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Fractal-to-nonfractal crossover for viscous fingers

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We propose a position-space renormalization-group approach to the problem of viscous fingering in the absence of surface tension, with an arbitrary viscosity ratio between the injected and displaced fluid. We find that there are only two fixed points, the Eden and the diffusion-limited aggregation (DLA) points. The Eden point, which corresponds to a compact cluster with a nonfractal surface, is stable in all directions, while the DLA fixed point is a saddle point. Hence if the viscosity of the injected fluid is not zero, the system *must eventually cross over to a compact cluster.* We also calculate the crossover exponent ϕ and crossover radius $R_\times$, and discuss possible experimental measurements.

How can patterns arise from processes which are random at the microscopic level? This question has recently occupied the attention of many investigators, ranging from physicists and mathematicians to materials scientists and embryologists. A problem that has come to serve as one paradigm of pattern formation is the viscous-fingering phenomenon.[1] In its simplest form, one injects a fluid of viscosity μ_1 into a fluid of higher viscosity μ_2, using a Hele-Shaw cell—a two-dimensional geometry in which the fluid is confined between two large transparent plates separated by a distance b on the order of a millimeter.[2] If the plates are circular disks and the injected fluid enters through a hole at the center, the viscous fingers have radial symmetry.

In the limiting case in which the interfacial tension between the injected and displaced fluid may be neglected and the viscosity ratio $\mu_1/\mu_2 \to 0$, one expects to find patterns that are isomorphic to diffusion-limited aggregation (DLA), since the Laplace equation underlies both the experimental phenomenon and the model system.[3] DLA patterns are relatively simple to recognize since they may be quantified by their fractal dimension d_f, and the value of d_f ($\simeq 1.7$) for two dimensions is far from the compact value of two.[4] However, experimental evidence on this point has been contradictory and no clear picture has emerged.[5-7]

There are many *possible* sources for the lack of agreement between experiments and the DLA model. In particular, the viscosity of the injected fluid is not zero, but rather is typically a factor of 10^2–10^4 smaller than that of the displaced fluid.[5-7] Recent computer simulations for viscous fingering with a nonzero viscosity ratio are inconclusive, displaying patterns that are rather difficult to analyze *quantitatively*; qualitatively, the "core" is compact, but the rest of the pattern is roughly as ramified as DLA.[8-10] Thus an open question concerns the asymptotic behavior of viscous fingering in real experimental situations in which the viscosity of the injected fluid is not strictly zero.

In this Rapid Communication we propose an answer to this question. Using a position-space renormalization-group (PSRG) approach, we find that for *any* nonzero viscosity ratio, the system must evolve into a compact cluster with a nonfractal surface. We also propose a scaling theory for the way the system evolves into a compact cluster, where the evolution is described by crossover radius $R_\times$ or crossover exponent ϕ.

First we consider the basic equations for the Hele-Shaw problem. In order to simplify the problem, we assume that both fluids are Newtonian with zero interfacial tension. We start with the Darcy law

$$\mathbf{v}_i = -k_i \nabla P_i \,. \tag{1a}$$

Here $\mathbf{v}_i$ is the velocity field, P_i the pressure field, and $k_i \equiv b^2/12\mu_i$ is the permeability coefficient, and the index $i=1,2$ indicates injected fluid and displaced fluid, respectively. Conservation of volume implies (for incompressible fluids) that

$$\nabla \cdot \mathbf{v}_i = 0 \,. \tag{1b}$$

From (1) we conclude that P_i should satisfy the Laplace equation, $\nabla^2 P_i = 0$.

The equation of motion for the interface is $v_n = -k_1 \hat{\mathbf{n}} \cdot \nabla P_1 = -k_2 \hat{\mathbf{n}} \cdot \nabla P_2$. Here v_n is the normal velocity of the interface, and $\hat{\mathbf{n}}$ is the unit vector normal to the interface. In the absence of surface tension, the pressure field must be continuous across the interface. These equations, together with the pressure field at the boundary, completely determine the time evolution of the system.

Consider now a system made of two different materials whose conductivities are s_1 and s_2. Connect an external current source to the system, and keep the electric poten-

tial at the junction fixed. Then steady electric current will pass through each material, with current density $\mathbf{j}_i$ satisfying Ohm's law

$$\mathbf{j}_i = -s_i \nabla \phi_i . \tag{2a}$$

Electric charge conservation implies

$$\nabla \cdot \mathbf{j}_i = 0 . \tag{2b}$$

Here ϕ_i is the electric potential, and the index i specifies the two materials. We apply the interface evolution equation $v_n = -s_1 \hat{\mathbf{n}} \cdot \nabla \phi_1 = -s_2 \hat{\mathbf{n}} \cdot \nabla \phi_2$. Due to the formal similarity between Eqs. (1) and (2), the fluid flow problem becomes isomorphic to the electric current problem.

The next step is to discretize the system. We cover the system with a square "net," where every bond is considered to be a resistor, and we assign conductances σ_1 and σ_2 to the bonds covering the injected and displaced fluid, and refer to them as "injected bond" and "displaced bond," respectively. The discretized rule for the interface evolution is that a bond with conductance σ_2 at the *interface* can be transformed to a bond with conductance σ_1. We refer to injected bonds in the "growth zone" as A bonds, and corresponding displaced bonds as B bonds. The probability of the transformation p_k for the kth bond in one time step (not a real time) is

$$p_k = \frac{(\nabla \phi)_k}{\sum_k (\nabla \phi)_k} , \tag{3}$$

where $(\nabla \phi)_k$ is the potential drop across the kth bond at the interface.[1]

Next we consider the small-cell PSRG. In the same spirit as the PSRG for percolation,[11] we only consider the y direction. Therefore, we apply periodic boundary conditions in the x direction of the cell. Without loss of generality, we set $\sigma_2 = 1$. Therefore, the conductances of the unrenormalized bonds of types A and B are $\sigma_A = \sigma_1$ and $\sigma_B = 1$, respectively. The renormalization rules are (i) if the injected bonds span the cell in the vertical direction, we replace the cell with a renormalized surface bond of conductance σ'_A. (ii) If the injected bonds do *not* span the cell in the vertical direction, we replace the cell with a renormalized surface bond of conductance σ'_B.

The renormalized conductances σ'_A and σ'_B are defined as the *average* vertical conductances of configurations satisfying rules (i) and (ii), respectively.[12] For a small cell, we can consider all the injected bonds as type-A bonds.

In order to calculate the renormalized conductances, we must know every possible cell configuration and its weight. Let C_α be the weight of configuration α (i.e., the probability of getting configuration α if we randomly choose one configuration from all possible configurations). The possible configurations and their weights are determined by the interface evolution rule discussed before.

As the cell size increases the number of possible configurations N_c becomes enormous ($N_c = 2.8 \times 10^{10}$ for the $L = 3$ cell). Thus N_c is much larger than for the analogous PSRG calculation for bond percolation, where the total number of configurations for a $L \times L$ cell is 2^{2L^2-L} (3.2×10^4 for $L = 3$). The large values N_c of the Hele-Shaw problem come from the fact that for a growth process there is a "history dependence." This point can be illustrated by considering two configurations that look alike. In the growth process, in contrast to percolation, these two configurations can be generated through different histories, which in general give different weights.

Since N_c is so vast, it is very difficult even by computer to enumerate all possible configurations. Moreover, we must solve the Laplace equation for every configuration in order to determine all the growth probabilities and total conductances, and we must do all these calculations for *each* renormalization step. Hence, we take an alternate approach. First, we find that some branches in the "growth tree structure" are identical. Therefore, instead of considering two branches separately, we can consider only one branch with symmetry factor 2. Motivated by this idea, we apply an algorithm[13] that recognizes these symmetries and produces a "reduced-growth tree structure," and thereby reduces N_c to 20 for $L = 2$ and to 3124 for $L = 3$.

We now proceed to define the renormalized conductances σ'_A and σ'_B. We denote by $\mathcal{S}_\alpha(\sigma_A, \sigma_B)$ the conductance in the vertical direction of configuration α. We consider two versions of the conductance renormalization. In the first version, the renormalized conductances are the *algebraic* averages of configurations satisfying the renormalization rules. Thus

$$\sigma'_A = \langle \mathcal{S}_\alpha(\sigma_A, \sigma_B) \rangle_{\alpha \in S} \tag{4a}$$

and

$$\sigma'_B = \langle \mathcal{S}_\alpha(\sigma_A, \sigma_B) \rangle_{\alpha \in U} , \tag{4b}$$

where S is the set of configurations that satisfy rule (i) and U is the set of configurations that satisfy rule (ii), and $\langle X_\alpha \rangle_{\alpha \in S} \equiv \sum_{\alpha \in S} C_\alpha X_\alpha / \sum_{\alpha \in S} C_\alpha$. Alternatively, we may use *geometric* averages,

$$\ln \sigma'_A = \langle \ln \mathcal{S}_\alpha(\sigma_A, \sigma_B) \rangle_{\alpha \in S} \tag{5a}$$

and

$$\ln \sigma'_B = \langle \ln \mathcal{S}_\alpha(\sigma_A, \sigma_B) \rangle_{\alpha \in U} . \tag{5b}$$

To find the fixed points of these renormalization equations, we randomly choose a point in the parameter space (ρ_B, ρ_A), where $\rho_B \equiv 1/\sigma_B$ and $\rho_A \equiv 1/\sigma_A$, and calculate the renormalized conductances using (4) or (5), to find a new point (ρ'_B, ρ'_A). We repeat this process to find another point (ρ''_B, ρ''_A), and continue until we approach a stable fixed point (ρ^*_B, ρ^*_A). We used 100 initial points and plot the renormalization flow. Figure 1(a) shows part of the flow for $L = 2$, obtained by using (5). The points on the $\rho_A = 0$ line (x axis) need special attention. For these points, the renormalization equations for σ_A, Eqs. (4a) and (5a), becomes trivial. Since σ'_A is simply an average conductance of the spanning configuration, and all spanning configurations have infinite conductance (note that $\sigma_A = \infty$), σ'_A also should be infinity. Therefore, we only need to consider Eqs. (4b) or (5b) with $\sigma_A = \infty$. Since there is a fixed point on the x axis which is found to be a saddle point, this decoupling of the equations is particularly important in order to converge to the fixed point. The renormalization flow on the $\rho_A = 0$ line obtained in

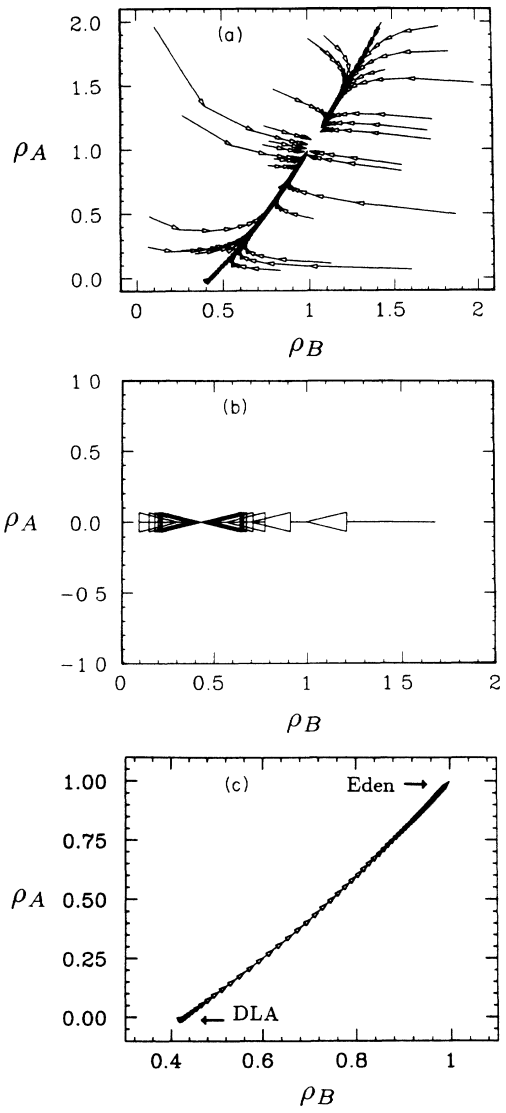

FIG. 1. (a) The renormalization flow for $L=2$ with the *geometric* average rule. Part (a) is for arbitrary (ρ_B,ρ_A), while part (b) is for the $\rho_A=0$ line. Note the stable fixed point at (1,1) in (a), and the saddle point on the x axis. (c) shows the crossover line from the DLA to the Eden fixed point.

this way is shown in Fig. 1(b).

From the renormalization flow, we identify two fixed points. One fixed point, which is on the x axis, corresponds to the case that the injected fluid has zero viscosity. The asymptotic behavior of the system at this point should be the same as DLA so we call this the DLA fixed point. The other point is $(\rho_B^*=1,\ \rho_A^*=1)$. For this point, the viscosity of the injected fluid is the same as the viscosity of the displaced fluid at the interface. We call this point the Eden fixed point, since every surface bond has equal probability to grow.[14] Therefore, we expect the system to evolve into a compact cluster with a nonfractal surface.

From the renormalization flow, we can determine the stabilities of these fixed points. The DLA fixed point is a saddle point: it is stable in the x direction, and unstable in other directions. The Eden point is stable in every direction. These results are confirmed by linearizing the renormalization equations and calculating the eigenvalues. When we change the cell size ($L=2,3$) and the conductance renormalization rules (4) and (5), the numerical values of the DLA fixed points are changed but the above results concerning the existence of the two fixed points and their stabilities are *not* changed[15] (Table I).

The main result from the flow diagram is that there exists a crossover from a fractal cluster (the DLA fixed point) to a compact cluster (the Eden fixed point). In order to *quantify* this crossover behavior, we define a crossover exponent ϕ and a crossover radius $R_\times$. Consider the flow diagram shown in Fig. 1(a), and note that eventually all the renormalization flows are sucked into the line joining the DLA point to the Eden point. Therefore, one can describe the crossover behavior of the system by concentrating only on this line. This crossover line can be determined by following the renormalization flow which starts from very close to the DLA fixed point. Figure 1(c) shows the crossover line for $L=2$ determined in this fashion. We propose the scaling ansatz in the vicinity of the DLA fixed point

$$M(R,u)=R^{d_f}F(uR^{\phi}), \tag{6a}$$

where M is the mass of the cluster, R is the radius of gyration, d_f is the fractal dimension of DLA, and $u\equiv\sigma_2/\sigma_A$ $(=\rho_A/\rho_B)$. Moreover, we assume

$$F(x)\sim\begin{cases}1 & \text{if } x\ll 1,\\ x^{a} & \text{if } x\gg 1.\end{cases} \tag{6b}$$

To find a relation between ϕ, d_f, and a, note that as R increases, $M(R,u)$ scales as $R^{d_f+\phi a}$. However, the cluster becomes compact, implying

$$d_f+\phi a=2. \tag{7}$$

Thus if one measures the scaling function $F(x)$ and hence a, one can calculate ϕ. Furthermore, for a given value of σ_A, we can estimate $R_\times$. From (6b), $R_\times$ should be the value of the radius which corresponds to $x=1$, that is $R_\times\sim u^{-1/\phi}$.

We estimate ϕ from the largest eigenvalue of the renormalization equation at the DLA fixed point. For $L=2$ with (4), we get $\phi\approx 0.5$.[15] A more precise value of ϕ could be calculated by other methods. Note that the data collapse predicted by Eq. (6a) may be put to direct experi-

TABLE I. The fixed points of the renormalization equations and their stabilities.

Cell size	R.G. rule	DLA point	Eden point
2	Algebraic	(0.321,0) Saddle	(1,1) Stable
	Geometric	(0.430,0) Saddle	(1,1) Stable
3	Algebraic	(0.349,0) Saddle	(1,1) Stable
	Geometric	(0.456,0) Saddle	(1,1) Stable

mental test; from the scaling function, one can also measure ϕ and $R_\times$. There are already some experimental indications that the broad outlines of our approach may be correct; e.g., the measured finger patterns are thicker near the injection point than near the periphery, which is consistent with the numerical simulations,[8] and recent experiments[6] demonstrate that radial viscous fingers have an apparent fractal dimension intermediate between DLA and Eden prediction, exactly what one would expect if the experimental conditions are in the crossover region.

In summary, based on the electrostatic analogy we developed a set of PSRG equations for the Hele-Shaw problem in the absence of surface tension with an arbitrary viscosity ratio between the injected and displaced fluid. We found that there are only two fixed points, the Eden and DLA points. The Eden point is stable in all directions, while the DLA fixed point is found to be a saddle point. Therefore, *for any nonzero viscosity ratio, the system must eventually cross over into a compact cluster with a nonfractal surface.* We also calculated the crossover exponent ϕ and crossover radius $R_\times$.

After submitting this work, we learned that our predicted crossover to the Eden universality class for nonzero viscosity ratio has received excellent numerical support.[16] We also learned that this prediction was to some extent anticipated by a linear stability calculation[17] for the analogous electrodeposition problem. Also, Nagatani has recently applied the present two-parameter renormalization group to DLA with variable sticking probabilities.[18]

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