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New Inequalities among the Critical-Point Exponents for the Spin-Spin and Energy-Energy Correlation Functions*†

Luke L. Liu‡ and Richard I. Joseph

Department of Electrical Engineering, The Johns Hopkins University, Baltimore, Maryland 20218

and

H. Eugene Stanley

Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Two new inequalities, (i) $\gamma' \geq 2\beta(2-\eta)/(d-2+\eta) + 2\phi[2\beta/(d-2+\eta) - \nu'_\phi]$ and (ii) $(\delta-1)/\delta \geq 2(2-\eta)/\delta(d-2+\eta) + 2\phi[2/\delta(d-2+\eta) - \mu_\phi]$, are derived among critical-point exponents that describe the behavior of the two-spin correlation function $C_2(T,H,\vec{r}) \equiv \langle s_0^z s_{\vec{r}}^z\rangle - \langle s_0^z\rangle\langle s_{\vec{r}}^z\rangle$, subject to plausible assumptions (rigorous for Ising magnets). Here ν'_ϕ and μ_ϕ describe the divergence as $T \to T_c^-$ and as $H \to 0^+$, respectively, of the "generalized correlation length" $\xi_\phi(T,H)$, defined as the 2ϕth root of the normalized 2ϕth spatial moment of $C_2(T, H, \vec{r})$. Also derived are the corresponding inequalities among exponents that describe the behavior of the energy-energy correlation function. Inequality (i) is shown to lead to an inequality between primed and unprimed exponents. Moreover, if ν'_ϕ is independent of ϕ, then (i) implies that $\nu' \geq 2\beta/(d-2+\eta)$ and $\gamma' \geq (2-\eta)\nu'$, while if μ_ϕ is independent of ϕ, then (ii) implies $\mu \geq 2/\delta(d-2+\eta)$ and $(\delta-1)/\delta \geq (2-\eta)\mu$.

I. INTRODUCTION

Rigorous inequalities among critical-point exponents[1] have served to assist in the interpretation of experimental data and, perhaps more significant historically, have contributed to the formulation of the static scaling hypothesis[2] (which has the feature that most inequalities are predicted to be satisfied as equalities).

These inequalities may be classified into two groups: (i) relations among critical-point exponents characterizing the behavior of thermodynamic functions,[1] and (ii) relations among exponents characterizing the behavior of the static correlation functions.[3–5]

Inequalities belonging to category (i) (e.g., $\alpha' + 2\beta + \gamma' \geq 2$) are frequently found to be satisfied as *equalities* by experimental results *and* by calculations on model systems.[6] On the other hand, certain of the inequalities belonging to category (ii) are almost invariably *not* obeyed as equalities, with the notable exceptions of the two-dimensional Ising model ($d=2$) and the three-dimensional spherical model. Thus, for example, $d\nu \geq 2-\alpha$ is satisfied as an equality for the $d=2$ Ising model ($\nu=1$, $\alpha=0$), but for the $d=3$ Ising model, numerical-approximation methods indicate that $d\nu$ is about 2% larger than $2-\alpha$.[7]

It has been pointed out[8] that certain of the category (ii) inequalities in fact do not become equalities on making the scaling hypothesis, providing the latter is formulated in terms of homogeneity assumptions. This result has motivated us to reexamine the correlation function exponent equalities with the hope of possibly finding additional inequalities that are somewhat "tighter" than the original inequalities. The extent to which this hope has been realized will be discussed in Sec. IV below—for now suffice it to say that if the original relations are of the form $A \geq B$, then certain of our new relations are of the form $A \geq B + C$, and there is no reason for C not being positive; here A, B, and C represent different combinations of critical-point exponents. It is in this sense that some of our new inequalities may be logically stronger than certain of the old inequalities.

II. INEQUALITIES AMONG SPIN-SPIN CORRELATION FUNCTION EXPONENTS

In order to state the main results of this work we must define certain additional critical-point exponents, and in order to do this we first define the two-spin correlation function

$$C_2(T, H, \vec{r}) \equiv \Gamma_2(T, H, \vec{r}) - \Gamma_1^2(T, H) = \Gamma_2(T, H, \vec{r}) - M^2(T, H), \qquad (1)$$

where $\Gamma_1(T, H) \equiv \langle s_0^z \rangle = M(T, H)$ and $\Gamma_2(T, H, \vec{r}) \equiv \langle s_0^z s_{\vec{r}}^z \rangle$. We have used Ising-model notation in our definitions because the inequalities that we shall discuss necessitate for their proof certain postulates which, although plausible for a wide class of magnetic systems, have thus far been proved[9] only for the Ising model:

(a) *Positivity*. For all finite T and $H > 0$, $\Gamma_1(T,H) \geq 0$ and $C_2(T, H, \vec{r}) \geq 0$.

(b) *Monotonicity*. Γ_1 and Γ_2 are monotonic *increasing* functions of H for fixed T and monotonic *decreasing* functions of T for fixed H.

Near the critical point the correlation function becomes long range, and this is reflected, for example, in the behavior of its moments. Of particular interest in this work are the quantities $\xi_\phi(T, H)$, defined by the relation

$$[\xi_\phi(T, H)]^{2\phi} \equiv \textstyle\sum_{\vec{r}} |\vec{r}|^{2\phi} C_2(T, H, \vec{r}) / \sum_{\vec{r}} C_2(T, H, \vec{r}) . \qquad (2)$$

We note that the quantity $\xi_\phi(T, H)$ defined in (2) has the dimension of length, and hence we shall call it the "generalized correlation length"; it reduces to previous definitions of the correlation length ξ for the special case $\phi = 1$.[5]

As the critical point is approached the generalized correlation length diverges to infinity, and we accordingly define the critical-point exponents ν_ϕ, ν'_ϕ, and μ_ϕ to describe these divergences:

$$\xi_\phi(T, 0) \sim (T - T_c)^{-\nu_\phi}, \quad T \to T_c^+$$
$$\xi_\phi(T, 0) \sim (T_c - T)^{-\nu'_\phi}, \quad T \to T_c^- \qquad (3a)$$

and

$$\xi_\phi(T_c, H) \sim H^{-\mu_\phi}, \quad H \to 0^+. \qquad (3b)$$

Various authors have presented derivations of certain inequalities for the case $\phi = 1$. These are the Josephson inequalities[3]

$$d\nu \geq 2 - \alpha, \quad d\nu' \geq 2 - \alpha', \qquad (4)$$

the Buckingham-Gunton-Stell (BGS) inequalities[4]

$$d\gamma'/(2\beta + \gamma') \geq 2 - \eta \qquad (5a)$$

and

$$d(\delta - 1)/(\delta + 1) \geq 2 - \eta, \qquad (5b)$$

and the Fisher inequalities[5]

$$d\alpha' \geq 2 - \eta_E, \qquad (6a)$$

$$d\alpha_c/(1 + \zeta + \alpha_c) \geq 2 - \eta_E, \qquad (6b)$$

and

$$(2 - \eta)\nu \geq \gamma, \qquad (6c)$$

where the exponents used are all defined in Ref. 5 (and $\nu_1 \equiv \nu$).

We shall generalize arguments of the sort used by Fisher in deriving Eqs. (5) and (6) to discuss the situation for general ϕ. In particular, we prove the inequalities

$$\gamma' \geq \frac{2\beta(2-\eta)}{d-2+\eta} + 2\phi\left(\frac{2\beta}{d-2+\eta} - \nu'_\phi\right) \qquad (7a)$$

and

$$\frac{\delta - 1}{\delta} \geq \frac{2(2-\eta)}{\delta(d-2+\eta)} + 2\phi\left(\frac{2}{\delta(d-2+\eta)} - \mu_\phi\right). \qquad (7b)$$

We then proceed to show that the tightness of Eqs. (5a) and (5b) hinges on the condition $B(0) = 0$, where $B(\phi)$ is the quantity in the large parentheses on the right-hand sides of Eqs. (7a) and (7b). That is, in (7a) we have

$$B(\phi) \equiv \frac{2\beta}{d-2+\eta} - \nu'_\phi, \qquad (8a)$$

while in (7b)

$$B(\phi) \equiv \frac{2}{\delta(d-2+\eta)} - \mu_\phi. \qquad (8b)$$

We begin by introducing a function

$$X(T, H, \phi, R) \equiv \sum_{|\vec{r}| \lesssim R} C_2(T, H, \vec{r}) |\vec{r}|^{2\phi}. \qquad (9)$$

Since $C_2(T, H, \vec{r})$ is assumed in postulate (a) to be a positive function, one may apply Hölder's inequality[10] to prove that $\xi_\phi(T, H)$ is a monotonic increasing function of ϕ.[5] One can thereby derive the following two lemmas[11]:

Lemma I. If $x \geq y$ and $xy \neq 0$, then $\nu_x \geq \nu_y$, $\nu'_x \geq \nu'_y$, and $\mu_x \geq \mu_y$.

Lemma II. $\eta_\phi = \eta$ providing $\phi \geq \frac{1}{2}\eta - 1$ and $2 \geq \eta \geq 2-d$. Here the exponent η_ϕ is defined by means of the relation

$$X(T_c,\ H=0,\ \phi,\ R) \sim R^{2-\eta_\phi+2\phi}, \quad R \to \infty . \tag{10}$$

Lemma II is proved in Appendix A.

Combining Eqs. (2) and (9) with postulate (a) (positivity) yields

$$\bar{\chi}(T,\ H)[\xi_\phi(T,\ H)]^{2\phi} \geq X(T,\ H,\ \phi,\ R) , \tag{11}$$

where $\bar{\chi}(T,\ H) \equiv \sum_{\vec{r}} C_2(T,\ H,\ \vec{r}) \equiv X(T,\ H,\ 0,\ \infty)$ is the *reduced* isothermal susceptibility related to χ_T by $\bar{\chi} \equiv (k_B T/g^2\mu_B^2)\chi_T$. Similarly, Eqs. (1) and (9) can be combined with postulate (b) (monotonicity) to show that for $T \leq T_c$

$$X(T,\ H,\ \phi,\ R) \geq X(T_c,\ 0,\ \phi,\ R) - N(\phi,\ R) M^2(T,\ H) , \tag{12}$$

where $N(\phi,\ R) \equiv \sum_{|\vec{r}| \leq R} |r|^{2\phi}$. Equations (11) and (12) then imply that for *all* $T \leq T_c$, $H \geq 0$, ϕ, and R

$$\bar{\chi}(T,\ H)[\xi_\phi(T,\ H)]^{2\phi} \geq X(T_c,\ 0,\ \phi,\ R) - N(\phi,\ R) M^2(T,\ H) . \tag{13}$$

We next derive a generalization of inequality (5a), and we accordingly set $H=0$ and allow $T \to T_c^-$ in (13). We shall consider R, which hitherto has been an arbitrary number, to depend upon temperature and to vary with T such that $R(T) \to \infty$ as $T \to T_c^-$. Following Fisher's derivation of (5a), we choose the function $R(T)$ such that

$$M(T,\ 0) \equiv 2^{-1}[X(T_c,\ 0,\ \phi,\ R(T))/N(\phi,\ R(T))]^{1/2} . \tag{14a}$$

Since $M(T,\ 0) \sim (T_c - T)^\beta$ and $N \sim R^{d+2\phi}$ for large R, we can combine (10) and (14a) to obtain

$$R(T) \sim (T_c - T)^{-2\beta/(d-2+\eta)} . \tag{15a}$$

Finally, we can substitute (14a) into (12) and utilize (10), (15a), (3a), and the definition $\bar{\chi} \sim (T_c - T)^{-\gamma'}$ to obtain the exponent inequality (7a). Note that when $\phi = 0$, (7a) reduces to (5a).[12]

To obtain a generalization of inequality (5b), we approach the critical point along the perpendicular path $T = T_c$, $H \to 0^+$. We choose $R = R(H)$ such that

$$M(T_c,\ H) \equiv 2^{-1}[X(T_c,\ 0,\ \phi,\ R(H))/N(\phi,\ R(H))]^{1/2} . \tag{14b}$$

Since $M(T_c,\ H) \sim H^{1/\delta}$, it follows from (10) and (14b) that

$$R(H) = H^{-2/\delta(d-2+\eta)} . \tag{15b}$$

Finally, we substitute (14b) into (13) and utilize (10), (15b), (3b), and the definition $\bar{\chi} \sim (\partial M/\partial H)_T \sim H^{-(\delta-1)/\delta}$ to obtain (7b).

We are now in a position to discuss the possible "weakness" of Eqs. (5). Since our new inequalities, Eqs. (7), are nothing but Eqs. (5) plus a "correction term" $2\phi B(\phi)$, we observe that (5) is in a sense weaker than (7) if there exists some value of ϕ, $\phi = \phi_0$, such that the correction term is positive $[2\phi_0 B(\phi_0) > 0]$. Now the only case in which we cannot find such a ϕ_0 is if

$$B(\phi) \geq 0 \quad \text{for } \phi < 0 \tag{16a}$$

and

$$B(\phi) \leq 0 \quad \text{for } \phi > 0 . \tag{16b}$$

Thus we conclude that unless $B(0) = 0$, Eqs. (5) are in some sense weaker than Eqs. (7). We may improve the weakness and get *optimum* inequalities by setting $\phi = \phi_{\max}$ in (7), where $2\phi_{\max} B(\phi_{\max})$ is the absolute maximum value of the correction term (in the range $\frac{1}{2}\eta - 1 \leq \phi < \infty$).

It is interesting to note with Fisher[5] that (6a) may be generalized to

$$(2-\eta)\,\nu_\phi \geq \gamma . \tag{17}$$

Hence from (17) and (7a) we have an inequality between *primed* (subcritical) and *unprimed* (supercritical) exponents,

$$\frac{2\phi\nu'_\phi + \gamma'}{2\beta} \geq \frac{2\phi\nu_\phi + \gamma}{d\nu_\phi - \gamma} . \tag{18}$$

III. INEQUALITIES AMONG ENERGY-ENERGY CORRELATION FUNCTION EXPONENTS

For the energy-energy correlation function, Fisher[5] derived (6a) and (6b) by extending postulates (a) and (b) to include, respectively, Γ_4 and C_4, where

$$\Gamma_4(T,\ H,\ \vec{r}_1,\ \vec{r}_2,\ \vec{r}_3) \equiv \langle s_0^z s_{\vec{r}_1}^z s_{\vec{r}_2}^z s_{\vec{r}_3}^z \rangle \tag{19a}$$

and

$$C_4(T,\ H,\ \vec{r}_1,\ \vec{r}_2,\ \vec{r}_3) \equiv \Gamma_4(T,\ H,\ \vec{r}_1,\ \vec{r}_2,\ \vec{r}_3) - \Gamma_2(T,\ H,\ \vec{r}_1)\Gamma_2(T,\ H,\ \vec{r}_3 - \vec{r}_2) . \tag{19b}$$

We find that inequalities (6b) and (6c) are weaker than they need be, a fact pointed out by Fisher.[5]

We may proceed in exactly the same fashion as our previous discussion, Eqs. (9)–(15), to obtain results analogous to Eqs. (7a) and (7b):

$$\alpha' \geq (2-\eta_E)\left(\frac{1-\alpha'}{d-2+\eta_E}\right) + 2\phi\left[\left(\frac{1-\alpha'}{d-2+\eta_E}\right) - \nu'_{E\phi}\right] , \tag{20a}$$

$$\alpha_c \geq (2-\eta_E)\left(\frac{1+\zeta}{d-2+\eta_E}\right) + 2\phi\left[\left(\frac{1+\zeta}{d-2+\eta_E}\right) - \delta\mu_{E\phi}\right] , \tag{20b}$$

where η_E, $\nu'_{E\phi}$, and $\mu_{E\phi}$ are defined precisely as were η, ν'_ϕ, and μ_ϕ in Eqs. (3) and (10), but with C_2 replaced by C_4. Here ζ and α_c are defined by

$$U \sim M^{\zeta+1}, \tag{21a}$$

$$C_H \sim M^{-\alpha_c}. \tag{21b}$$

It then follows (in analogy with the discussion in Sec. II) that (6a) and (6b) are, in general, weak inequalities, except for the special cases when the square brackets on the right-hand sides of (20a) and (20b) are zero for $\phi = 0$.

IV. DISCUSSION

Although we have indicated the possible weakness of inequalities (5a), (5b), (6a), and (6b), we have not proved that they *are* weak. In fact, we have only demonstrated that their possible weakness depends upon whether the correction terms may be positive, and we cannot assess this in general. However, our new inequalities can furnish other information.

For example, consider the $d = 2$ Ising model, for which $\nu_1' = 1$, $\beta = \frac{1}{8}$, $\eta = \frac{1}{4}$, and $\gamma' = \frac{7}{4}$. Equation (5a) is obeyed as an equality. Now in the discussion following (16), we saw that (5a) is an equality only if $B(0) = 0$. Thus we have

$$\frac{2\beta}{d-2+\eta} = \nu_0' = \nu_1', \tag{22}$$

where the first equality follows from (8a) and the second from the numerical values of the exponents. Hence from Lemma I we have

$$\nu_\phi' = \frac{2\beta}{d-2+\eta} \tag{23}$$

for $0 \le \phi \le 1$. Equation (23) has hitherto been obtained only by making the scaling hypothesis (cf. Appendix B), which leads to the prediction that ν_ϕ' is independent of ϕ.

If γ' is finite, then by letting $\phi \to \infty$ in (7a) we have

$$\nu_\infty' \ge \frac{2\beta}{d-2+\eta}. \tag{24}$$

If ν_ϕ' is independent of ϕ (as predicted by scaling), Eq. (24) becomes

$$\nu' \ge \frac{2\beta}{d-2+\eta}. \tag{25}$$

Now (25) may or may not be obeyed as an equality. If it is, then the correction term in (7a) is always zero, and hence one may expect (5a) to be obeyed as an equality. However, if (25) is *not* an equality, then the optimum inequality is obtained by setting $\phi = \phi_{\max} = \frac{1}{2}\eta - 1$ in (7a), with the result

$$\gamma' \ge \nu'(2-\eta). \tag{26}$$

TABLE I. Summary of general results and some special cases. Also shown are the simplified forms one obtains if the exponents ν_ϕ, μ_ϕ are independent of ϕ.

Path 1: $H=0$, $T \to T_c^-$	Path 2: $T = T_c$, $H \to 0$
Two-spin correlation function exponents General result: $\gamma' \ge \frac{2\beta(2-\eta)}{d-2+\eta} + 2\phi\left(\frac{2\beta}{d-2+\eta} - \nu_\phi'\right)$. In particular $\nu_\infty' \ge \frac{2\beta}{d-2+\eta}$, $\gamma' \ge (2-\eta)\nu'_{(\eta/2)-1}$, $\nu_\phi' = \nu' \Rightarrow \gamma' \ge (2-\eta)\nu'$.	Two-spin correlation function exponents General result: $\frac{\delta-1}{\delta} \ge \frac{2(2-\eta)}{\delta(d-2+\eta)} + 2\phi\left(\frac{2}{\delta(d-2+\eta)} - \mu_\phi\right)$. In particular $\mu_\infty \ge \frac{2/\delta}{d-2+\eta}$, $\frac{\delta-1}{\delta} \ge (2-\eta)\mu_{(\eta/2)-1}$, $\mu_\phi = \mu \Rightarrow \frac{\delta-1}{\delta} \ge (2-\eta)\mu$.
Four-spin correlation function exponents General result: $\alpha' \ge (2-\eta_E)\left(\frac{1-\alpha'}{d-2+\eta_E}\right) + 2\phi\left[\left(\frac{1-\alpha'}{d-2+\eta_E}\right) - \nu_{E\phi}'\right]$. In particular $\nu_{E\infty}' \ge \frac{1-\alpha'}{d-2+\eta_E}$, $\alpha' \ge (2-\eta_E)\nu'_{E,(\eta/2)-1}$, $\nu_{E\phi}' = \nu_E' \Rightarrow \alpha' \ge (2-\eta_E)\nu_E'$.	Four-spin correlation function exponents General result: $\alpha_c \ge (2-\eta_E)\left(\frac{1+\zeta}{d-2+\eta_E}\right) + 2\phi\left[\left(\frac{1+\zeta}{d-2+\eta_E}\right) - \delta\mu_{E\phi}\right]$. In particular $\delta\mu_{E\infty} \ge \frac{1+\zeta}{d-2+\eta_E}$, $\alpha_c \ge (2-\eta_E)\delta\mu_{E,(\eta/2)-1}$, $\mu_{E\phi} = \mu_E \Rightarrow \alpha_c \ge (2-\eta_E)\delta\mu_E$.

Note that inequality (26) is similar to (6c), but the direction of the inequality sign is reversed! Of course, (26) is far from rigorous—but the above conditions are at least plausible.

In summary, then, by considering Eqs. (2) and (11) for general ϕ, we have derived certain additional inequalities among the critical-point exponents describing the spin-spin and energy-energy correlation functions: (7a) and (7b), (17), (18), (20a), (20b), and (24). If we further make the assumption that $\nu'_\phi = \nu'$ for all ϕ, then the inequality $\gamma' \geq \nu'(2-\eta)$ is obtained. The new inequalities are summarized in Table I.

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APPENDIX A: PROOF OF $\eta_\phi = \eta$

To show η_ϕ is independent of ϕ, we need to use postulate (a) of the text,

$$C_{2c}(\vec{r}) \equiv C_2(T_c,\ H=0,\ \vec{r}) \geq 0, \tag{A1}$$

and we need to assume that the exponent η_ϕ defined in Eq. (10) has meaning,

$$X_c(\phi,\ R) \equiv \sum_{|\vec{r}| \lesssim R} |\vec{r}|^{2\phi} C_{2c}(\vec{r}) \sim R^{2-\eta_\phi+2\phi}. \tag{A2}$$

We begin by considering two values of ϕ, a and b, and we define $c \equiv a-b$, which we take to be positive. First we show $\eta_a \geq \eta_b$ and then we shall show $\eta_b \geq \eta_a$.

From (A1) we have

$$X_c(a,\ R) \equiv \sum_{|\vec{r}| \lesssim R} |\vec{r}|^{2b+2c} C_{2c}(\vec{r}) \leq \sum_{|\vec{r}| \lesssim R} |\vec{r}|^{2b} C_{2c}(\vec{r}) R^{2c}. \tag{A3}$$

Thus

$$X_c(a,\ R) \leq R^{2c} X_c(b,\ R) \tag{A4}$$

and we have, from (A2), that

$$\eta_a \geq \eta_b . \tag{A5}$$

To obtain $\eta_b \geq \eta_a$, we use (A1) to write

$$X_c(b,\ R) \equiv X_c(b,\ R_0) + \sum_{R_0 \lesssim |\vec{r}| \lesssim R} |\vec{r}|^{2a-2c} C_{2c}(\vec{r}) \leq X_c(b,\ R_0) + R_0^{-2c} X_c(a,\ R) . \tag{A6}$$

Since (A6) holds for any value of R_0 in the range $1 \leq R_0 \leq R$, we let $R_0 \to \infty$ with R by means of the relation

$$X_c(b,\ R) \equiv 2X_c(b,\ R_0) . \tag{A7}$$

Note that since $C_{2c}(r)$ is positive definite, $R_0 \leq R$ and $R_0 \sim R$ as $R \to \infty$. Hence on substituting (A7) into (A6), we obtain

$$X_c(b,\ R_0) \leq R_0^{-2c} X_c(a,\ R) . \tag{A8}$$

Since $R_0 \sim R$, (A8) leads to

$$2-\eta_b+2b \leq -2c+2-\eta_a+2a, \tag{A9}$$

which simplifies to

$$\eta_b \geq \eta_a . \tag{A10}$$

Combining (A5) and (A10), we have

$$\eta_a = \eta_b , \tag{A11}$$

and the proof is complete.

It is now clear from (A2) that Lemma II is meaningful only for $2-\eta+2\phi \geq 0$ or

$$\phi \geq \tfrac{1}{2}\eta - 1 . \tag{A12}$$

APPENDIX B: PROOF THAT THE HOMOGENEITY HYPOTHESIS IMPLIES $\nu_\phi = \nu$ FOR ALL ϕ

The following proof, while straightforward, appears nowhere in the literature. Accordingly, we feel it might be worthwhile including it here.

Let us assume that $C_2(T,\ H,\ \vec{r})$ is asymptotically a generalized homogeneous function—i.e., that we can find three numbers b_τ, b_H, and b_r such that for all positive λ

$$C_2(\lambda^{b_\tau}\tau,\ \lambda^{b_H}H,\ \lambda^{b_r}\vec{r}) = \lambda C_2(\tau,\ H,\ \vec{r}) , \tag{B1}$$

where here we measure temperature in the units $\tau \equiv T - T_c$. Then it follows from simple properties of generalized homogeneous functions that the generalized correlation length of Eq. (2) obeys the functional equation[13]

$$\xi_\phi(\lambda^{b_\tau}\tau,\ \lambda^{b_H}H) = \lambda^{b_r}\xi_\phi(\tau,\ H) , \tag{B2}$$

and one sees that

$$\nu_\phi = -b_r/b_\tau , \tag{B3a}$$

$$\mu_\phi = -b_r/b_H , \tag{B3b}$$

completing the proof.

Similarly, from (9) and (10) we have

$$d-2+\eta = -1/b_r . \tag{B4}$$

Now (B1) implies that the spatial Fourier transform of $C_2(\tau,\ H,\ \vec{r})$ obeys[13]

$$C_2(\lambda^{b_\tau}\tau,\ \lambda^{b_H}H,\ \lambda^{b_q}\vec{q}) = \lambda^{1-db_q} C_2(\tau,\ H,\ \vec{q}) , \tag{B5}$$

with $b_q = -b_r$. Since $\chi(\tau,\ H) = C_2(\tau,\ H,\ q=0)$, we have

$$-\gamma' = (1+db_r)/b_\tau \tag{B6a}$$

and

$$-(\delta-1)/\delta=(1+db_r)/b_H . \tag{B6b}$$

Combining (B3), (B4), and (B6), we obtain

$$\gamma'=\nu_\phi(2-\eta), \tag{B7a}$$

$$(\delta-1)/\delta=\mu_\phi(2-\eta). \tag{B7b}$$

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‡Present address: Physics Department, Room 13-2134, MIT, Cambridge, Mass. 02139.

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[10]See G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge U. P., London, 1959), 2nd ed., Theorem 16.

[11]Note that an unfortunate reversal of the inequality sign occurs in Fisher's statement of Lemma I (Ref. 5). We wish to thank Professor Fisher for having called this to our attention.

[12]Note that although Fisher (Ref. 5) uses $\phi=1$ in Eq. (2) [his Eq. (25)] to define the correlation length, he actually starts his derivation of the inequalities (5) from Eq. (11) [his Eq. (27)] by taking $\phi=0$. Hence although our ν_ϕ reduces to ν for $\phi=1$, our new inequalities reduce to old inequalities by taking $\phi=0$.

[13]A. Hankey and H. E. Stanley, see Ref. 8.

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Neutron-Diffraction Study of the Magnetic Structure of the Trirutile $LiFe_2F_6$

G. Shachar, J. Makovsky, and H. Shaked
Nuclear Research Center-Negev, P. O. Box 9001, Beer Sheva, Israel
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A neutron-diffraction study of polycrystalline $LiFe_2F_6$ was performed. This compound is of the trirutile structure ($D_{4h}^{14}-P4_2/mnm$) and is paramagnetic at room temperature. The previously reported transition to antiferromagnetism at $T_N\sim 105\,^\circ\mathrm{K}$ is confirmed. The magnetic structure was found to be collinear, with the spins parallel to the tetragonal axis. Nearest and next-nearest neighbors are coupled ferro and antiferromagnetically, respectively. The magnetic space group is $P4_2'/mnm'$. It is argued that this structure is to be expected on the basis of known data for the rutiles: MnF_2, FeF_2, and Fe-doped MnF_2. It was also found that at low temperatures the iron and fluorine ions are shifted from their room-temperature positions.

I. INTRODUCTION

The compound $LiFe_2F_6$ is a member of the family of compounds whose chemical formula can be written as $LiA^{+2}B^{+3}F_6$ (A, B, = transition metals). Several compounds of this family were investigated by Viebahn *et al.*[1,2] and were found to be of the trirutile structure. This structure belongs to the tetragonal space group $D_{4h}^{14}-P4_2/mnm$. X-ray studies by Portier *et al.*[3] showed that also $LiFe_2F_6$ is of the trirutile structure with $a=4.673$ Å and $c=9.290$ Å. Susceptibility measurements[3] showed that this compound undergoes a para to anti-ferromagnetic transition at $T_N\sim 105\,^\circ\mathrm{K}$. In the present work we report on the low-temperature magnetic structure and crystallographic distortion in