5

Ultra-Slow Convergence to a Gaussian: The Truncated Lévy Flight

Rosario N. Mantegna[1] and H. Eugene Stanley

Center for Polymer Studies and Department of Physics
Boston University, 590 Commonwealth Av., Boston, MA 02215, USA

Abstract. We introduce a class of *quasi*-stable stochastic process, the *truncated* Lévy Flight (TLF). A TLF is a stochastic process with finite variance. We show theoretically and numerically that the convergence of the sum of n independent TLF to a Gaussian process is usually extremely slow. In fact a remarkably large value of n can be required to ensure the convergence to a Gaussian process. We also investigate the statistical properties of the S&P 500 (a financial index) and we show that they are qualitatively in agreement with the one of a TLF.

Lévy flights, i.e. stochastic processes with jumps being Lévy stable non-Gaussian stochastic variables [1,2,3], have infinite variance. An infinite variance is not expected in physical systems. In spite of this Lévy, or power-law, density functions have been observed in physical [4,5] and biological systems [6,7]. The observation of Lévy-like density functions in physical or biological systems is then paradoxical.

One resolution of the paradox of infinite variance is provided by a stochastic process called a Lévy walk [8]. A Lévy walk is a random walk performed by visiting the same sites of a Lévy flight. However in a Lévy walk instantaneous jumps, which are responsible for the infinite variance, are not allowed and a time cost is introduced so that long steps are penalized. In other words, a spatio-temporal coupling memory is present in a Lévy walk. It is worth to point out that the space-time coupling memory is mathematically essential to avoid the divergence of the moments in the Lévy walk, divergence which is present in the underlying Lévy flight.

In this paper we present an alternative resolution of the paradox of infinite variance in physical systems which is also valid in the absence of a spatio-temporal coupling. We introduce a *quasi*-stable stochastic process with a *finite* variance. We define the the *Truncated Lévy Flight* (TLF), a stochastic process $\{x\}$ characterized by the probability density

$$T(x) \equiv \begin{cases} 0 & x > \ell \\ c_1 L(x) & -\ell \le x \le \ell \\ 0 & x < -\ell \end{cases} , \qquad (1)$$

[1] Present address: Dipartimento di Energetica ed Applicazioni di Fisica, Viale delle Scienze, I-90128 Palermo, Italia

where c_1 is a normalizing constant, ℓ is the cutoff length and

$$L(x) \equiv \frac{1}{\pi} \int_0^{+\infty} \exp(-\gamma q^\alpha) \cos(qx) dq \tag{2}$$

is the symmetrical Lévy stable distribution of index α $(0 < \alpha \leq 2)$ and scale factor γ $(\gamma > 0)$.

We study the dynamics of a discrete random walk $\{z\}$ in which the successive jumps are independent stochastic processes $\{x\}$. We show that by analyzing the stochastic process $\{z\}$ a Lévy regime can be observed for a huge number of steps n. The crossover between the Lévy regime and the Gaussian regime of the stochastic process

$$z_n \equiv \sum_{i=1}^{n} x_i \tag{3}$$

is a function of the cut-off length ℓ. In Fig. 1a, we plot the probability density function (PDF) of a TLF characterized by $\alpha = 1.5$, $\gamma = 1$ and $\ell = 20$. In Fig. 1a the difference between the TLF and the Lévy stable distribution of the same index cannot be noticed. However the truncation of the PDF becames evident when we plot the density function in a semilogarithmic plot (Fig. 1b). In a TLF the "rare events" $(|x| > \ell)$ are forbidden.

We study the stochastic process z_n as a function of n. In Eq. (3) x_i is a TLF and

$$< x_i \; x_j >= k \; \delta_{ij} \tag{4}$$

Since z_n is by definition a sum of n independent stochastic variables with finite variance, the central limit theorem implies that for $n \approx \infty$, z_n is a Gaussian stochastic process. Hence two distinct regimes are expected for the stochastic process z_n: (i)A regime where the stochastic process is *quasi*-stable and the PDF is

$$P(z_n) \approx L_\alpha(z) \qquad n \approx 1 \tag{5}$$

and (ii) a Gaussian regime observed for high values of n,

$$P(z_n) \approx G(z) \qquad n \gg 1 \tag{6}$$

The key question is: how fast is the convergence to the Gaussian? Or, in other words, for how long a *quasi*-stable stochastic process is observed?

We answer this question by studying the probability of return to the origin $P(z_n = 0)$ of the stochastic process z_n as a function of n. Our choice is motivated by two observations: (i) the maximal distance between $P(z_n)$ and the Gaussian distribution with the same variance σ_n is detected at $z_n = 0$ for any n. (ii) an analytical relation between $P(z_n)$ and n is known for Lévy stable processes when $z_n = 0$.

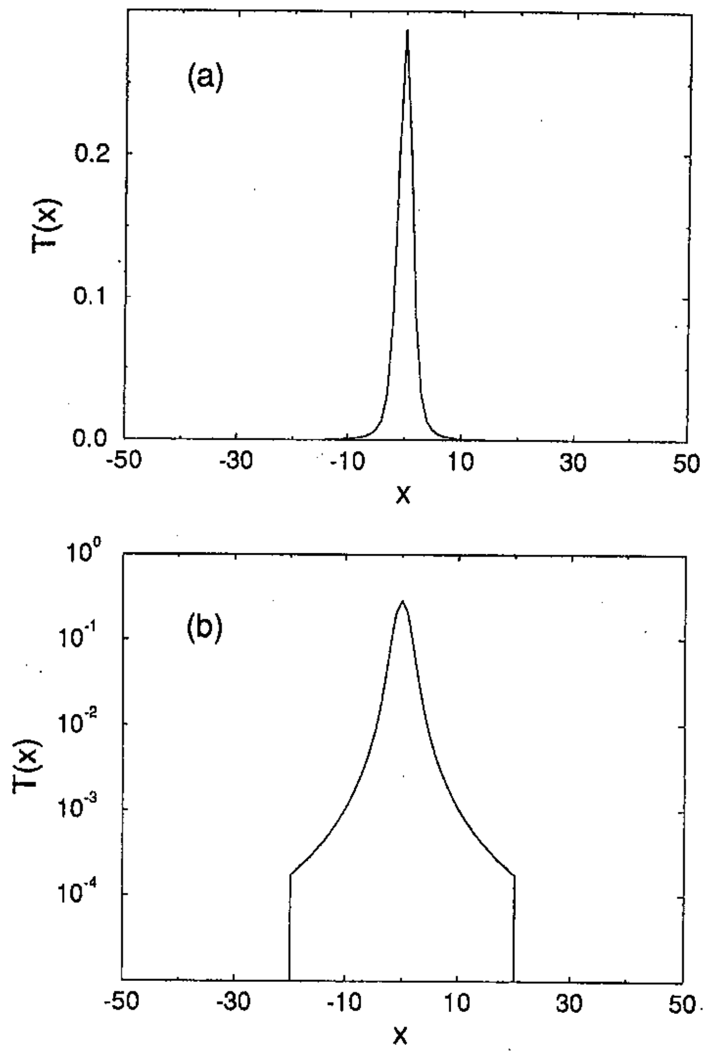

Figure 1: (a) Probability density function of a TLF characterized by $\alpha = 1.5$, $\gamma = 1.0$ and cut-off length $\ell = 20$; (b) the same PDF as in (a) in a semilogarithmic plot. The truncation of jumps longer than ℓ is evident in the semilogarithmic plot.

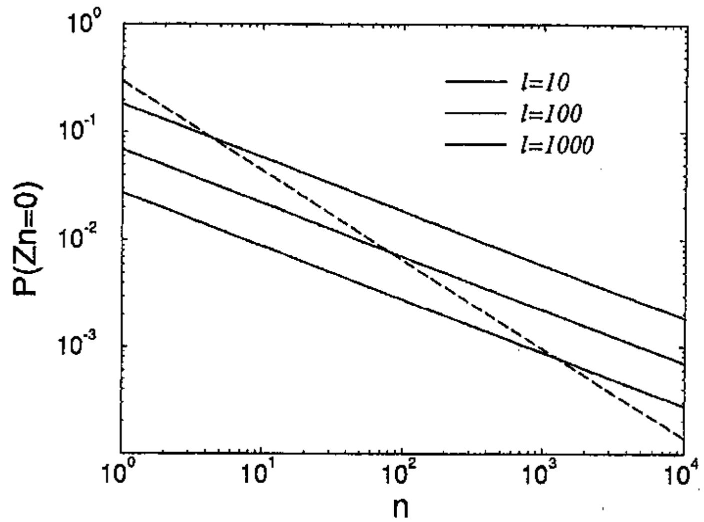

Figure 2: Logarithm of the probability of return to the origin for a Lévy flight of index $\alpha = 1.2$ (dashed line) and for three different Gaussian processes having the same standard deviation as a TLF of the same index and cut-off length $\ell = 10, 100$ and 1000 (solid lines from top to the bottom respectively).

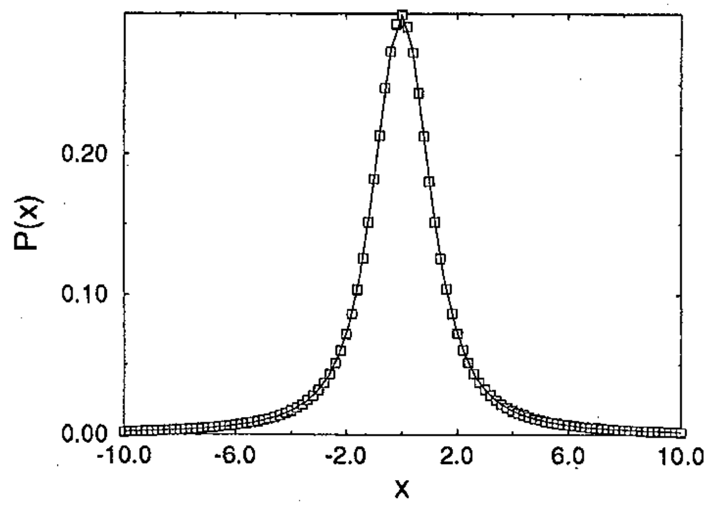

Figure 3: (a) Probability density function of the jumps of a 1-dimensional Lévy flight of index $\alpha = 1.20$ and $\gamma = 1.0$. The PDF is obtained by simulating the process (10^5 realizations) with the algorithm of ref. [9]. The symmetrical Lévy stable distribution of the same index and scale factor is shown for comparison (solid line).

The probability of return to the origin in the stable non-Gaussian regime is given by

$$P(z_n = 0) \simeq L(z_n = 0) = \frac{\Gamma(1/\alpha)}{\pi\alpha n^{1/\alpha}}. \tag{7}$$

whereas in the Gaussian regime the expected behavior is

$$P(z_n = 0) \simeq G(z_n = 0) = \frac{1}{\sqrt{2\pi}\sigma_o(\alpha,\ell) n^{1/2}}, \tag{8}$$

where $\sigma_o(\alpha,\ell)$ is the standard deviation of the TLF $\{x\}$ and we set $\gamma = 1$ for the sake of simplicity.

The crossover between the two regimes occurs for the value $n_\times$, which is the solution solution of the equation

$$L(z_n = 0) = G(z_n = 0) \tag{9}$$

The only implicit term in Eq.(9) is the TLFs standard deviation. By using the first term of the series expansion

$$L(z) \simeq -\frac{1}{\pi}\sum_{k=1}^{m} \frac{(-1)^k}{k!} \frac{\Gamma(\alpha k+1)}{z^{\alpha k+1}} \sin\left(\frac{k\pi\alpha}{2}\right) + R(z), \tag{10}$$

valid for a symmetrical Lévy distribution in the interval $1 < \alpha < 2$, we write the approximate relation

$$\sigma_o(\alpha,\ell) \simeq \left[\frac{2\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi(2-\alpha)}\right]^{1/2} \ell^{(2-\alpha)/2}. \tag{11}$$

By using Eq. (11), we solve Eq. (9) and we determine the crossover $n_\times$. Under the approximations used to determine Eq. (11), the crossover between the two regimes is given by

$$n_\times \simeq A\,\ell^\alpha, \tag{12}$$

where

$$A = \left[\frac{\pi\alpha}{2\Gamma(1/\alpha)[\Gamma(1+\alpha)\sin(\pi\alpha/2)/(2-\alpha)]^{1/2}}\right]^{2\alpha/(\alpha-2)}. \tag{13}$$

A geometric interpretation of the determination of $n_\times$ is given in Fig. 2. The dashed line of Fig. 2 is the probability of return $P(z_n = 0)$ as a function of n of a Lévy flight of index $\alpha = 1.2$ and the three solid lines are the $P(z_n = 0)$ calculated for a Gaussian process having the same standard deviation of a TLF with the same index and cut-off length $\ell = 10, 100$ and 1000 (top to bottom respectively). For each value of ℓ, a crossover between the two regimes occurs when the probability of return to the origin of the associated Gaussian process exceded the probability of return expected for the Lévy flight. In addition to the

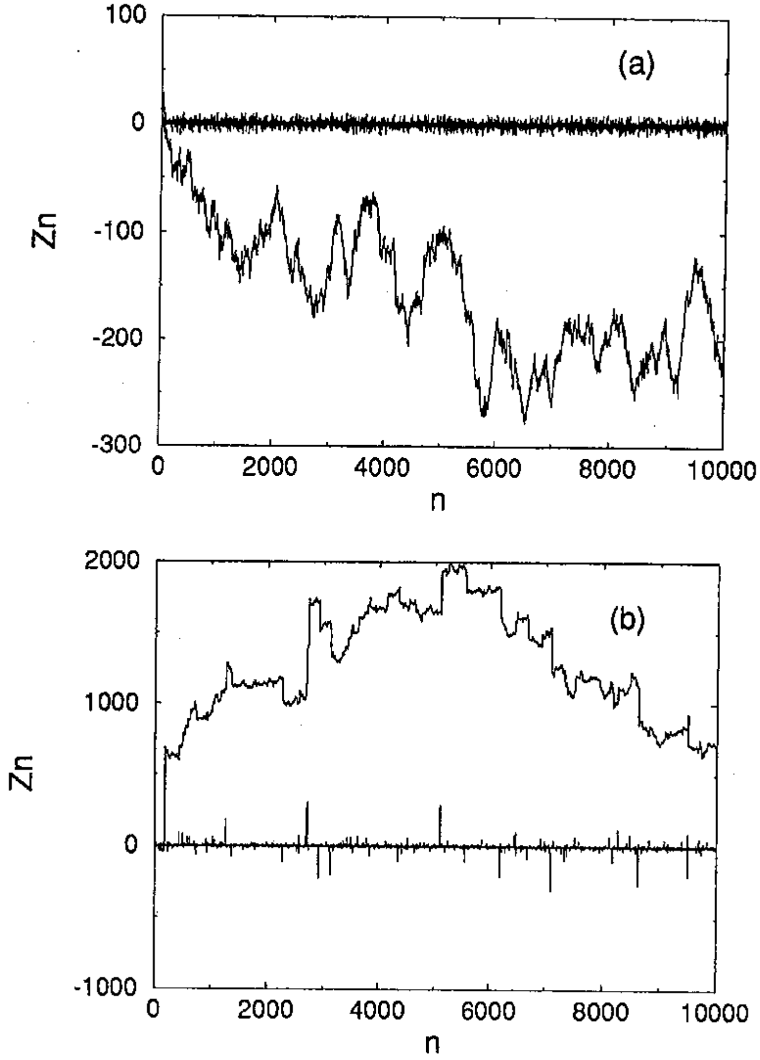

Figure 4: (a) z_n is the sum of n independent jumps distributed as a TLF of index $\alpha = 1.2$ and $\gamma = 1.0$ and cut-off length $\ell = 10$ (also in the figure as noise close the origin $z_n \approx 0$). The profile of the z_n walk is analogous to a Brownian random walk. From [14] .(b) Same as in (a) but with cut-off length $\ell = 1000$. The profile of the z_n walk is now close to the one observed for a Lévy flight of the same index (abrupt jumps are observed quite frequently). From [14].

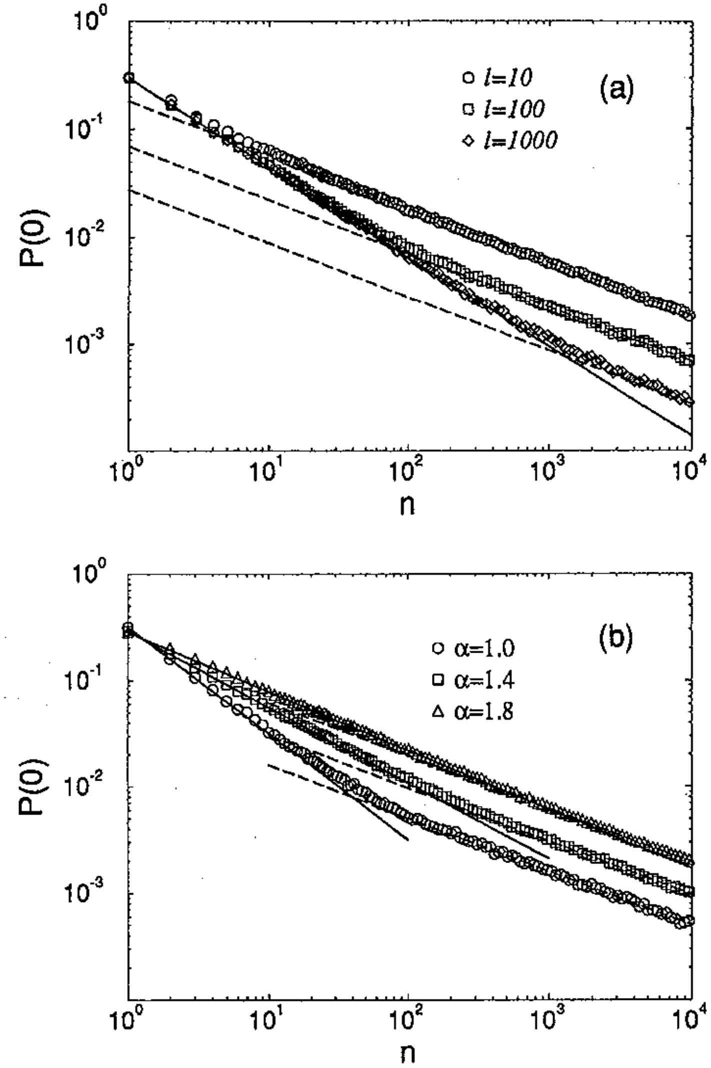

Figure 5: (a) Probability of return to the origin of z_n as a function of n for $\alpha = 1.2$ and $\ell = 10, 100$ and 1000. The simulations (circles, squares and diamonds) are compared with the Lévy regime (solid line) and the Gaussian regimes calculated for $\ell = 10, 100$ and 1000 (dotted lines from top to bottom respectively). From [14]. (b) Same as in (a) but for a fixed value of the cut-off length $\ell = 100$ and different values of α (1.0,1.4 and 1.8). The simulations (circles squares and triangles) are compared with the corresponding Lévy (solid lines) and Gaussian (dotted lines) regimes.

crossover $n_\times$ another interesting quantity is the "distance" between the probability of return to the origin of a TLF and the rescaled asymptotic Gaussian process

$$G_s(z_n = 0) \equiv G(z_n = 0) n^{1/2} \tag{14}$$

To be precise, we define

$$\Delta \equiv \log_{10} \frac{T(0)}{G_s(0)}, \tag{15}$$

to be the "distance" between the probability of return to the origin of a TLF and the Gaussian process with the same standard deviation $\sigma_o(\alpha, \ell)$. In Fig. 2 the "distance" is the separation between the two regimes observed at $n = 1$. Under the same assumptions used to determine the crossover $n_\times$, we obtain

$$\Delta \simeq \log_{10} \frac{2\Gamma(1/\alpha)}{\pi\alpha} + \frac{1}{2} \log_{10} \left(\frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{2-\alpha} \right) + \frac{2-\alpha}{2} \log_{10} \ell. \tag{16}$$

We test the accuracy of the results of Eqs. (12) and (16) by performing several numerical simulations of TLFs. In particular, we investigate the probability of return to the origin of the stochastic process z_n as a function of n. Since we investigate $P(z_n = 0)$, we need an algorithm which is accurate all over the definition range, including the origin. In our simulations we use a simple, fast and accurate algorithm proposed recently [9]; other algorithms can be found in the mathematical literature [10]. The algorithm is accurate over the entire range of the stochastic variable $\{x\}$. As an example, in Fig. 3 we show the PDF of jumps of a simulated Lévy flight together with the theoretical Lévy stable distribution of the same index. The index α is 1.2 and the scale factor γ is 1.0 . The PDF (black boxes) is measured by analyzing an ensemble of 10^5 independent realizations. It is in very good agreement with the theoretical distribution of the same index and scale factor (solid line).

In Fig. 4 we show single realizations of a 1-dimensional TLF random walk of index $\alpha = 1.2$ and cut-off length $\ell = 10$ and 1000 (Figs. 4a and 4b). In each figure, z_n is the sum of n independent jumps $\{x\}$ characterized by the PDF of Eq. (1). The variables x_n are also shown; they are the noise plotted around the origin ($z_n \approx 0$). In the Figure we can observe the role played by the cut-off length ℓ. In fact, the pattern of Fig. 4a is similar to the pattern observed in a Brownian random walk. In Fig. 4b a Lévy like pattern is observed due to the fact that a significant number of rare events are allowed.

We perform a quantitative analysis of the statistical properties of z_n as a function of n by investigating the probability of return to the origin ($P(z_n = 0)$). In Fig. 5 we show typical results of our simulations. In Fig. 5a we show $P(z_n = 0)$ when $\alpha = 1.2$ and $\ell = 10, 100$ and 1000. The solid line is the

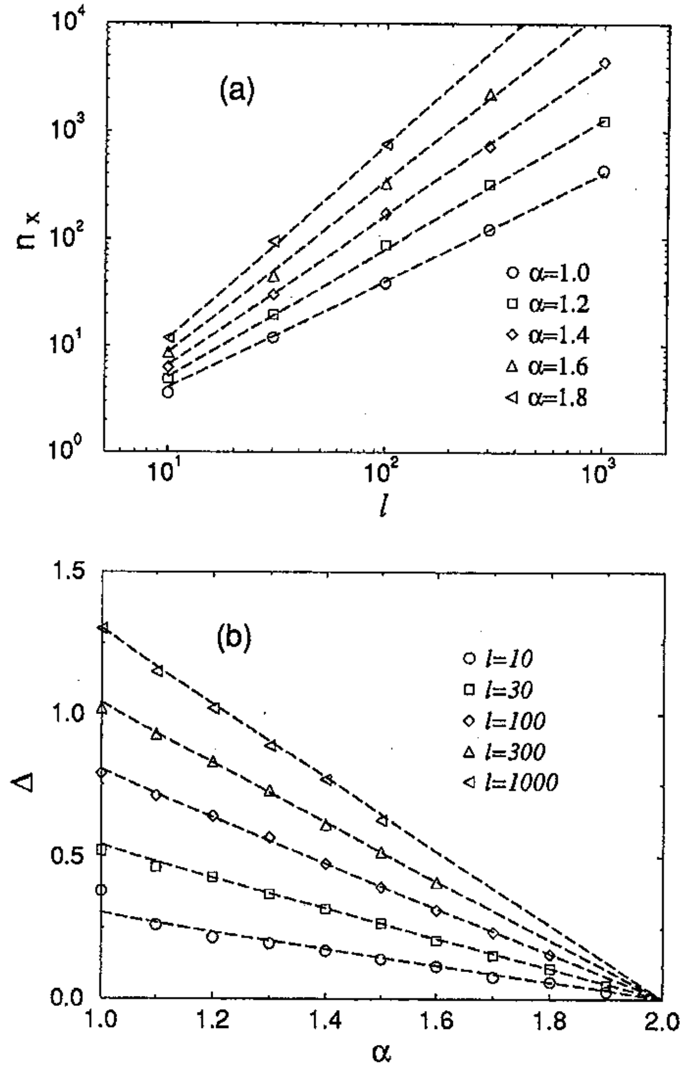

Figure 6: (a) Crossover between the Lévy and the Gaussian regime obtained from numerical simulations of z_n. The dotted lines are the theoretical predictions of Eqs. (12) and (13). From [14]. (b) *Distance* between the associated Lévy and the rescaled Gaussian process measured in numerical simulations of z_n. The dotted lines are the theoretical predictions of Eq. (16). From [14].

theoretical prediction for a Lévy flight of index $\alpha = 1.2$ (Eq. (7)) and the dotted lines are the predicted asymptotic Gaussian regimes of Eq. (8) calculated for the three different values of ℓ. In Fig. 5b we show the simulated $P(z_n = 0)$ obtained by setting $\ell = 100$ and $\alpha = 1.0$, 1.4 and 1.8. In the same Figure we show the theoretical prediction for the Lévy flights of the same indices and the associated asymptotic Gaussian regimes. We note that the crossover between the two regimes is observed for larger values of n when α increases at fixed values of ℓ, in spite of the fact that the distance Δ between the two regimes decreases. The agreement between numerical simulations and the theoretical prediction for the crossover between the Lévy and Gaussian regimes is very good for all the investigated values of α and ℓ.

We summarize the results of our numerical simulations in Fig. 6. In Fig. 6a we show the measured crossover between the Lévy and Gaussian regimes as a function of the cut-off length ℓ for different values of the index α. The dotted lines are obtained by using Eqs (12) and (13) with the same value of α used in numerical simulations. In Fig. 6b we show the "distance" Δ measured in numerical simulations, and we compare the values obtained from simulations with the theoretical prediction of Eq.(16) (dotted lines). The agreement between simulations and the theoretical predictions of Eqs. (12) (13) and (16) is quite good.

Before concluding, we show that a stochastic process having statistical properties similar to that of a TLF is the time evolution of the S&P 500. The study of economic time series as stochastic processes was pioneered by L.Bachelier at the beginning of this century [11]. The relation between economic series end Lévy (or Pareto) distribution was pointed out for the first time by Mandelbrot in the sixties [12]. Here we analize the time series as a common random process and study the successive differences of the time series without performing a nonlinear transformation.

The S&P 500 is one of the most important financial indices of the *New York Stock Exchange*, and has been recorded for between time intervals as short as 15 seconds. We analyze the dynamics of the S&P 500 with a very high temporal resolution (1 minute) in a time interval of 6 years. By studying the PDF of successive non-overlapping differences of the index $V(\Delta t)$, we measure a non-Gaussian scaling of the probability of return to the origin. In Fig. 7 we show the probability of return to the origin as a function of the time interval between the data. The investigated period is Jan 84 - Dec 89. The number of non-overlapping successive differences used in this analysis decreases from nearly 500,000 ($\Delta t = 1$ minute) to nearly 500 ($\Delta t = 1000$ minutes). In Fig. 7 we also show the best linear fit of the data performed in a double logarithmic plot. The functional form of the probability of return is a power-law $P(V = 0) = k/(\Delta t)^{0.712}$. The value of the exponent is different from the one expected for a Gaussian process, 0.5, so that experimental data are compatible with a Lévy walk or flight of index $\alpha = 1/0.712 = 1.40$. We check the hypothesis of Lévy stable PDF for the successive variations of the S&P 500 by comparing

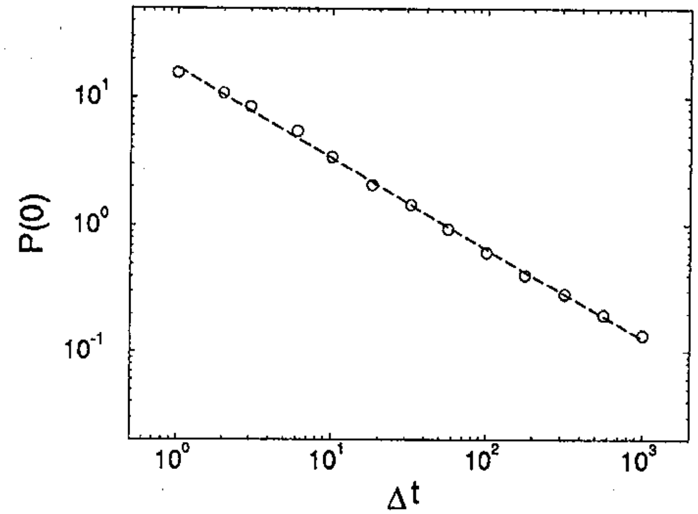

Figure 7: Probability of return to the origin of the successive changes of the S&P 500 measured between non-overlapping intervals ranging from 1 minute to 1000 minutes. The data are collected during the period Jan 84-Dec 89. The dotted line is the best linear fit of the experimental results. The slope of the fit is -0.712 a value different from the one expected for a normal diffusion (-0.5).

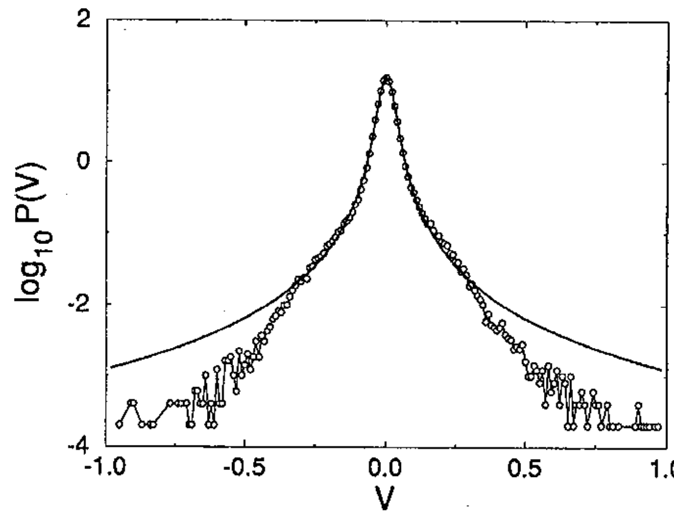

Figure 8: (a) Probability density function of the 1 minute changes of the S&P500 index measured in the period Jan 84-Dec 89. The solid line is a symmetrical Lévy stable distribution of index $\alpha = 1.40$ and $\gamma = 0.00375$. The agreement between the experimental PDF (circles) and the Lévy distribution is very good for $|V| \leq 6\sigma$. The observed exponential (or stretched exponential) wings ensure a finite variance for the observed stochastic process.

the PDF measured when $\Delta t = 1$ minute with the Lévy stable PDF of index $\alpha = 1.40$ and scale factor $\gamma = 0.00375$. The scale factor is determined by using the experimental values $P(0) = 15.66$ and $\alpha = 1.40$ and the theoretical relation $P(0) = \Gamma(1/\alpha)/(\pi\alpha\gamma^{1/\alpha})$ valid for a Lévy flight of scale factor γ ($\Delta t = 1$). In Fig. 8 we show the experimental PDF (circles) together with the Lévy stable distribution of the determined index and scale factor (solid line). The agreement between the experimental and Lévy distribution is excellent when the absolute value of the jumps V is less then 6 standard deviations σ ($|V| < 6\sigma$). Conversely, when $|V| > 6\sigma$ an approximately exponential fall-off of the wings is observed. Although the cut-off is not abrupt as in a TLF, exponential fall-off ensures a finite variance for the investigated stochastic process. Another similarity with the TLF is that the stochastic process $\{V\}$ is diffusive. In fact in the interval $\Delta t = 10,, 1000$ minutes, the measured variance $< V^2 >$ is fitted by the relation $< V^2 >= k_1 \Delta t^{1.08}$. The measured diffusion exponent, 1.08, is very close to the value (1) observed in normal diffusive processes. The absence of a super diffusive behavior of the variance of V rules out the possibility that a Lévy walk can describe the dynamics of the S&P 500.

In conclusion, our theoretical and numerical results show that the TLF, a *quasi*-stable stochastic process, can show a ultra-slow convergence to the asymptotic associated Gaussian process. In the interval $n \ll n_\times$ a TLF is a stochastic process well described by Lévy stable distribution, with the exception of the most rare events, but having a *finite* variance. This is a *practical* violation of the central limit theorem. With the previous statement we mean one can expect to observe experimentally a stochastic process with the characteristic of a Lévy flight for a long time (for example $n \approx 1O^4$ time steps), even if the stochastic process has indeed a finite variance. A TLF provides a resolution of the paradox of the experimental observation of a Lévy-like stochastic processes in physical systems. This resolution is different and complementary to the resolution provided by Lévy walks. It is worthwhile to compare the general properties of Lévy walks and TLFs. Similarities: Lévy walks and TLFs have PDFs which are Lévy distributions, with the exception of the most rare events and finite variance, moreover Lévy walks and TLFs (in the Lévy regime) show a non-Gaussian scaling of the probability of return to the origin. Differences between the two processes are observed in the asymptotic basin of attraction and in the time evolution of the variance of the process. Lévy walks converge to Lévy flights whereas TLFs converge to Gaussian processes. Moreover, Lévy walks are superdiffusive whereas the diffusion of a TLF walk is normal. The empirical observation of a Lévy profile of the PDF together with a non-Gaussian scaling of the probability of return to the origin is not sufficient to discriminate between a Lévy walk and a TLF. Additional information about the properties of the spectral density (or the time evolution of the variance) is needed to discriminate between the two modeling processes. We provide as a possible example of TLF the time evolution of the S&P 500.

We thank S. V. Buldyrev, C. K. Peng, and F. Sciortino for helpful discussions and NSF for financial support. After this work was completed, the result (12) was derived by M. F. Shlesinger [13] using the Berry-Esseen theorem, which applies to the convergence to a Gaussian for a symmetric random walk whose jump probabilities have a finite third moment.

References

1. P. Lévy, *Théorie de l'Addition des Variables Aléatoires* (Gauthier-Villars, Paris, 1937).
2. W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1971).
3. B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
4. T. H. Solomon, E. R. Weeks, and H. L. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).
5. F. Bardou, J.-P. Bouchaud, O. Emile, A Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. **72**, 203 (1994).
6. A. Ott, J.-P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
7. C.-K. Peng, J. Mietus, J.M. Hausdorff, S. Havlin, H.E. Stanley, and A.L. Goldberger, Phys. Rev. Lett. **70**, 1343 (1993).
8. M.F. Shlesinger, G.M. Zaslavsky, and J.Klafter, Nature **363**, 31 (1993), and references therein.
9. R. N. Mantegna, Phys. Rev. E **49**, 4677 (1994).
10. G. Samorodnitsky and M.S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance* (Chapman and Hall, NY, 1994).
11. L. J. B. Bachelier, *Théorie de la Speculation* (Gauthier-Villars, Paris, 1900); this article is reproduced in P. H. Cootner (ed.), *The Random Character of Stock Market Prices* (MIT Press, Cambridge MA, 1964).
12. B. Mandelbrot, J. Business **36**, 394 (1963).
13. M. F. Shlesinger, preprint
14. R. N. Mantegna and H. E. Stanley, Phys. Rev. Letters **73**, 2946 (1994).