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Dynamics on Complex Networks and Applications

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Anomalous electrical and frictionless flow conductance in complex networks

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Abstract

We study transport properties such as electrical and frictionless flow conductance on scale-free and Erdős–Rényi networks. We consider the conductance G between two arbitrarily chosen nodes where each link has the same unit resistance. Our theoretical analysis for scale-free networks predicts a broad range of values of G, with a power-law tail distribution $\Phi_{\text{SF}}(G) \sim G^{-g_G}$, where $g_G = 2\lambda - 1$, where λ is the decay exponent for the scale-free network degree distribution. We confirm our predictions by simulations of scale-free networks solving the Kirchhoff equations for the conductance between a pair of nodes. The power-law tail in $\Phi_{\text{SF}}(G)$ leads to large values of G, thereby significantly improving the transport in scale-free networks, compared to Erdős–Rényi networks where the tail of the conductivity distribution decays exponentially. Based on a simple physical 'transport backbone' picture we suggest that the conductances of scale-free and Erdős–Rényi networks can be approximated by $ck_Ak_B/(k_A + k_B)$ for any pair of nodes A and B with degrees k_A and k_B. Thus, a single quantity c, which depends on the average degree $\overline{k}$ of the network, characterizes transport on both scale-free and Erdős–Rényi networks. We determine that c tends to 1 for increasing $\overline{k}$, and it is larger for scale-free networks. We compare the electrical results with a model for frictionless transport, where conductance is defined as the number of link-independent paths between A and B, and find that a similar picture holds. The effects of distance on the value of conductance are considered for both models, and some differences emerge. Finally, we use a recent data set for the AS (autonomous system) level of the Internet and confirm that our results are valid in this real-world example.
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Keywords: Complex networks; Transport; Diffusion; Conductance; Scaling

1. Introduction

Transport in many random structures is 'anomalous', i.e. fundamentally different than that in regular space [1–3]. The anomaly is due to the random substrate on which transport is constrained to take place. Random structures are found in many places in the real world, from oil reservoirs to the Internet, making the study of anomalous transport properties a far-reaching field. In this problem, it is paramount to relate the structural properties of the medium with the transport properties.

An important and recent example of random substrates is that of complex networks. Research on this topic has uncovered their importance for real-world problems as diverse as the World Wide Web and the Internet to cellular networks and sexual-partner networks [4].

Two distinct models describe the two limiting cases for the structure of the complex networks. The first of these is the classic Erdős–Rényi model of random networks [5], for which sites are connected with a link with probability p and disconnected (no link) with probability $1 - p$ (see Fig. 1). In this case the degree distribution $P(k)$, the probability of a node to have k connections, is a Poisson:

$$P(k) \sim \frac{\left(\overline{k}\right)^k e^{-\overline{k}}}{k!}, \tag{1}$$

where $\overline{k} \equiv \sum_{k=1}^{\infty} kP(k)$ is the average degree of the network. Mathematicians discovered critical phenomena through this model. For instance, just as in percolation on lattices, there is a critical value $p = p_c$ above which the largest connected

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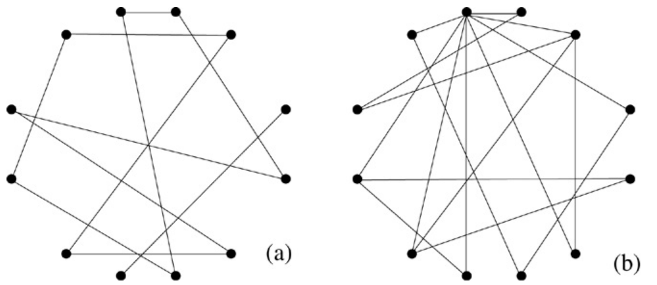

Fig. 1. (a) Schematic of an Erdős–Rényi network of $N = 12$ and $p = 1/6$. Note that in this example ten nodes have $k = 2$ connections, and two nodes have $k = 1$ connections. This illustrates the fact that for Erdős–Rényi networks, the range of values of degree is very narrow, typically close to $\overline{k}$. (b) Schematic of a scale-free network of $N = 12$, $k_{\min} = 2$ and $\lambda \approx 2$. We note the presence of a hub with $k_{\max} = 8$ which is connected to many of the other links of the network.

component of the network has a mass that scales with the system size N, but below p_c, there are only small clusters of the order of $\log N$. Another characteristic of an Erdős–Rényi network is its 'small-world' property which means that the average distance d (or radius) between all pairs of nodes of the network scales as $\log N$ [6]. The other model, recently identified as characterizing the topological structure of many real world systems, is the Barabási–Albert scale-free network and its extensions [7–9], characterized by a scale-free degree distribution (see Fig. 1(b)):

$$P(k) \sim k^{-\lambda} \quad [k_{\min} \leq k \leq k_{\max}]. \tag{2}$$

The cutoff value $k_{\min}$ represents the minimum allowed value of k on the network ($k_{\min} = 2$ except when otherwise noted), and $k_{\max} \equiv k_{\min} N^{1/(\lambda-1)}$, the typical maximum degree of a network with N nodes [10,11]. The scale-free feature allows a network to have some nodes with a large number of links ('hubs'), unlike the case for the Erdős–Rényi model of random networks [5,6]. Scale-free networks with $\lambda > 3$ have $d \sim \log N$, while for $2 < \lambda < 3$ they are 'ultra-small-world' since the radius scales as $d \sim \log \log N$ [4,10].

Here we extend our recent study of transport in complex networks [12,13]. We find that for scale-free networks with $\lambda \geq 2$, transport properties characterized by conductance display a power-law tail distribution that is related to the degree distribution $P(k)$. The origin of this power-law tail is due to pairs of nodes of high degree which have high conductance. Thus, transport in scale-free networks is better because of the presence of large degree nodes (hubs) that carry much of the traffic, whereas Erdős–Rényi networks lack hubs and the transport properties are controlled mainly by the average degree $\overline{k}$ [6,14]. Also, we present a simple physical picture of transport in scale-free and Erdős–Rényi networks and test it through simulations. Additionally, we study a form of frictionless transport, in which transport is measured by the number of independent paths between source and destination. These later results are similar to those in [15]. The results of our study are relevant to problems of diffusion in scale-free and Erdős–Rényi networks, given that conductivity and diffusivity are related by the Einstein relation [1–3].

The paper is structured as follows. Section 2 concentrates on the numerical calculation of the electrical conductance of networks. In Section 3 a simple physical picture gives a theoretical explanation of the results. Section 4 deals with the number of link-independent paths as a form of transport. In Section 5 we present the conclusions and summarize the results in a coherent picture.

2. Transport in complex networks

Most of the work done so far regarding complex networks has concentrated on static topological properties or on models for their growth [4,10,8,16]. Transport features have not been extensively studied with the exception of random walks on specific complex networks [17–19]. Transport properties are important because they contain information about network function [20]. Here we study the electrical conductance G between two nodes A and B of Erdős–Rényi and scale-free networks when a potential difference is imposed between them. We assume that all the links have equal resistances of unit value [21].

To construct an Erdős–Rényi network, we begin with N nodes and connect each pair with probability p. To generate a scale-free network with N nodes, we use the Molloy–Reed algorithm [22], which allows for the construction of random networks with arbitrary degree distribution. We generate k_i copies of each node i, where k_i is a random number taken from a distribution of the form $P(k_i) \sim k_i^{-\lambda}$. We then randomly pair these copies of the nodes in order to construct the network, making sure that two previously-linked nodes are not connected again, and also excluding links of a node to itself [23].

We calculate the conductance G of the network between two nodes A and B using the Kirchhoff method [24], where entering and exiting potentials are fixed to $V_A = 1$ and $V_B = 0$. We solve the set of $N - 2$ linear equations

$$\sum_{j=1, j\neq i}^{N} \frac{V_j - V_i}{r_{ij}} = 0, \quad \forall i \neq A, B \tag{3}$$

representing the conservation of current at the nodes. The resistances r_{ij} are 1 if nodes i and j are connected, and infinite if i and j are not connected. Finally, the total current $I \equiv G$ entering at node A and exiting at node B is computed by adding the outgoing currents from A to its nearest neighbours through $\sum_j (V_A - V_j)$, where j runs over the neighbours of A.

First, we analyze the probability density function (pdf) $\Phi(G)$ which comes from $\Phi(G)\mathrm{d}G$, the probability that two nodes on the network have conductance between G and $G + \mathrm{d}G$. To this end, we introduce the cumulative distribution $F(G) \equiv \int_G^\infty \Phi(G')\mathrm{d}G'$, shown in Fig. 2(a) for the Erdős–Rényi and scale-free ($\lambda = 2.5$ and $\lambda = 3.3$, with $k_{\min} = 2$) cases. We use the notation $\Phi_{\mathrm{SF}}(G)$ and $F_{\mathrm{SF}}(G)$ for scale-free, and $\Phi_{\mathrm{ER}}(G)$ and $F_{\mathrm{ER}}(G)$ for Erdős–Rényi. The function $F_{\mathrm{SF}}(G)$ for both $\lambda = 2.5$ and 3.3 exhibits a tail region well fit by the power law

$$F_{\mathrm{SF}}(G) \sim G^{-(g_G - 1)}, \tag{4}$$

and the exponent $(g_G - 1)$ increases with λ. In contrast, $F_{\mathrm{ER}}(G)$ decreases exponentially with G.

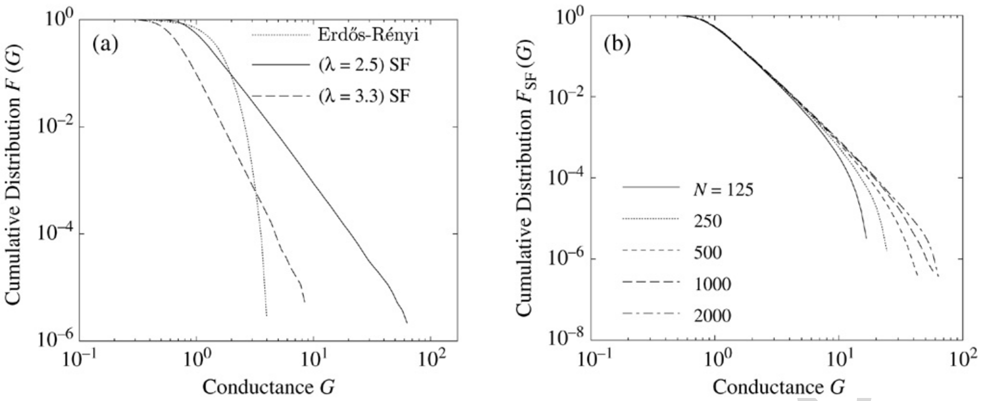

Fig. 2. (a) Comparison for networks with $N = 8000$ nodes between the cumulative distribution functions of conductance for the Erdős–Rényi and the scale-free cases (with $\lambda = 2.5$ and 3.3). Each curve represents the cumulative distribution $F(G)$ vs. G. The simulations have at least 10^6 realizations. (b) Effect of system size on $F_{\rm SF}(G)$ vs. G for the case $\lambda = 2.5$. The cutoff value of the maximum conductance $G_{\max}$ progressively increases as N increases.

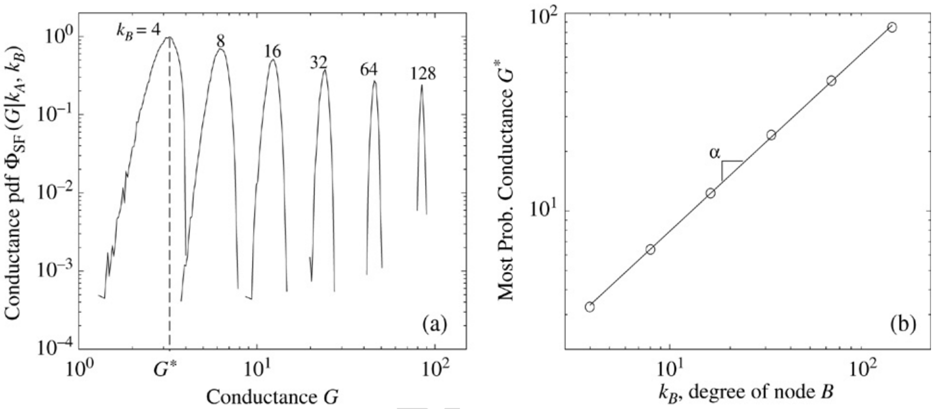

Fig. 3. (a) The pdf $\Phi_{\rm SF}(G|k_A, k_B)$ vs. G for $N = 8000$, $\lambda = 2.5$ and $k_A = 750$ (k_A is close to the typical maximum degree $k_{\max} = 800$ for $N = 8000$). (b) Most probable values G^*, estimated from the maxima of the distributions in Fig. 3(a), as a function of the degree k_B. The data support a power law behavior $G^* \sim k_B^\alpha$ with $\alpha = 0.96 \pm 0.05$.

Increasing N does not significantly change $F_{\rm SF}(G)$ (Fig. 2(b)) except for an increase in the upper cutoff $G_{\max}$, where $G_{\max}$ is the typical maximum conductance, corresponding to the value of G at which $\Phi_{\rm SF}(G)$ crosses over from a power law to a faster decay. We observe no change of the exponent g_G with N. The increase of $G_{\max}$ with N implies that the average conductance $\overline{G}$ over all pairs also increases slightly.

We next study the origin of the large values of G in scale-free networks and obtain an analytical relation between λ and g_G. Larger values of G require the presence of many parallel paths, which we hypothesize arise from the high degree nodes. Thus, we expect that if either of the degrees k_A or k_B of the entering and exiting nodes is small (e.g. $k_A > k_B$), the conductance G between A and B is small since there are at most k different parallel branches coming out of a node with degree k. Thus, a small value of k implies a small number of possible parallel branches, and therefore a small value of G. To observe large G values, it is therefore necessary that both k_A and k_B be large.

We test this hypothesis by large scale computer simulations of the conditional pdf $\Phi_{\rm SF}(G|k_A, k_B)$ for specific values of the entering and exiting node degrees k_A and k_B. Consider first $k_B \ll k_A$, and the effect of increasing k_B, with k_A fixed. We find that $\Phi_{\rm SF}(G|k_A, k_B)$ is narrowly peaked (Fig. 3(a)) so that it is well characterized by G^*, the value of G when $\Phi_{\rm SF}$ is a maximum. We find similar results for Erdős–Rényi networks. Further, for increasing k_B, we find [Fig. 3(b)] G^* increases as $G^* \sim k_B^\alpha$, with $\alpha = 0.96 \pm 0.05$ consistent with the possibility that as $N \to \infty$, $\alpha = 1$ which we assume henceforth.

For the case of $k_B \gtrsim k_A$, G^* increases less fast than k_B, as can be seen in Fig. 4 where we plot G^*/k_B against the scaled degree $x \equiv k_A/k_B$. The collapse of G^*/k_B for different values of k_A and k_B indicates that G^* scales as:

$$G^* \sim k_B f\left(\frac{k_A}{k_B}\right). \tag{5}$$

Below we study the possible origin of this function.

3. Transport backbone picture

The behaviour of the scaling function $f(x)$ can be interpreted using the following simplified 'transport backbone' picture [Fig. 4 inset], for which the effective conductance G between nodes A and B satisfies:

$$\frac{1}{G} = \frac{1}{G_A} + \frac{1}{G_{\rm tb}} + \frac{1}{G_B}, \tag{6}$$

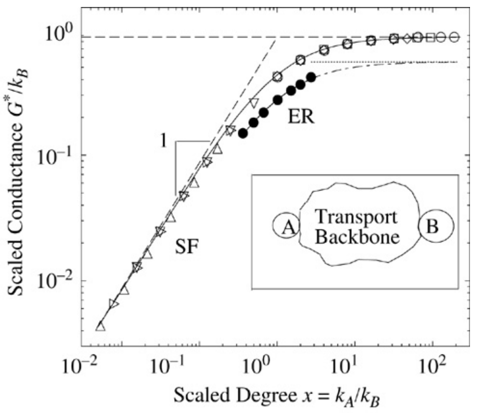

Fig. 4. Scaled most probable conductance G^*/k_B vs. scaled degree $x \equiv k_A/k_B$ for system size $N = 8000$ and $\lambda = 2.5$, for several values of k_A and k_B: □ ($k_A = 8$, $8 \le k_B \le 750$), ◇ ($k_A = 16$, $16 \le k_B \le 750$), △ ($k_A = 750$, $4 \le k_B \le 128$), ○ ($k_B = 4$, $4 \le k_A \le 750$), ▽ ($k_B = 256$, $256 \le k_A \le 750$), and ▷ ($k_B = 500$, $4 \le k_A \le 128$). The curve crossing the symbols is the predicted function $G^*/k_B = f(x) = cx/(1+x)$ obtained from Eq. (8). We also show G^*/k_B vs. scaled degree $x \equiv k_A/k_B$ for Erdős–Rényi networks with $\bar{k} = 2.92$, $4 \le k_A \le 11$ and $k_B = 4$ (symbol •). The curve crossing the symbols represents the theoretical result according to Eq. (8), and an extension of this line to represent the limiting value of G^*/k_B (dotted-dashed line). The probability of observing $k_A > 11$ is extremely small in Erdős–Rényi networks, and thus we are unable to obtain significant statistics. The scaling function $f(x)$, as seen here, exhibits a crossover from a linear behavior to the constant c ($c = 0.87 \pm 0.02$ for scale-free networks, horizontal dashed line, and $c = 0.55 \pm 0.01$ for Erdős–Rényi, dotted line). The inset shows a schematic of the 'transport backbone' picture, where the circles labeled A and B denote nodes A and B and their associated links which do not belong to the 'transport backbone'.

where $1/G_{\text{tb}}$ is the resistance of the 'transport backbone' while $1/G_A$ (and $1/G_B$) are the resistances of the set of links near node A (and node B) not belonging to the 'transport backbone'. It is plausible that G_A is linear in k_A, so we can write $G_A = ck_A$. Since node B is equivalent to node A, we expect $G_B = ck_B$. Hence:

$$G = \frac{1}{1/ck_A + 1/ck_B + 1/G_{\text{tb}}} = k_B \frac{ck_A/k_B}{1 + k_A/k_B + ck_A/G_{\text{tb}}}, \tag{7}$$

so the scaling function defined in Eq. (5) is

$$f(x) = \frac{cx}{1 + x + ck_A/G_{\text{tb}}} \approx \frac{cx}{1+x}. \tag{8}$$

The second equality follows if there are many parallel paths on the 'transport backbone' so that $1/G_{\text{tb}} \ll 1/ck_A$ [25]. The prediction (8) is plotted in Fig. 4 for both scale-free and Erdős–Rényi networks and the agreement with the simulations supports the approximate validity of the transport backbone picture of conductance in scale-free and Erdős–Rényi networks.

The agreement of (8) with simulations has a striking implication: the conductance of a scale-free and Erdős–Rényi networks depends on only one quantity c. Further, since the distribution of Fig. 3(a) is sharply peaked, a single measurement of G for any values of the degrees k_A and k_B of the entrance and exit nodes suffices to determine G^*, which then determines c and hence through Eq. (8) the conductance for all values of k_A and k_B.

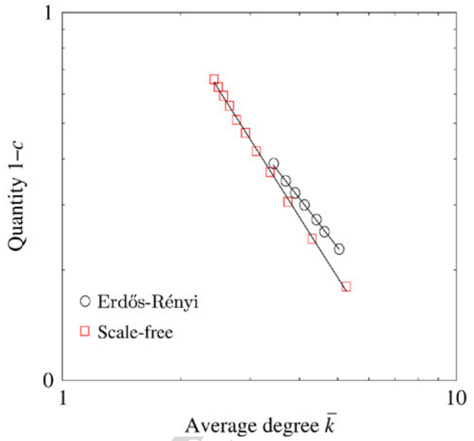

Fig. 5. Parameter $1 - c$ vs. $\bar{k}$ for scale-free and Erdős–Rényi networks with $N = 8000$. The scale-free networks display a power-law decay with exponent -1.69 ± 0.02, whereas the Erdős–Rényi networks exhibit a decay exponent of -1.37 ± 0.02.

With regards to quantity c, first note it should grow, up to its upper limit 1, as the number of connections increases. For instance, a complete graph has conductance $N/2$ which, if compared to Eq. (7), indicates that indeed $c \to 1$. This suggests testing c as a function of the average degree $\bar{k}$. In Fig. 5 we present results for both scale-free and Erdős–Rényi networks. The most important feature is that there seems to be a power-law decay of $1 - c$ with respect to $\bar{k}$. We find that the dependence is of the form $1 - c \sim \bar{k}^q$, with $q = -1.37 \pm 0.02$ for Erdős–Rényi and $q = -1.69 \pm 0.02$ for scale-free. Also, we observe that c for Erdős–Rényi networks, at least in the region of $\bar{k}$ studied, is lower than for scale-free networks. As $\bar{k}$ increases, transport on scale-free networks becomes increasingly better than in Erdős–Rényi networks, because c is closer to one for the same $\bar{k}$.

Within this 'transport backbone' picture, we can analytically calculate $F_{\text{SF}}(G)$. The key insight necessary for this calculation is that $G^* \sim k_B$, when $k_B \le k_A$, and we assume that $G \sim k_B$ is also valid given the narrow shape of $\Phi_{\text{SF}}(G|k_A, k_B)$. This implies that the probability of observing conductance G is related to k_B through $\Phi_{\text{SF}}(G)\text{d}G \sim M(k_B)\text{d}k_B$, where $M(k_B)$ is the probability that, when two nodes A and B are chosen at random, k_B is the minimum degree. This can be calculated analytically through:

$$M(k_B) \sim P(k_B) \int_{k_B}^{k_{\max}} P(k_A)\text{d}k_A. \tag{9}$$

Performing the integration we obtain for $G < G_{\max}$

$$\Phi_{\text{SF}}(G) \sim G^{-g_G} \quad [g_G = 2\lambda - 1]. \tag{10}$$

Hence, for $F_{\text{SF}}(G)$, we have $F_{\text{SF}}(G) \sim G^{-(2\lambda-2)}$. To test this prediction, we perform simulations for scale-free networks and calculate the values of $g_G - 1$ from the slope of a log–log plot of the cumulative distribution $F_{\text{SF}}(G)$. From Fig. 6(b) we find that

$$g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13). \tag{11}$$

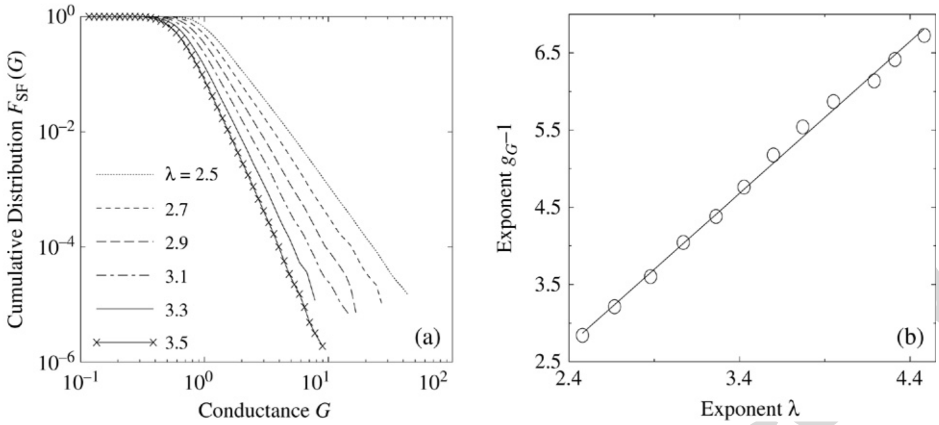

Fig. 6. (a) Simulation results for the cumulative distribution $F_{\mathrm{SF}}(G)$ for λ between 2.5 and 3.5, consistent with the power law $F_{\mathrm{SF}} \sim G^{-(g_G-1)}$ (cf. Eq. (10)), showing the progressive change of the slope $g_G - 1$. (b) The exponent $g_G - 1$ from simulations (circles) with $2.5 < \lambda < 4.5$; shown also is a least square fit $g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13)$, consistent with the predicted expression $g_G - 1 = 2\lambda - 2$ [cf. Eq. (10)].

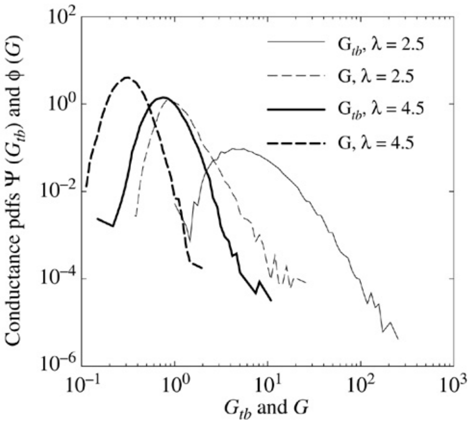

Fig. 7. Comparison of pdf $\Psi(G_{\mathrm{tb}})$ and $\Phi(G)$ for networks of $N = 8000$ for two values of λ.

Thus, the measured slopes are consistent with the theoretical values predicted by Eq. (10) [26].

The transport backbone conductance G_{tb} of scale-free networks can also be studied through its pdf Ψ_{SF} (see Fig. 7). To determine G_{tb}, we consider the contribution to the conductance of the part of the network with paths between A and B, excluding the contributions from the vicinities of nodes A and B, which are determined by the quantity c. The most relevant feature in Fig. 7 is that, for a given λ value, both Ψ_{SF} and $\Phi(G)$ have equal decay exponents, suggesting that they are also related to λ as Eq. (11). Fig. 7 also shows that the values of G_{tb} are significantly larger than G.

4. Number of link-independent paths: Transport without friction

In many systems, it is the nature of the transport process that the particles flowing through the network links experience no friction. For example, this is the case in an electrical system made of super-conductors [27], or water flow along pipes, if frictional effects are minor. Other examples are flow of cars along traffic routes, and perhaps most important, the transport of information in communication networks. Common to all these processes is that, the quality of the transport is determined by the number of link-independent paths leading from the source to the destination (and the capacity of each path), and not by the length of each path (as is the case for simple electrical conductance). In this section, we focus on nonweighted networks, and define the conductance, as the number of link-independent paths between a given source and destination A and B. We name this transport process as the *max-flow model*, and denote the conductance as G_{MF}. Fast algorithms for solving the max-flow problem, given a network and a pair (A, B) are well known within the computer science community [28]. We apply those methods to random scale-free and Erdős–Rényi networks, and observe similarities and differences from the electrical conductance transport model. Max-flow analysis has been applied recently for complex networks in general [15,29], and for the Internet in particular [30], where it was used as a significant tool in the structural analysis of the underlying network.

We find that in the max-flow model, just as in the electrical conductance case, scale-free networks exhibit a power-law decay of the distribution of conductances with the same exponent (and thus very high conductance values are possible), while in Erdős–Rényi networks, the conductance decays exponentially (Fig. 8(a)). In order to better understand this behaviour, we plot the scaled-flow G^*_{MF}/k_B as a function of the scaled-degree $x \equiv k_A/k_B$ (Fig. 8(b)). It can be seen that the transition at $x = 1$ is sharp. For all $x < 1 (k_A < k_B)$, $G^*_{\mathrm{MF}} = x$ (or $G^*_{\mathrm{MF}} = k_A$), while for $x > 1 (k_B < k_A)$, $G^*_{\mathrm{MF}} = 1$ (or $G^*_{\mathrm{MF}} = k_B$). In other words, the conductance simply equals the minimum of the degrees of A and B. In the symbols of Eq. (7), this also implies that $c \to 1$; i.e. scale-free networks are optimal for transport in the max-flow sense. The derivation leading to Eq. (10) becomes then exact, so that the distribution of conductances is given again by $\Phi_{\mathrm{MF,SF}}(G_{\mathrm{MF}}) \sim G_{\mathrm{MF}}^{-(2\lambda-1)}$.

This picture of the transport is seen when the minimum degree in the network is $k_{\min} = 2$. When the minimum degree is allowed to take values in the range between 1 and 2 [31], we find that $G_{\mathrm{MF}} \propto \min\{k_A, k_B\}$, but the two quantities are no longer equal. This reflects the fact that as the minimum network

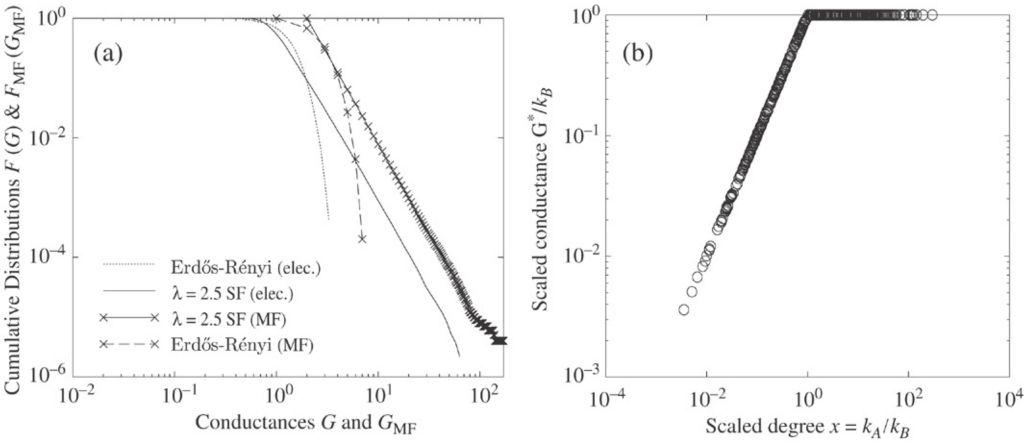

Fig. 8. (a) Cumulative distribution of link-independent paths (conductance) $F_{MF}(G_{MF})$ vs. G_{MF} compared with the electrical conductance distributions taken from Fig. 2. We see that the scaling is indeed the same for both models, but the proportionality constant of $F_{MF}(G_{MF})$ vs. G_{MF} is larger for the frictionless problem. (b) Scaled most probable number of independent paths G^*_{MF}/k_B as a function of the scaled degree k_A/k_B for scale-free networks of $N = 8000$, $\lambda = 2.5$ and $k_{min} = 2$. The behaviour is sharp, and shows how G^*_{MF} is a function of only the minimum k.

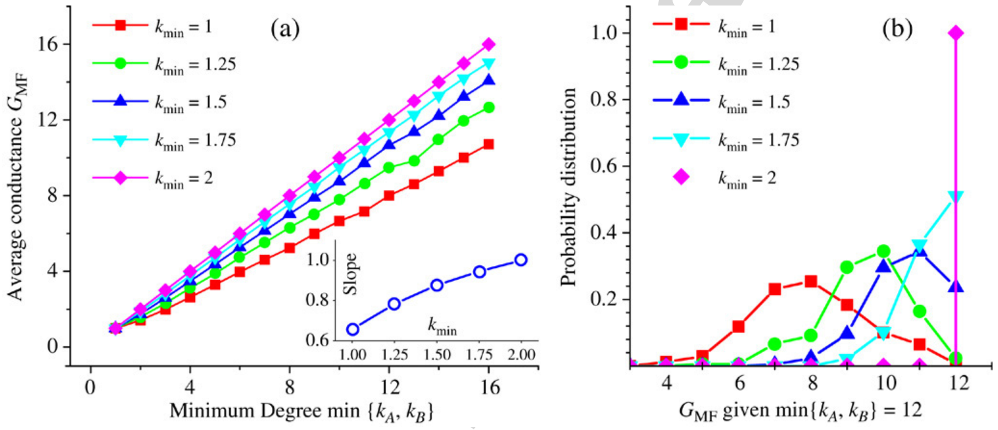

Fig. 9. (a) Average conductance $\overline{G}_{MF}$ vs. minimum degree of the source and sink A and B for different values of k_{min}, the minimum degree in the network. All curves show the behavior $\overline{G}_{MF} \propto k$, as the proportionality coefficient gradually increases (see inset), until eventually becomes 1 as k_{min} approaches 2. (b) The same concept is illustrated by plotting the probability to find a specific conductance G_{MF} when the minimum degree is 12, for few values of k_{min}.

degree is lowered, the network becomes more dilute, such that two paths starting at the source might intersect at some link inside the backbone. In other words, the conductance of the backbone is still high, but no longer infinite. This is illustrated in Fig. 9(a), where we plot the average conductance $\overline{G}_{MF}$ vs. the minimum degree of the source and sink min$\{k_A, k_B\}$, and find that while the relation between the two variables is linear, the slope is not necessarily 1. Nevertheless, as k_{min} approaches 2, the slope becomes 1, which indicates that a sufficient condition for the network to have infinite backbone conductivity is $k_{min} \geq 2$. This is illustrated again in Fig. 9(b), where the distribution of conductance values G_{MF} for fixed min$\{k_A, k_B\}$ is plotted.

We have so far observed that the max-flow model is quite similar to electrical conductance, by means of having a finite possibility of finding very high values of conductance. Also, the fact that the minimum degree plays a dominant role in the number of link-independent paths makes the scaling behaviour of the electrical and frictionless problems similar. Only when the conductances are studied as a function of distance, some differences between the electrical and frictionless cases begin to emerge. In Fig. 10(a), we plot the dependence of the average conductance $\overline{G}_{MF}$ with respect to the minimum degree min(k_A, k_B) of the source and sink, for different values of the shortest distance ℓ_{AB} between A and B, and find that $\overline{G}_{MF}$ is independent of ℓ_{AB} as the curves for different ℓ_{AB} overlap. This result is a consequence of the frictionless character of the max-flow problem. However, when we consider the electrical case, this independence disappears. This is illustrated in Fig. 10(b), where $\overline{G}$ is also plotted against the minimum degree min(k_A, k_B), but in this case, curves with different ℓ_{AB} no longer overlap. From the plot we find that $\overline{G}$ decreases as the distance increases. This is explained using the observation of [32], that the average shortest distance between the source and the sink is inversely proportional to the (logarithm of the product) of their degrees. Thus, on average, shorter distances are attributed to higher degrees, which in turn are connected by larger conductance.

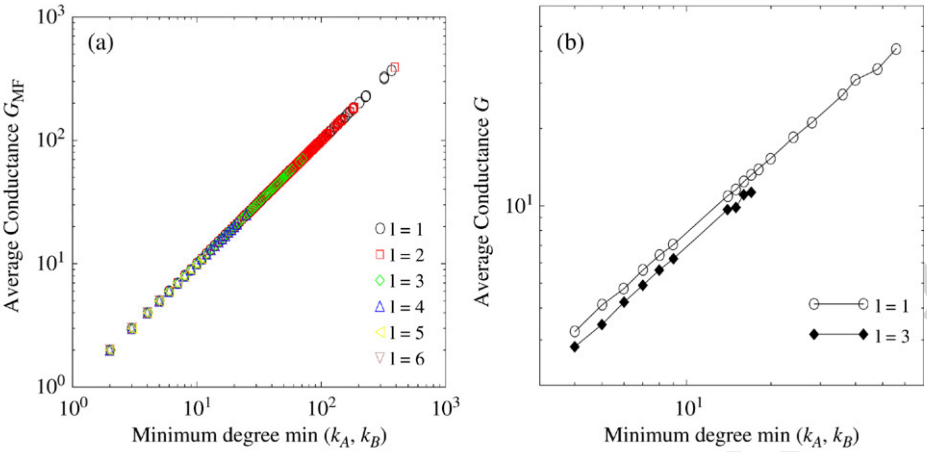

Fig. 10. (a) Average conductance $\overline{G}_{\mathrm{MF}}$ vs. minimum degree $\min(k_A, k_B)$ of the source and sink A and B for different values of the shortest distance ℓ_{AB}. The relation is independent of ℓ_{AB} indicating the independence of $\overline{G}_{\mathrm{MF}}$ on the distance. The network has $N = 8000$, $\lambda = 2.5$, $k_{\mathrm{min}} = 2$. (b) Average conductance $\overline{G}$ vs. minimum degree $\min(k_A, k_B)$ of the source and sink A and B for different values of distance ℓ_{AB}. The independence of $\overline{G}$ with respect to ℓ_{AB} breaks down and, as ℓ_{AB} increases, $\overline{G}$ decreases. Once again, $N = 8000$ and $\lambda = 2.5$, but the average has been performed for various $k_B < k_A$ and $k_A = 750$.

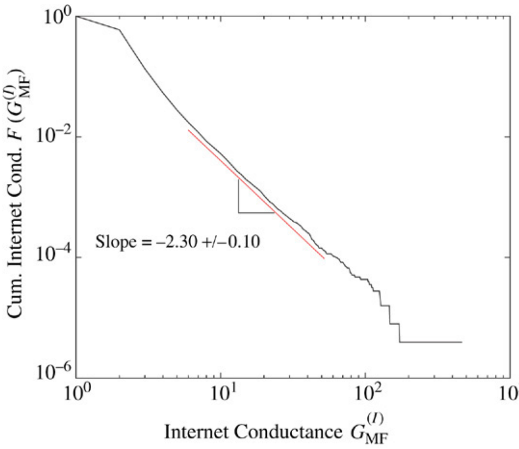

Fig. 11. Cumulative distribution $F(G^{(I)}_{\mathrm{MF}})$ of $G^{(I)}_{\mathrm{MF}}$ for the Internet. This data set is consistent with the scale-free structure that has been observed for the Internet (see text).

In order to test the validity of our results in real networks, we measured the conductance $G^{(I)}_{\mathrm{MF}}$ on the most up to date map of the Autonomous Systems (AS) level of the Internet structure [33]. From Fig. 11 we find that the slope of the plot, which corresponds to $g_G - 1$ from Eq. (10), is approximately 2.3, implying that $\lambda \approx 2.15 \pm 0.05$. This value of λ is in good agreement with the value of the degree distribution exponent for the Internet observed in [33].

5. Summary

In summary, we find that the conductance of scale-free networks is highly heterogeneous, and depends strongly on the degree of the two nodes A and B. Our results suggest that the transport constants are also heterogeneous in these networks, and depend on the degrees of the starting and ending nodes. We also find a power-law tail for $\Phi_{\mathrm{SF}}(G)$ and relate the tail exponent g_G to the exponent λ of the degree distribution $P(k)$. This power-law behaviour makes scale-free networks better for transport. Our work is consistent with a simple physical picture of how transport takes place in scale-free and Erdős–Rényi networks. This, so called ‘transport backbone’ picture consists of the nodes A and B and their vicinities, and the rest of the network, which constitutes the transport backbone. Because of the great number of parallel paths contained in the transport backbone, transport takes place inside with very small resistance, and therefore the dominating effect of resistance comes from the vicinity of the node (A or B) with the smallest degree. This scenario appears to be valid for both the electrical and frictionless models, as clearly indicated by the similarity in the results. The quantity c, which characterizes transport for a complex network exhibits a behaviour of the form $1 - \overline{k}^q$ for both scale-free and Erdős–Rényi networks in the electrical model, and in the frictionless model $c = 1$ in most cases. We observe that as $\overline{k}$ increases, scale-free networks become progressively better than Erdős–Rényi networks in electrical transport.

Finally, we point out that our study can be extended further. For instance, it has been found recently that many real-world scale-free networks possess fractal properties [34]. However, random scale-free and Erdős–Rényi networks, which are the subject of this study, do not display fractality. Since fractal substrates also lead to anomalous transport [1–3], it would be interesting to explore the effect of fractality on transport and conductance in fractal networks. This case is expected to have anomalous effects due to both the heterogeneity of the degree distribution and to the fractality of the network. Furthermore, the effect on conductivity and transport of the correlation between distance of two nodes and their degree [32] should be further investigated.

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References

[1] S. Havlin, D. ben-Avraham, Adv. Phys. 36 (1987) 695–798.

[2] D. ben-Avraham, S. Havlin, Diffusion and Reactions in Fractals and Disordered Systems, Cambridge, New York, 2000.

[3] A. Bunde, S. Havlin (Eds.), Fractals and Disordered Systems, Springer, New York, 1996.

[4] R. Albert, A.-L. Barabási, Rev. Modern Phys. 74 (2002) 47–97;
R. Pastor-Satorras, A. Vespignani, Structure and Evolution of the Internet: A Statistical Physics Approach, Cambridge University Press, Cambridge, 2004;
S.N. Dorogovsetv, J.F.F. Mendes, Evolution of Networks: From Biological Nets to the Internet and WWW, Oxford University Press, Oxford, 2003.

[5] P. Erdős, A. Rényi, Publ. Math. Debreccen 6 (1959) 290–297.

[6] B. Bollobás, Random Graphs, Academic Press, Orlando, 1985.

[7] A.-L. Barabási, R. Albert, Science 286 (1999) 509–512.

[8] P.L. Krapivsky, S. Redner, F. Leyvraz, Phys. Rev. Lett. 85 (2000) 4629–4632.

[9] H.A. Simon, Biometrika 42 (1955) 425.

[10] R. Cohen, K. Erez, D. ben-Avraham, S. Havlin, Phys. Rev. Lett. 85 (2000) 4626–4628.

[11] In principle, a node can have a degree up to $N - 1$, connecting to all other nodes of the network. The results presented here correspond to networks with upper cutoff $k_{\max} = k_{\min} N^{1/(\lambda-1)}$ imposed. We also studied networks for which $k_{\max}$ is not imposed, and found no significant differences in the pdf $\Phi_{\mathrm{SF}}(G)$.

[12] E. López, S.V. Buldyrev, S. Havlin, H.E. Stanley, Phys. Rev. Lett. 94 (2005) 248701.

[13] S. Havlin, E. López, S.V. Buldyrev, H.E. Stanley, Anomalous conductance and diffusion in complex networks, in: J. Kärger, F. Grinberg, P. Heitjans (Eds.), Diffusion Fundamentals, Universitätsverlag, Leipzig, 2005, pp. 38–48.

[14] G.R. Grimmett, H. Kesten, J. London Math. Soc. 30 (1984) 171–192. math/0107068.

[15] D.-S. Lee, H. Rieger, Europhys. Lett. 73 (2006) 471–477.

[16] Z. Toroczkai, K. Bassler, Nature 428 (2004) 716.

[17] J.D. Noh, H. Rieger, Phys. Rev. Lett. 92 (2004) 118701.

[18] V. Sood, S. Redner, D. ben-Avraham, J. Phys. A 38 (2005) 109–123.

[19] L.K. Gallos, Phys. Rev. E 70 (2004) 046116.

[20] The dynamical properties we study are related to transport on networks and differ from those which treat the network topology itself as evolving in time [7,8].

[21] The study of community structure in social networks has led some authors (M. E. J. Newman and M. Girvan, Phys. Rev. E 69 (2004) 026113, and F. Wu and B. A. Huberman, Eur. Phys. J. B 38 (2004) 331–338) to develop methods in which networks are considered as electrical networks in order to identify communities. In these studies, however, transport properties have not been addressed.

[22] M. Molloy, B. Reed, Random Structures Algorithms 6 (1995) 161–179.

[23] We performed simulations with the node copies randomly matched, and also matched in order of degree from highest to lowest and obtained similar results.

[24] G. Kirchhoff, Ann. Phys. Chem. 72 (1847) 497;
N. Balabanian, Electric Circuits, McGraw-Hill, New York, 1994.

[25] Flux starts at node A, and it is controlled by the conductance of the bonds in the vicinity of A. This flux passes into the “transport backbone”, which is comprised of many parallel paths and hence has a high conductance. Finally, flux ends at node B, controlled by the conductance of the bonds in the vicinity of B (see inset of Fig. 4). This is similar to traffic around a major freeway. Most of the limitations to transport occur in getting to the freeway (“node A”) and then after leaving it (“node B”), but flow occurs easily on the freeway (“transport backbone”).

[26] The λ values explored here are limited by computer time considerations. Most real-world examples of scale-free networks exhibit exponents that satisfy $2 \le \lambda \le 4$ and are thus within the range of values we simulate. In the case of very large λ, and for $k_{\min}$ large enough, scale-free networks gradually become similar to Erdős–Rényi networks, and Eq. (7) continues to hold, but the power-law behavior for $\Phi(G)$ disappears. When $k_{\min}$ is small (close to 1), large λ induces a behavior close to percolation, where Eq. (7) no longer holds due to the tree-like structure of the connected components.

[27] S. Kirkpatrick, Percolation thresholds in granular films—non-universality and critical current, in: W. Va, S.A. Wolf, D.U. Gubser (Eds.), Proceedings of Inhomogeneous Superconductors Conference, Berkeley Springs, A.I.P. Conf. Procs. 58 (1979) 79.

[28] B.V. Cherkassky, Algorithmica 19 (1997) 390.

[29] Z. Wu, L.A. Braunstein, S. Havlin, H.E. Stanley, Phys. Rev. Lett. 96 (2006) 148702.

[30] S. Carmi, S. Havlin, S. Kirkpatrick, Y. Shavitt, E. Shir, MEDUSA—New model of internet topology using k-shell decomposition. arXiv.cond-mat/0601240.

[31] In order to obtain a value of mimimum degree between 1 and 2, realizations of networks are generated that have a probability of having mimimum degree either 1 or 2. Statistically, one can do this in such a way as to control the average value of the mimimum degree by controling the proportion of networks created with mimimum degree 1 compared to those created with mimimum degree 2, as is done here.

[32] J.A. Holyst, J. Sienkiewicz, A. Fronczak, P. Fronczak, K. Suchecki, Physica A 351 (2005) 167–174.

[33] Y. Shavitt, E. Shir, ACM SIGCOMM, Comput. Commun. Rev. 35 (5) (2005) 71.

[34] C.M. Song, S. Havlin, H.A. Makse, Nature 433 (2005) 392–395.