terms of a truncated Lévy flight (TLF). A TLF is a finite variance quasistable stochastic process (Mantegna and Stanley 1994). This stochastic process shows a Lévy stable PDF for short-time horizons and a Gaussian PDF for very long-time horizons. The transition from the 'Lévy' regime to the Gaussian regime can be extremely slow depending on the parameters of the TLF process. The TLF model is able to describe several of the features observed in the empirical analyses of the Standard & Poor's 500 (Mantegna and Stanley 1996, 1997) and of the (deutschmark–US dollar) DM-USD exchange rate (Arneodo *et al.* 1996). However, the TLF fails to describe the fluctuations of the scale factor of the PDF observed in empirical data (see below for the definition of scale factor of a Lévy distribution).

The fact that there is no single description accepted by the majority of scholars interested in this problem manifests itself in the fact that the precise stochastic nature of the stock price (or index) dynamics is still an unsolved problem. What can we say about it? Can we reach some conclusion by using the limit theorems of probability theory?

§4. Limit theorems and price changes statistics

The most widely known limit theorem is the central limit theorem (for example Feller (1971)) which states that the sum of n independent identically distributed stochastic variables with finite variance is characterized by a Gaussian PDF when n tends to infinity (indeed in almost all cases the convergence in probability to a Gaussian process is reached already for rather low values of n). A generalized limit theorem also exists (for example Gnedenko and Kolmogorov (1954)) and is related to the definition of stable stochastic processes. During the 1930s, Lévy and Kolmogorov showed that the sum of n independent identically distributed stochastic variables converges to a family of PDFs called the Lévy stable distribution (Samorodnitsky and Taqqu 1994) (those proposed by Mandelbrot to describe the price changes of speculative prices). The stable symmetrical PDFs are characterized by an index $0 < \alpha \leqslant 2$ and a scale factor $\gamma \geqslant 0$. When $\alpha < 2$ the stable distributions are characterized by infinite moments of order $\mu \geqslant \alpha$. The stable PDF with $\alpha = 2$ is the Gaussian PDF which is the only one with a finite second (and higher) moment.

At first sight, one can conclude that the result of empirical analyses show that index increments are not identically distributed. Hence limit theorems should provide an approximate description of empirical data only. The above statement is correct; however, a more general limit theorem exists which is valid for a sum of independent but not necessarily identically distributed stochastic variables. This theorem is due to Khintchine (Gnedenko and Kolmogorov 1954). The theorem states that the sum of n independent random variables converges in probability to a PDF belonging to the class of infinitely divisible PDFs (for example Feller (1971)). The theorem relies on rather weak constraints, namely that the independent random variables x summands of the random variable y ($y = x_1 + x_2 + \cdots + x_n$) must be infinitesimal that is there is not a single stochastic variable x_i dominating the sum y. An infinitely divisible PDF is defined by a characteristic function $\varphi(q)$ obeying the functional equation

$$\varphi(q) = [\varphi_n(q)]^n \tag{2}$$

for all n and by the additional properties $\varphi_n(0) = 1$ and $\varphi_n(q)$ a continuous function.

The class of infinitely divisible stochastic process is a broad class. Examples are Gaussian, Poisson and stable processes. However, this class selects a subsets of all the possible stochastic processes. An example of stochastic processes with PDFs outside the class of infinitely divisible PDFs is a stochastic process characterized by a stretched-exponential PDF.

§5. Discussion

In conclusion, we recall that under the assumptions of the absence of arbitrage opportunities in a financial market and of martingale representation of stock price (or index), the Khintchine limit theorem allows us to conclude that the PDF of price changes in financial markets must be expected to be infinitely divisible.

We have tried to give a concrete example of how and why physicists may consider financial systems to be very interesting.

It is true that, at the moment, it may seem to be an unusual challenge for a physicist to investigate *economic* systems using tools and paradigms developed to describe *physical* phenomena. Physical phenomena and economic systems are, of course, rather different. In physical phenomena, one often may apply conservation laws and/or find equilibrium states characterized by the maximization of some (extremely relevant) extensive function, for example entropy. In financial systems, nothing similar has been discovered yet. This makes extremely challenging the modelization of financial '*complex systems*'.

On the other hand, physicists are today increasingly involved in research projects devoted to obtain theoretical, numerical and experimental descriptions of many-body non-equilibrium disordered (in the considered space and/or in time) systems (included non-ergodic systems). For these scholars an interdisciplinary approach to financial problems might provide a set of new problems connoted by either fundamental or applied aspects.

Acknowledgements

We thank Istituto Nazionale per la Fisica della Materia, Ministero dell'Università e della Ricerca Scientifica e Tecnologica and National Science Foundation for financial support.

References

Arneodo, A., *et al.* 1996, preprint, COND-MAT 9607120.
Black, F., and Scholes, M., 1973, *J. Political Econ.* **81**, 637.
Bollerslev, T., Chou, R. Y., and Kroner, K. F., 1992, *J. Econometrics,* **52**, 5.
Clark, P. K., 1973, *Econometrica*, **41**, 135.
Cootner, P. H., (editor) 1964, *The Random Character of Stock Market Prices* (Cambridge, Massachusetts: MIT Press).
Doob, J. L., 1953, *Stochastic Processes* (New York: Wiley).
Engle, R. F., 1982, *Econometrica,* **50**, 987.
Feller, W., 1971, *Introduction to Probability Theory and Its Applications* (New York: Wiley).
Gnedenko, B. V., and Kolmogorov, A., 1954, *Limit Distribution for Sums of Independent Random Variables* (Cambridge, Massachusetts: Addison-Wesley).
Ingersoll, J. E., 1983, *Theory of Financial Decision Making* Savage, Maryland: (Rowman & Littlefield).
Lo, A. W., 1991, *Econometrica,* **59**, 1279.
Mandelbrot, B. B., 1963, *J. Business,* **36**, 394.
Mantegna, R. N., and Stanley, H. E., 1994, *Phys. Rev. Lett.*, **73**, 2946; 1995, *Nature*, **376**, 46; 1996, *ibid.*, **383**, 587; 1997 *Physica* A, **239**, 255.
Samorodnitsky, G., and taqqu, M. S., 1994, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance* (London: Chapman & Hall).
Samuelson, P. A., 1965, *Ind. Management Rev.,* **6**, 41; 1972, *Collected Scientific Papers*, Vol. III (Cambridge, Massachusetts: MIT Press), chapter 198.
Shiller, R. J., 1989, *Market Volatility* (Cambridge, Massachusetts: MIT Press).