SOME RIGOROUS RESULTS CONCERNING THE CROSSOVER BEHAVIOR OF THE ISING MODEL WITH LATTICE ANISOTROPY *

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Received 15 June 1972

The following rigorous relations are established for the Ising model with interaction strengths J in some lattice directions and RJ in other directions: $\gamma_1 = 2\gamma, \gamma_2 \geqslant 3\gamma$, and $\gamma_3 \geqslant 4\gamma$, where $\chi_n^{(0)} \equiv (\partial^n\chi/\partial R^n)_{R=0} \sim \epsilon^{-\gamma_n}$, and $\gamma_0 = \gamma$ is the susceptibility exponent for the lattice when R=0. These results disagree with recently-reported numerical estimates of certain of the γ_n.

There has recently been considerable interest [1–10] in systems with "lattice anisotropy" (different coupling strengths in different lattice directions). Consider, e.g., the d-dimensional nearest-neighbor (nn) Ising system

$$\mathcal{H} = -J \sum_{v_i = v_j}^{\mathrm{nn}} s_i s_j - RJ \sum_{u_i = u_j}^{\mathrm{nn}} s_i s_j \equiv \mathcal{H}_0 + R\mathcal{H}_1 , \tag{1}$$

where $r_i \equiv (x_1, x_2, \ldots x_{\bar{d}}) \equiv (u_i, v_i)$ where $u_i \equiv (x_1, \ldots, x_d)$ and $v_i \equiv (x_{d+1}, \ldots, x_{\bar{d}})$. For example, very recently there have been extensive calculations [1, 2] concerning the case $\bar{d}$=3, d=2, corresponding to a "square to simple cubic crossover". Henceforth we shall consider this system for the purpose of specificity and clarity; thus $r_i \equiv (x_i, y_i, z_i) \equiv (u_i, z_i)$ and $R \equiv J_z/J_{xy}$. Our approach is, however, more general.

According to the generalized scaling hypothesis, for which the parameter R is scaled (as well as ϵ, H, ...), the "crossover" exponent ϕ is the only exponent that one needs to describe the crossover behavior [8]. In particular,

$$\gamma_n = \gamma + n\phi , \tag{2}$$

where the new exponent γ_n is defined by

$$\chi_n(R=0) \equiv (\partial^n \chi/\partial R^n)_{R=0} \sim |T - T_c(0)|^{-\gamma_n} . \tag{3}$$

Here χ is the reduced zero-field magnetic susceptibility and $\gamma_0 = \gamma$ is the susceptibility exponent of the d-dimensional system.

The exponents γ_n cannot be calculated exactly but they can be estimated by extrapolations based upon high-temperature series expansions. There presently exists a dispute [1, 3–5] in the literature concerning numerical values of γ_n, and the most recent work claims that for sq → sc Ising model,

$$\gamma_1 = 3.5 , \quad \gamma_2 = 5.0 \pm 0.1 , \quad \gamma_3 = 6.5 \pm 0.2 , \quad \gamma_4 = 8.0 \pm 0.3 . \tag{4}$$

In this note we shall report the following rigorous results:

$$\gamma_1 = 2\gamma \tag{5a}$$

$$\gamma_2 \geqslant 3\gamma \tag{5b}$$

$$\gamma_3 \geqslant 4\gamma . \tag{5c}$$

Since γ=1.75 for a sq Ising model, the numerical estimates of (4) violate (5). Our results also lend support for the predictions (2) and $\gamma_n = (n+1)\gamma$.

As a demonstration, we shall here outline the proof of (5b). Details of the analysis will be published elsewhere.

* Supported by NSF Grant GP-15428.

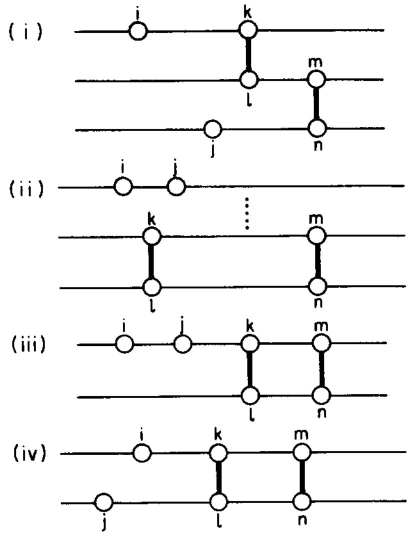

Fig. 1. Conformations of lattice sites which correspond to non-zero contributions to $\langle s_i s_j \mathcal{H}_1^2 \rangle$ at $R=0$. Sites in the same plane are joined by a horizontal line. A heavy vertical line indicates that the sites are coupled with strength RJ.

For a lattice of $N+1$ layers with M^2 spins in each layer, we have:

$$\beta^{-2}(N+1)M^2 \chi_2(0) = \sum_{r_i, r_j} [\langle s_i s_j \mathcal{H}_1^2 \rangle - \langle s_i s_j \rangle \langle \mathcal{H}_1^2 \rangle]_{R=0} \,. \quad (6)$$

At R=0, we observe that spins on different layers (with different z_k's) are not coupled. Since $\mathcal{H}_1$ consists only of products of $s_k s_l$ with $z_k \neq z_l$, there are in fact only four possible topological conformations (cf. fig.1) of the lattice-sites i, j, k, l, m, n which make a non-zero contribution to the six-spin thermal average $\langle s_i s_j s_k s_l s_m s_n \rangle_{R=0}$. The contribution of conformations (i) – (iv) of fig. 1 are respectively,

$$\langle s_i s_k \rangle_0 \langle s_l s_m \rangle_0 \langle s_j s_n \rangle_0 \,, \quad (7a)$$

$$\langle s_i s_j \rangle_0 \langle s_k s_m \rangle_0 \langle s_l s_n \rangle_0 \quad (7b)$$

$$\langle s_i s_j s_k s_m \rangle_0 \langle s_l s_n \rangle_0 \quad (7c)$$

$$\langle s_i s_k s_m \rangle_0 \langle s_j s_l s_n \rangle_0 \,, \quad (7d)$$

where $\langle \ldots\ldots \rangle_0$ denotes a thermal average for R=0.

The expressions (7a) – (7d) are weighted by factors $4(N-1)$, $N(N-1)$, $2N$, and $2N$ respectively, arising from the fact that we can make interchanges of the form $i \Leftrightarrow j$ etc. in fig. 1.

The second term in (6) has two factors,

$$\sum_{r_i, r_j} \langle s_i s_j \rangle_{R=0} = (N+1)M^2 \chi_0(0) \quad (8)$$

and

$$\langle \mathcal{H}_1^2 \rangle = J^2 N M^2 \sum_u \langle s_0 s_u \rangle_0^2 \,. \quad (9)$$

Thus (6) becomes

$$(\beta J)^{-2}(N+1)M^2 \chi_2(0) = 4(N-1)M^2 [\chi_0(0)]^3 - 2NM^2 \chi_0(0) M^2 \sum_u \langle s_0 s_u \rangle_0{}^2 + 2NM^2 \sum_u \left\langle \Big(\sum_{u_i} s_i\Big)^2 s_0 s_u \right\rangle_0 \langle s_0 s_u \rangle_0 + 2NM^2 \sum_u \left\langle s_0 s_u \Big(\sum_{u_i} s_i\Big) \right\rangle^2 . \quad (10)$$

The Griffiths inequality [11],

$$\langle s_{u_i} s_{u_j} s_0 s_u \rangle \geq \langle s_0 s_u \rangle \langle s_{u_i} s_{u_j} \rangle \,,$$

permits us to "cancel" the second and third terms on the right-hand side of eq. (10), and noting that the fourth term is positive, we have

$$\chi_2(0) \geq 4(\beta J)^2 \{\chi_0(0)\}^3 \quad (11)$$

where we have neglected $O(1/N)$ with respect to unity, inequality (5b) follows from (11).

In conclusion, we have shown rigorously that $\gamma_1 = 2\gamma, \gamma_2 \geq 3\gamma$, and $\gamma_3 \geq 4\gamma$. If the scaling hypothesis is valid (so that $\gamma_n = \gamma + n\phi$), our work furnishes a simple but rigorous proof of $\phi = \gamma$. Moreover, our results (5b) and (5c) indicate that reported values of γ_2 and γ_3 are unreliable [1–3]. A detailed study of these (and other [12]) high-temperature series for the lattice anisotropy problem is now underway, and preliminary numerical results indicate that $\gamma_n = (n+1)\gamma$ for $n = 1, 2, 3, 4$.

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