

Scale-free networks emerging from weighted random graphs

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We study Erdős-Rényi random graphs with random weights associated with each link. We generate a “supernode network” by merging all nodes connected by links having weights below the percolation threshold (percolation clusters) into a single node. We show that this network is scale-free, i.e., the degree distribution is $P(k) \sim k^{-\lambda}$ with $\lambda=2.5$. Our results imply that the minimum spanning tree in random graphs is composed of percolation clusters, which are interconnected by a set of links that create a scale-free tree with $\lambda=2.5$. We suggest that optimization causes the percolation threshold to emerge spontaneously, thus creating naturally a scale-free supernode network. We discuss the possibility that this phenomenon is related to the evolution of several real world scale-free networks.

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Scale-free topology is very common in natural and man-made networks. Examples vary from social contacts between humans to technological networks such as the World Wide Web or the Internet [1–3]. Scale-free (SF) networks are characterized by a power-law distribution of connectivities $P(k) \sim k^{-\lambda}$, where k is the degree of a node and the exponent λ controls the broadness of the distribution. Many networks are observed to have values of λ around 2.5. For values of $\lambda < 3$ the second moment of the distribution $\langle k^2 \rangle$ diverges, leading to several anomalous properties [4].

In many real world networks there is a “cost” or a “weight” associated with each link, and the larger the weight on a link, the harder it is to traverse this link. In this case, the network is called a weighted network [5]. Examples can be found in communication and computer networks, where the weights represent the bandwidth or delay time, in protein networks where the weights can be defined by the strength of interaction between proteins [6,7] or their structural similarity [8], and in sociology where the weights can be chosen to represent the strength of a relationship [9,10].

In this paper we present a simple process that generates random scale-free networks with $\lambda=2.5$ from weighted Erdős-Rényi graphs [11]. We further show that the minimum spanning tree (MST) on an Erdős-Rényi graph is related to this network, and is composed of percolation clusters, which we regard as “supernodes,” interconnected by a scale-free tree. We will see that due to optimization, this scale-free tree is dominated by links having high weights—significantly higher than the percolation threshold p_c . Hence, the MST naturally distinguishes between links below and above the percolation threshold, leading to a scale-free “supernode network.” Our results may explain the origin of scale-free degree distribution in some real world networks.

Consider an Erdős-Rényi (ER) graph with N nodes and an average degree $\langle k \rangle$, thus having a total of $N\langle k \rangle/2$ links. To each link we assign a weight chosen randomly and uniformly from the range $[0, 1]$. We define black links to be those links with weights below a threshold $p_c = 1/\langle k \rangle$ [11]. Two nodes belong to the same cluster if they are connected by black links [Fig. 1(a)]. From percolation theory [12] follows that the number of clusters of s nodes scales as a power law, $n_s \sim s^{-\tau}$, with $\tau=2.5$ for ER networks [13]. Next, we merge all nodes inside each cluster into a single supernode. We define a new supernode network [Fig. 1(b)] of N_{sn} supernodes [14]. The links between two supernodes [see Figs. 1(a) and 1(b)] have weights *larger* than p_c .

The degree distribution $P(k)$ of the supernode network can be obtained as follows. Every node in a supernode has the same (finite) probability to be connected to a node outside the supernode. Thus we assume that the degree k of each supernode is proportional to the cluster size s , which obeys $n_s \sim s^{-2.5}$. Hence $P(k) \sim k^{-\lambda}$, with $\lambda=2.5$, as supported by simulations shown in Fig. 2. Furthermore, we also see that if the threshold for obtaining the clusters that are merged into supernodes is changed slightly, the degree distribution still remains scale-free with $\lambda=2.5$, but with an exponential cut-off. This is an indication of the fact that there are still supernodes of high degree that are connected to many other (small) supernodes by links with weights significantly higher than p_c ; if this was not the case, a small change in the threshold would cause many clusters to merge and destroy the power law in the supernode network degree distribution.

We next show that the minimum spanning tree (MST) on an ER graph is related to the supernode network, and therefore also exhibits scale-free properties. The MST on a weighted graph is a tree that reaches all nodes of the graph and for which the sum of the weights of all the links (total weight) is minimal. Also, each path between two sites on the MST is the optimal path in the “strong disorder” limit

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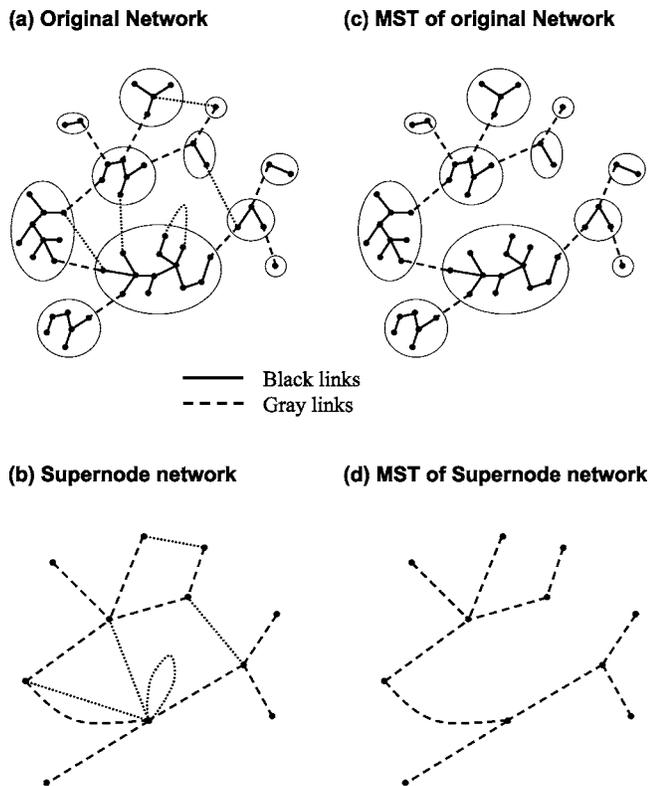


FIG. 1. Sketch of the supernode network. (a) The original ER network, partitioned into percolation clusters whose sizes s are power-law distributed, with $n_s \sim s^{-\tau}$ where $\tau=2.5$ for ER graphs. The black links are the links with weights below p_c , the dotted links are the links that are removed by the bombing algorithm, and the gray links are the links whose removal will disconnect the network (and therefore are not removed even though their weight is above p_c). (b) The supernode network: the nodes are the clusters in the original network and the links are the links connecting nodes in different clusters (i.e., dotted and gray links). The supernode network is scale-free with $P(k) \sim k^{-\lambda}$ and $\lambda=2.5$. Notice the existence of self loops and of double connections between the same two supernodes. (c) The minimum spanning tree (MST), composed of black and gray links only. (d) The MST of the supernode network (gray tree), which is obtained by bombing the supernode network (thereby removing the dotted links), or equivalently, by merging the clusters in the MST to supernodes. The gray tree is scale-free, with $\lambda=2.5$.

[15,16], meaning that along this path the maximum *barrier* (weight) is the smallest possible [14,16,17].

Standard algorithms for finding the MST [18] are Prim's algorithm, which resembles invasion percolation, and Kruskal's algorithm, which resembles percolation. An equivalent algorithm to find the MST is the bombing optimization algorithm [16,17]. We start with the full ER network and remove links in order of descending weights. If the removal of a link disconnects the graph, we restore the link and mark it gray [19]; otherwise the link [shown dotted in Fig. 1(a)] is removed. The algorithm ends and a MST is obtained when no more links can be removed without disconnecting the graph.

In the bombing algorithm, only links that close a loop can be removed. Below criticality loops are negligible for ER

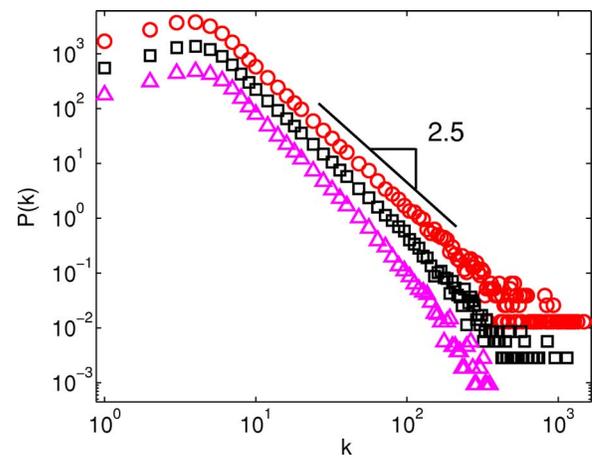


FIG. 2. (Color online) The degree distribution of the supernode network of Fig. 1(b), where the supernodes are the percolation clusters, and the links are the links with weights larger than p_c (\circ). The distribution exhibits a scale-free tail with $\lambda \approx 2.5$. If we choose a threshold less than p_c , we obtain the same power-law degree distribution with an exponential cutoff. The different symbols represent slightly different threshold values: $p_c - 0.03$ (\square) and $p_c - 0.05$ (\triangle). The original ER network has $N=50\,000$ and $\langle k \rangle=5$. Note that for $k \approx \langle k \rangle$ the degree distribution has a maximum.

networks ($d \rightarrow \infty$) [11,12]. Therefore, bombing does not modify the percolation clusters—where the links have weights below p_c . Thus bombing modifies only links outside the clusters, so actually it is only the links of the supernode network that are bombed. Hence the MST resulting from bombing is composed of percolation clusters connected by gray links [Fig. 1(c)].

From the MST of Fig. 1(c) we now generate a new tree, the MST of the supernode network, which we call the “gray tree,” whose nodes are the supernodes and whose links are the gray links connecting them [see Fig. 1(d)]. Note that bombing the original ER network to obtain the MST of Fig. 1(c) is equivalent to bombing the supernode network of Fig. 1(b) to obtain the gray tree, because the links inside the clusters are not bombed. We find [Fig. 3(a)] that the gray tree has also a scale-free degree distribution $P(k)$, with $\lambda=2.5$ —the same as the supernode network [20]. We also find [Fig. 3(b)] the average path length ℓ_{gray} scales as $\ell_{\text{gray}} \sim \ln N_{\text{sn}} \sim \ln N$ [14,21]. Note that even though the gray tree is scale-free, it is not ultrasmall [4], since the length does not scale as $\log N$.

Next we show that the bombing optimization, which leads to the MST, yields a significant separation between the weights of the links inside the supernodes and the links connecting the supernodes. As explained above, the MST is optimal in two senses: (i) the total weight of all links is minimal and (ii) any path between any two nodes on the MST will encounter the smallest maximal barrier (weight) between these nodes. The last property is common to many physical systems [e.g., the protein folding network (see below)]. Accordingly, we study the weights encountered when traveling along a typical path on the MST.

We consider all pairs of nodes in the original MST of N nodes [Fig. 1(c)] and calculate the typical path length ℓ_{typ} ,

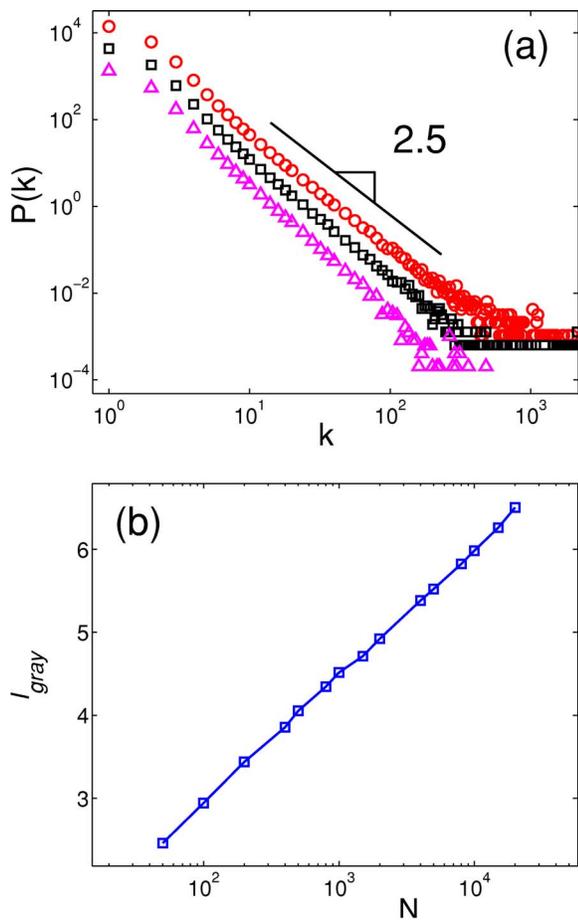


FIG. 3. (Color online) (a) The degree distribution of the gray tree [the MST of the supernode network, shown in Fig. 1(d)], in which the supernodes are percolation clusters and the links are the gray links. Different symbols represent different threshold values: p_c (\circ), $p_c+0.01$ (\square), and $p_c+0.02$ (\triangle). The distribution exhibits a scale-free tail with $\lambda \approx 2.5$, and is relatively insensitive to changes in p_c . (b) The average path length ℓ_{gray} on a the gray tree as a function of original network size. It is seen that $\ell_{\text{gray}} \sim \ln N_{\text{sn}} \sim \ln N$.

which is the most probable path length on the MST. For each path of length ℓ_{typ} we rank the weights on its links in descending order. For the largest weights (“rank 1 links”), we calculate the average weight $\bar{w}_{r=1}$ over all paths. Similarly, for the next largest weights (“rank 2 links”) we find the average $\bar{w}_{r=2}$ over all paths, and so on up to $r = \ell_{\text{typ}}$. Figure 4(a) shows \bar{w}_r as a function of rank r for three different network sizes $N=2000$, 8000, and 32 000. It can be seen that weights below p_c (black links inside the supernodes) are uniformly distributed and approach one another as N increases. As opposed to this, weights above p_c (“gray links”) are *not* uniformly distributed, due to the bombing algorithm, and are independent of N . Actually, weights above p_c encountered along the optimal path (such as the largest weights w_1 , w_2 , and w_3) are significantly higher than those below p_c . Figure 4(b) shows that the links with the highest weights on the MST can be associated with gray links from very small clusters [Figs. 1(a) and 1(c)] (similar results have been obtained along the optimal path).

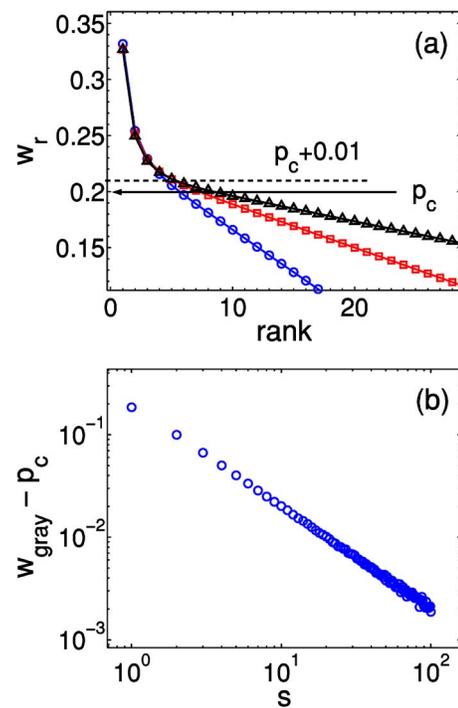


FIG. 4. (Color online) (a) The average weights \bar{w}_r along the optimal path of an ER graph with $\langle k \rangle = 5$, sorted according to their rank. Different symbols represent different system sizes: $N=2000$ (\circ), $N=8000$ (\square), and $N=32\,000$ (\triangle). Below $p_c=0.2$, the weights are uniformly distributed, while weights above p_c are significantly higher and independent of N . (b) Cluster size vs the minimal gray link emerging from each cluster, for ER graphs with $\langle k \rangle = 5$ and $N=10\,000$. Small clusters are associated with higher weights because they have a small number of exits and thus cannot be optimized.

As mentioned earlier, this property is present also in the original supernode network and hence the change in the threshold used to obtain the supernodes does not destroy the power-law degree distribution but only introduces an exponential cutoff. We thereby obtain a scale-free supernode network with $\lambda=2.5$, which is not very sensitive to the precise value of the threshold used for defining the supernodes. For example, the scale-free degree distribution shown in Fig. 3(a) for a threshold of $p_c+0.01$ corresponds to having only four largest weights on the optimal paths [see Fig. 4(a)]. However, even for $p_c+0.02$ the degree distribution is well approximated by a scale-free distribution with $\lambda=2.5$ [see Fig. 3(a)]. This means that mainly very small clusters, connected with high-weight links to large clusters, dominate the scale-free distribution $P(k)$ of the MST of the supernode network (gray tree). Hence, the bombing optimization process on an ER graph causes a significant separation between links below and above p_c to emerge spontaneously in the system, and by merging nodes connected with links of low weights, a scale-free network can arise.

The process described above may be related to the evolution of some real world networks. Consider a homogeneous network with many components whose average degree $\langle k \rangle$ is well defined. Suppose that the links between the components have different weights, and that some optimization process

separates the network into nodes that are well connected (i.e., connected by links with low weights) and nodes connected by links having much higher weights. If the well-connected components merge into a single node, this results in a new heterogeneous supernode network with components that vary in size, and thus in number of outgoing connections.

An example of a real world network whose evolution may be related to this model is the protein folding network, which was found to be scale-free with $\lambda \approx 2.3$ [7]. The nodes are the possible physical configurations of the system and the links between them describe the possible transitions between the different configurations. We assume that this network is optimal because the system chooses the path with the smallest energy barrier from all possible trajectories in phase space. It is possible that the scale-free distribution evolves through a similar procedure as described above for random graphs: Adjacent configurations with close energies (nodes in the same cluster) cannot be distinguished and are regarded as a single supernode, while configurations (clusters) with high barriers between them belong to different supernodes.

A second example is computer networks. Strongly interacting computers (such as computers belonging to researchers from the same company or research institution) are likely to converge into a single domain, and thus domains with various sizes and connectivities are formed. This network

might be also optimal, because packets destined to an external domain are presumably routed through the router that has the best connection to the target domain.

To summarize, we have seen that any weighted random network hides an inherent scale-free supernode network [22]. We showed that the minimum spanning tree, generated by the bombing algorithm, is composed of percolation clusters connected by a scale-free tree of gray links. Most of the gray links connect small clusters to large ones, thus having weights well above the percolation threshold that do not change with the original size of the network. Thus the optimization in the process of building the MST distinguishes between links with weights below and above the threshold, leading to a spontaneous emergence of a scale-free supernode network. We raise the possibility that in some naturally optimal real-world networks, nodes connected well merge into one single node, and thus a scale-free network emerges.

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