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New class of screened growth aggregates with a continuously tunable fractal dimension

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A new family of fractals is investigated. The fractal dimension D_f is found to be equal to a variable parameter of the model characterizing the strength of the screening. Thus we can make fractals with *arbitrary* D_f, and study anomalous diffusion as a function of D_f. Our data support a generalization we propose of the recent Aharony-Stauffer conjecture based on the spatial distribution of "growth sites" of a fractal.

It is of considerable general interest to discover how the familiar laws of physics are modified for fractals, in part because of the numerous examples of fractal structures in nature.[1–4] Some studies have focused on regular fractals —such as the Sierpinski gasket—for which the fractal dimension D_f is usually known exactly.[5,6] Recently it has become increasingly apparent that the physical systems of interest are not describable by regular fractals, and hence many studies of random fractals have been undertaken. A major problem that plagues these studies is that D_f is not generally known exactly, even for simple $d=2$ systems.

Here we develop a family of random fractal structures for which D_f *is* known exactly. Moreover, one can continuously tune D_f in order to test laws that may not be readily tested using the discrete values of D_f available from the above-mentioned fractals; these fractals thereby provide an ideal testing ground for properties of random fractals in general. More important, perhaps, is the conceptual rationale for this model. *It bears the same relation to the Rikvold model*[7] (or any other model with a discrete value of D_f) *that the Fisher-Ma-Nickel model*[8] *of spin-spin interactions that decay as a power law bears to the ordinary Ising model with short-range forces.*

We are concerned with clusters generated by starting from a seed and successively adding new sites at the perimeter. The probability for adding a new site at the vacant perimeter site x is given by[7]

$$p(x)=K(x)\Big/\sum_{y\,\in\,\text{perimeter}}K(y)\ , \tag{1a}$$

where

$$K(x)=\prod_{y\,\in\,\text{cluster}}\exp(-|x-y|^{-\epsilon})\ . \tag{1b}$$

Here ϵ is a free parameter. Thus we grow a cluster by successive addition of new sites on the perimeter with a *long-range* screening effect as a result of the *nature* of the dependence of $p(x)$ on the existing cluster sites. We have three main objectives: (i) to give a compelling argument that $D_f=\epsilon$, (ii) to put this prediction to a searching test by means of very large scale numerical simulations, and (iii) to investigate the properties of random walks on these clusters, thereby testing the relative validity of two competing theories of fractal dynamics.[9,10]

Fractal dimension. To find D_f it is more convenient to visualize the cluster being generated by growing sites at a *rate* $K(x)$ given by Eq. (1b), so that $\sum_y K(y)$ is the average number of sites created per unit time. Now consider $K(x)$ as a function of D_f. Changing from sums to integrals and going over to polar coordinates,

$$\sum_{y\,\in\,\text{cluster}}|x-y|^{-\epsilon}\simeq\int_a^R dr\,r^{D_f-1}r^{-\epsilon}=O(R^{D_f-\epsilon})+O(1)\ , \tag{2}$$

where a is a short-distance cutoff on the order of the lattice length. First suppose that $D_f>\epsilon$, so that

$$K(x)\simeq\exp(-AR^{D_f-\epsilon})\ , \tag{3a}$$

$$\sum_{y\,\in\,\text{perimeter}}K(y)\lesssim R^{D_f}\exp(-AR^{D_f-\epsilon})<<1\ . \tag{3b}$$

Equation (3) then implies that the total growth rate decreases dramatically as R becomes large. However, it is natural to assume that such a "blocked" process would be unstable against the formation of branches which effectively decrease D_f, thus allowing the process to continue growing. This means that eventually D_f will be less than or equal to ϵ. Assume now $D_f < \epsilon$. Then from (3a) it follows that the rate of growth $K(x)$ is essentially a constant independent of x. This would mean that the model belongs in the same universality class as the Eden model, implying $D_f = d$. This, however, would only be consistent if $\epsilon \geq d$. Therefore,

$$D_f = \min(\epsilon, d) \quad . \tag{4}$$

It should be pointed out that the above reasoning totally excludes $D_f \geq \epsilon$ for any reasonable cluster size, but allows for some finite-size effects if $D_f \leq \epsilon$, since at finite size it is clearly not true to say that $K(x)$ is essentially a constant if $D_f \leq \epsilon$. This may explain why the observed values of D_f are systematically somewhat lower than ϵ.

Computer simulations. The generation of clusters using the screened growth model has been described previously.[7] The screening effect of a single occupied lattice site x_n (the nth site) is to reduce the growth probability at site x_n by a factor of S_n where

$$S_n(x) = \exp(-|x - x_n|^{-\epsilon}) \quad . \tag{5}$$

Since our model assumes that the screening effects of more than one site are multiplicative, the growth probability is given by $K(x)$ of Eq. (1b). The growth simulation was carried out on a 1001×1001 lattice with a "seed" or growth site at the center of the lattice and the simulation was stopped when the growing cluster either reached an edge of the lattice or attained a size of 25 000 occupied lattice sites.

Fractal dimension of substrate: simulation results. Simulations were carried out with the screening exponent ϵ set to the values of $\frac{5}{4}$, $\frac{4}{3}$, $\frac{3}{2}$, and $\frac{7}{4}$. Typical clusters are shown in Fig. 1. The fractal dimensionality was estimated from both the density-density correlation function $C(r)$ and the dependence of the radius of gyration on cluster size. Our first estimate of D_f can be obtained from the density-density correlation function by using the relationship

$$C(r) \sim r^{D_f^{(1)} - d} \quad . \tag{6}$$

In practice, we use the dependence of $C(r)$ on r at intermediate length scales (larger than a few lattice units but

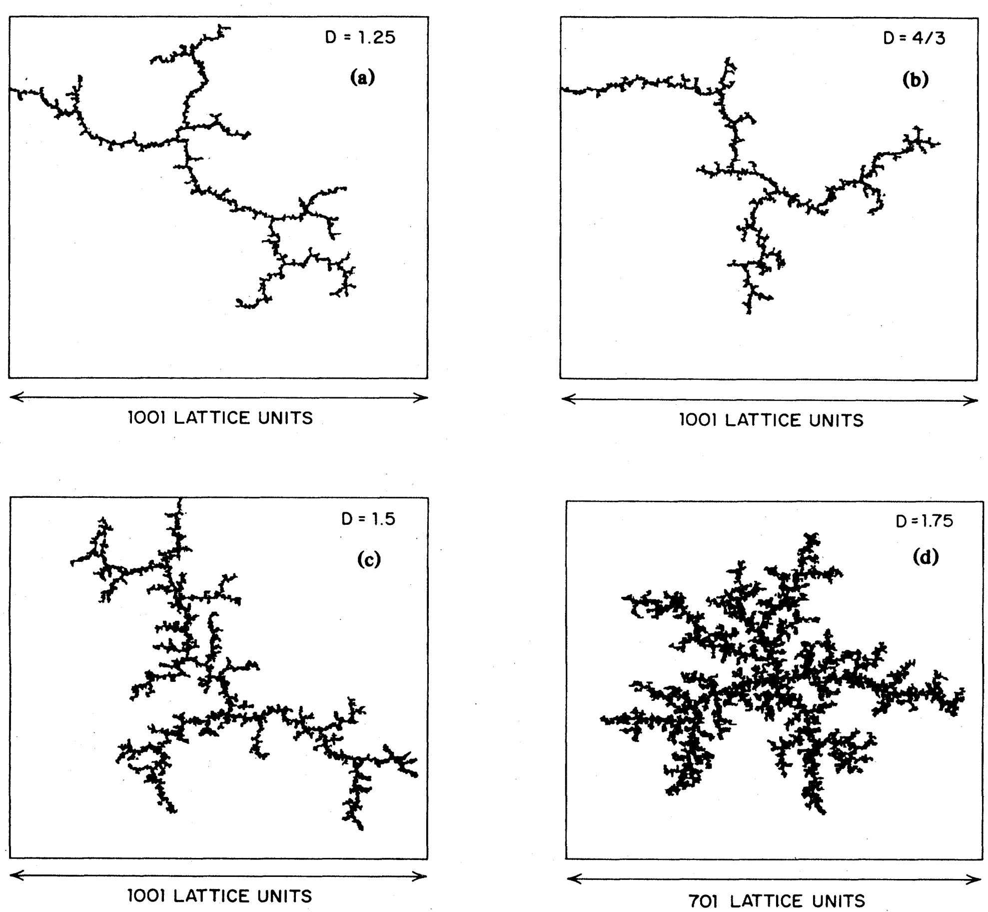

FIG. 1. Typical clusters grown on a 1001×1001 lattice using the screening function given in Eq. (5). The screening exponents (ϵ) used to generate Figs. 1(a), 1(b), 1(c), and 1(d) are $\frac{5}{4}$, $\frac{4}{3}$, $\frac{3}{2}$, and $\frac{7}{4}$, respectively. The number of sites occupied by the clusters in 9347 [Fig. 1(a)], 9536 [Fig. 1(b)], 20695 [Fig. 1(c)], and 25000 [Fig. 1(d)].

smaller than the overall size of the cluster), where $\ln[C(r)]$ depends approximately linearly on $\ln(r)$. A second effective dimension $D_f^{(2)}$ can be obtained assuming that[11]

$$R_g \sim N^{1/D_f^{(2)}} \quad . \tag{7}$$

Both estimates for the fractal dimension are consistent with the result that $D_f = \epsilon$. In our earlier work[7] with smaller clusters, we found that D_f was substantially smaller than ϵ for $\epsilon \geq 1.5$. We still find that D_f is smaller than ϵ for $\epsilon = 1.75$ but now the difference between D_f and ϵ is much smaller, and we believe that even better agreement would be obtained with larger clusters ($>>$ 25 000 occupied lattice sites).

Random walks on screened fractals. Random walks were simulated on all 28 clusters. For each cluster we carried out 4000 walks, each consisting of 2^{15} (32 768) steps. Each walk was started out on sites randomly chosen from those sites which were occupied when the growth process was 5%–25% complete. Ideally, the random walks should sample as large a region of the cluster as possible, avoiding the (small) region near the origin, which may be anomalous, and the outer regions of the cluster which may be subject to further growth. Clearly, these requirements cannot be satisfied simultaneously and the choice of the origin and length of the walks was made in order to achieve a reasonable compromise. The quantities measured during the walks were $\langle r^2 \rangle$ (the mean displacement from the origin of the walk), $\langle r^4 \rangle$, $\langle s \rangle$ (the mean number of sites visited by the walk), and $\langle s^2 \rangle$. The quantities $\langle r^2 \rangle$ and $\langle r^4 \rangle$ allow us to estimate the fractal dimensionality of the walk (D_w) from their relationship to N_w (the number of steps in the walk)

$$\langle r^{2k} \rangle \sim N_w^{2k/D_w} \quad (k=1,2) \quad . \tag{8a}$$

The fracton or spectral dimensionality D_s is obtained using Eq. (1). A more direct way of estimating D_s is to use the dependence of $\langle s \rangle$ and $\langle s^2 \rangle$ on N_w

$$\langle s^k \rangle \sim N_w^{kD_s/2} \quad (k=1,2) \quad . \tag{8b}$$

The results for D_s and D_w are displayed in Table I. The exponents are estimated in two ways, using either $k=1$ or $k=2$. It is seen that both are well compatible.

The third column displays ϵ/D_w, which should be equal to $\frac{1}{2}D_s$. We notice that for $\epsilon = 1.5$ there is a small—and for $\epsilon = 1.75$ a larger—discrepancy. A closer examination of the data shows, however, that the exponent D_s has not saturated to its final value but is growing as the number of steps increases, whereas D_w does not show any such tendency.

TABLE I. Fractal properties of screened growth model. The top row arises from (8a) and (8b) with $k=1$, while the bottom row is from (8a) and (8b) with $k=2$.

ϵ	D_s	$2/D_w$	ϵ/D_w	D_w
1.25	1.10 ± 0.02	0.85 ± 0.05	0.53 ± 0.03	2.25 ± 0.05
	1.10 ± 0.01	0.85 ± 0.05	0.53 ± 0.03	2.25 ± 0.05
$\frac{4}{3}$	1.11 ± 0.014	0.84 ± 0.03	0.56 ± 0.02	2.38 ± 0.09
	1.11 ± 0.015	0.85 ± 0.04	0.57 ± 0.03	2.36 ± 0.12
1.5	1.17 ± 0.02	0.81 ± 0.03	0.61 ± 0.025	2.47 ± 0.09
	1.17 ± 0.02	0.78 ± 0.04	0.59 ± 0.03	2.54 ± 0.12
1.75	1.22 ± 0.03	0.78 ± 0.02	0.67 – 0.70	2.57 ± 0.07
	1.22 ± 0.03	0.76 ± 0.01	0.66 – 0.67	2.63 ± 0.04

As a final remark, note that the Alexander-Orbach (AO) result[9]

$$D_w = \tfrac{3}{2} D_f; \quad D_s = \tfrac{4}{3} \quad , \tag{9a}$$

which is valid to high accuracy for percolation clusters and frequently close to measured values for other fractals, is here definitely invalid (at least for ϵ less than 1.75). Since it is well known that $D_w \geq 2$, Eq. (9a) cannot hold for $\epsilon < \frac{4}{3}$. This is borne out by our results. However the region where AO breaks down extends probably to $\epsilon = 1.5$. Indeed there is no reason to think that it should hold for any particular range of values of ϵ but for $\epsilon = 1.75$ the uncertainty about D_s is too large to permit any meaningful statement.

However, the values of D_s can be interpreted using a recent idea of Aharony and Stauffer (AS).[10] If the *number of growth sites* (adjacent sites to the sites visited by the walk) scales as $[\langle s \rangle]^{1/2}$, then one obtains the AO conjecture.[12] On the other hand, AS assume that the growth sites lie on a narrow ring around the perimeter of the walk, and argue that the AO rule must fail for $D_f < D_f^-$, where $D_f^- = 2$ is a lower critical dimension. For $D_f < D_f^-$, they find

$$D_w = 1 + D_f; \quad D_s = 2D_f/(1+D_f) \quad . \tag{9b}$$

This formula agrees well with our data for *all* ϵ.

An explanation of this agreement follows from the general approach of Ref. 13. Define the "chemical distance" l as the length of the shortest path connecting two points, and define D_f^l and D_w^l by[13,14]

$$N \sim l^{D_f^l}, \quad R^2 \sim t^{2/D_w^l} \quad . \tag{10}$$

However, if we define $\tilde{\zeta}_l$ by $\rho(l) \sim l^{\tilde{\zeta}_l}$, where $\rho(l)$ is the resistance between two points connected by a path of length l, then one obtains from the Einstein relation[13]

$$\tilde{\zeta}_l = D_w^l - D_f^l \quad . \tag{11}$$

But if large loops do not occur, then two points are connected by one path only and hence $\tilde{\zeta}_l = 1$. Since D_s is independent of the metric used, one obtains in this case[13]

$$D_s = 2D_f^l / D_w^l = 2D_f^l / (D_f^l + 1) \quad . \tag{12a}$$

Since $D_f^l < D_f$, we find

$$D_s \leq 2D_f^l / (D_f^l + 1) \quad . \tag{12b}$$

Thus, the AS value of D_s, (9b), is a rigorous upper bound for any fractal for which loops are irrelevant. *Moreover, the* AS *result becomes exact if* $D_f^l = D_f$. Since our present results agree well with the AS conjecture, we may conjecture that $D_f^l = D_f$ and work is underway to test this conjecture.

Summarizing, we have studied the properties of a certain class of screened growth clusters. A convincing argument has been given to show that they are fractals and that their fractal dimension is identical to the parameter ϵ involved in their definition. This hypothesis has been tested numerically and shown to hold within the accuracy of the measurement. Random walks were then generated and the exponents D_w and D_s measured. The relationship $D_s = 2D_f/D_w$ was found to hold to a good accuracy except for $\epsilon = 1.75$, where the measured value of D_s was unreliable. This data do not support the Alexander-Orbach conjecture, but *do* support the recent AS conjecture based on the spatial distribution of "growth sites" of a random walk.

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